

## 4. INTERSECTIONS OF PLANES

### I Main Topics

A Equation of a plane

B Pole to a plane using cross-products

C Intersection of two planes in a line  
(apparent dip problem)

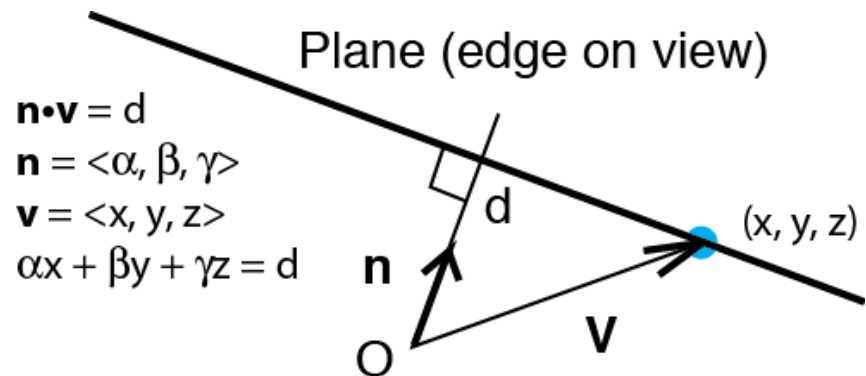
D Intersection of three planes in a point  
(solution of simultaneous linear equations)

# 4. INTERSECTIONS OF PLANES

## II Equation for plane: $\mathbf{n} \cdot \mathbf{v} = d$

### A Meaning

- 1 The distance from a known reference point to a plane, as measured in the direction of a unit vector  $\mathbf{n}$  normal to the plane, is  $d$ .
- 2  $d > 0$  if  $\mathbf{n}$  points from the reference point towards the plane;
- 3  $d < 0$  if  $\mathbf{n}$  points from the plane towards the reference point.



## 4. INTERSECTIONS OF PLANES

II Equation for plane:  $\mathbf{n} \bullet \mathbf{v} = d$  (cont.)

B Solution for  $\mathbf{n} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} = \langle \alpha, \beta, \gamma \rangle$

1 If x=north, y=east, z=down

$$\alpha = \cos\phi\cos\theta, \beta = \cos\phi\sin\theta, \gamma = \sin\phi$$

2 If x=east, y=north, z=up

$$\alpha = \cos\phi\sin\theta, \beta = \cos\phi\cos\theta, \gamma = -\sin\phi$$

where  $\theta$  is the pole trend and  $\phi$  is the pole plunge

3 Matlab: `[alpha,beta,gamma]=sph2cart(trend,plunge,1)`

Trend and plunge must be in radians.

4 Matlab: `[trend,plunge,R]=cart2sph(alpha,beta,gamma)`

5 **Good for z = down**; trend and plunge must be in radians.

## 4. INTERSECTIONS OF PLANES

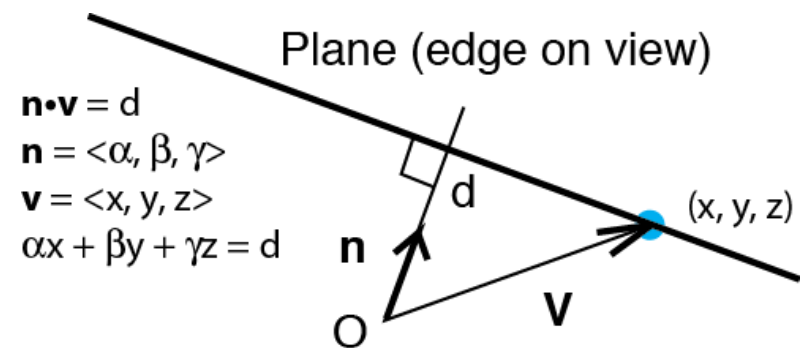
II Equation for plane:  $\mathbf{n} \cdot \mathbf{v} = d$  (cont.)

C  $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \langle x, y, z \rangle$

D Solution for  $d$

1  $d = \alpha x + \beta y + \gamma z$   
(Normal form)

2  $d = \text{dot}(\mathbf{n}, \mathbf{v})$



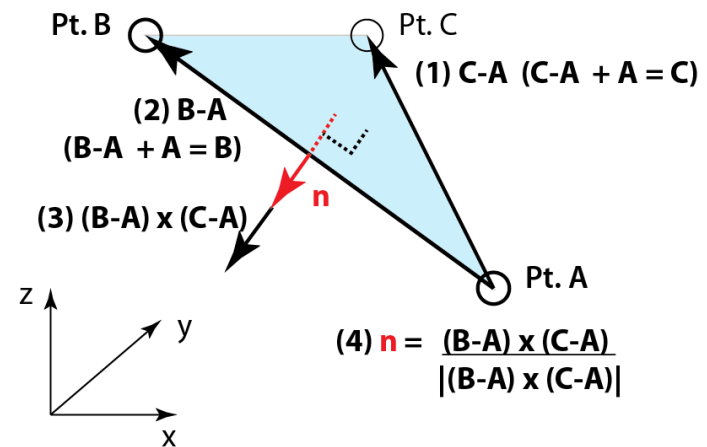
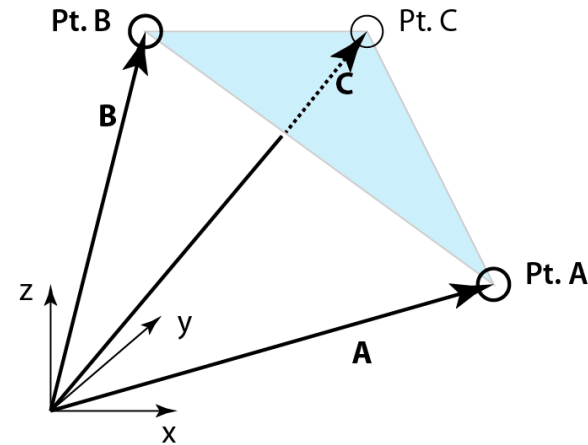
# 4. INTERSECTIONS OF PLANES

## III Pole to a plane using cross-products

A Consider three non-coplanar points A, B, C that form a clockwise circuit as viewed from above

B  $(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A})$  is normal to the plane

C  $\mathbf{n} = \frac{(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A})}{|(\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A})|}$ ,  
unit normal to the plane.

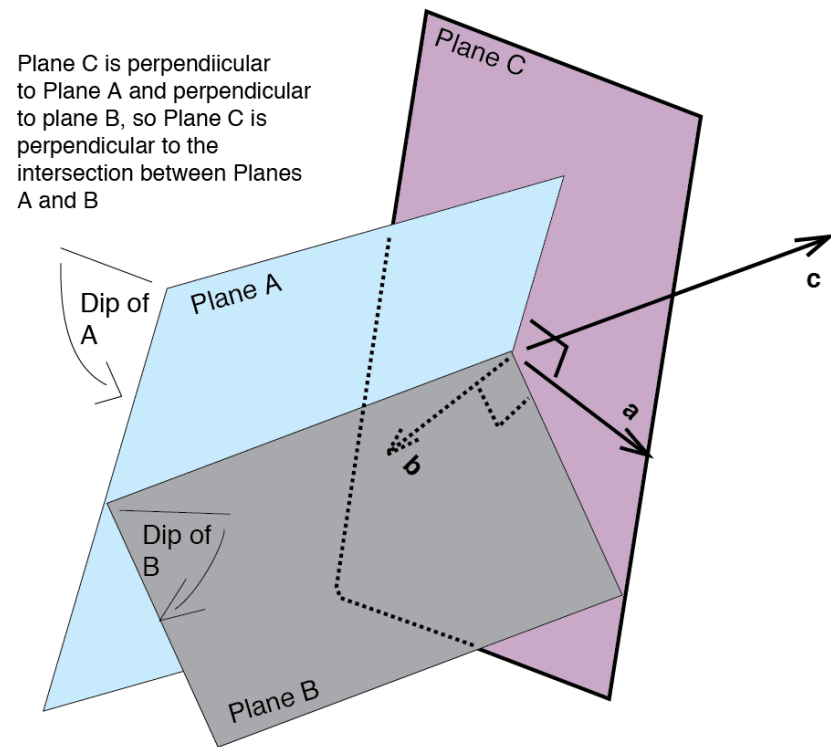


# 4. INTERSECTIONS OF PLANES

## IV Intersection of two planes in a line

- A Two planes A and B intersect in a line. The equations of those two planes define the line.
- B The direction of intersection is along the vector  $\mathbf{a} \times \mathbf{b}$ , where  $\mathbf{a}$  is normal to plane A, and  $\mathbf{b}$  is normal to plane B

Direction of Intersection of Two Planes



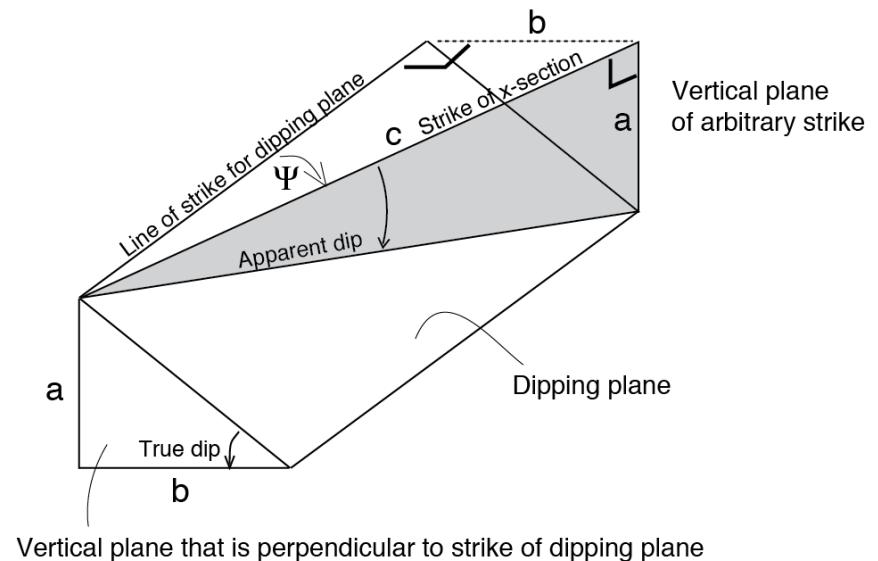
$\mathbf{a}$  is unit normal to Plane A and lies in Plane C  
 $\mathbf{b}$  is unit normal to Plane B and lies in Plane C  
 $\mathbf{c}$  is unit normal to Plane C  
 $\mathbf{c} = \mathbf{a} \times \mathbf{b} / |\mathbf{a} \times \mathbf{b}|$

# 4. INTERSECTIONS OF PLANES

## IV Intersection of two planes in a line (cont.)

### C Apparent dip

- 1 How a geologic plane appears to dip in a vertical cross section of arbitrary strike
- 2 Plunge of the line of intersection between a geologic plane and a vertical cross section plane of arbitrary strike
- 3 Plunge of the cross product of two vectors
  - a Vector normal to a geologic plane
  - b Vector normal to vertical cross section plane of arbitrary strike
  - c Make sure cross product points down to get a pole



# Apparent Dip

- 1)  $a/b = \tan(\text{true dip})$ .
- 2)  $a = b \tan(\text{true dip}) = c \tan(\text{app. dip})$ . So
- 3)  $b \tan(\text{true dip}) = c \tan(\text{app. dip})$

Now consider the angle  $\Psi$  between the line of strike of the cross section and the line of strike for the dipping plane.

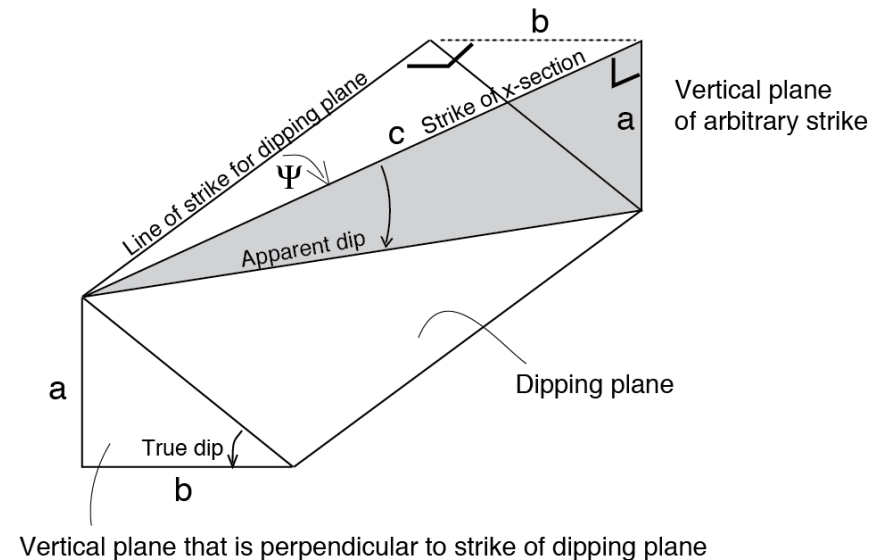
- 4)  $b/c = \sin\Psi$ , so
- 5)  $c = b/\sin\Psi$ .

Now insert (5) into the right side of (3)

- 6)  $b \tan(\text{true dip}) = (b/\sin\Psi) \tan(\text{app. dip})$
- 7)  $\text{app. dip} = \tan^{-1}\{[\tan(\text{true dip})][\sin\Psi]\}$

Since  $\sin\Psi \leq 1$ ,  $\text{app. dip} \leq \text{true dip}$ .

If  $\Psi = 90^\circ$ , then  $\text{app. dip} = \text{true dip}$ .



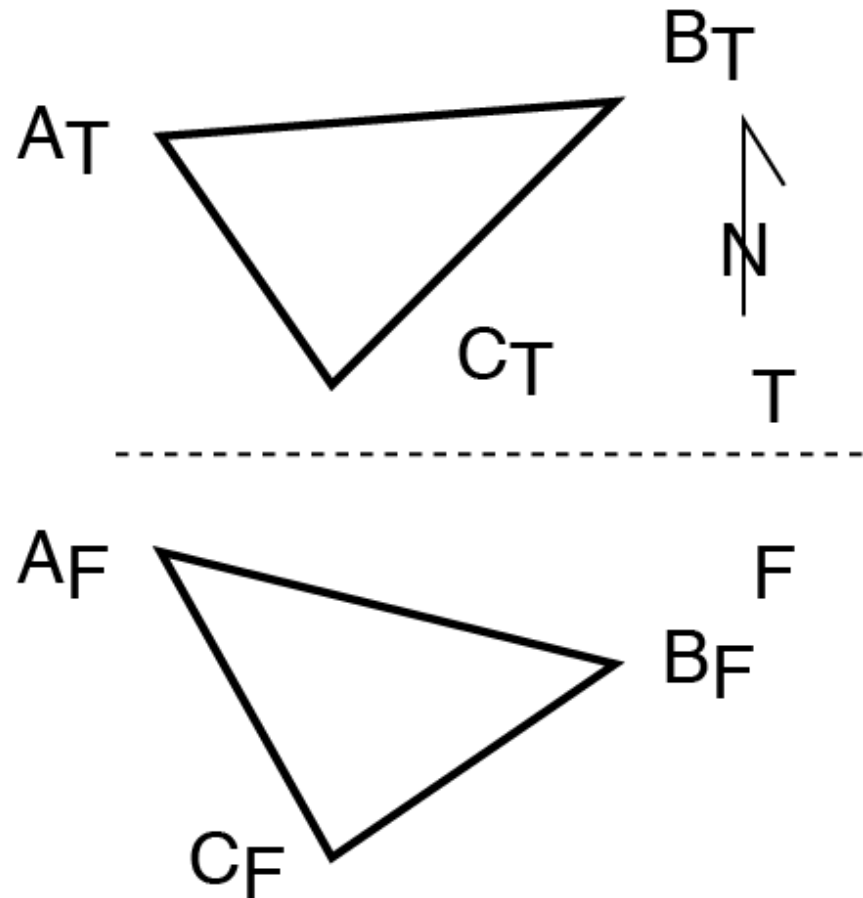


## 4. INTERSECTIONS OF PLANES

IV Intersection of two planes  
in a line (cont.)

D Apparent dip and  
orthographic projection  
example

Find the apparent dip  
of plane ABC in a  
vertical cross  
section that strikes N45W.



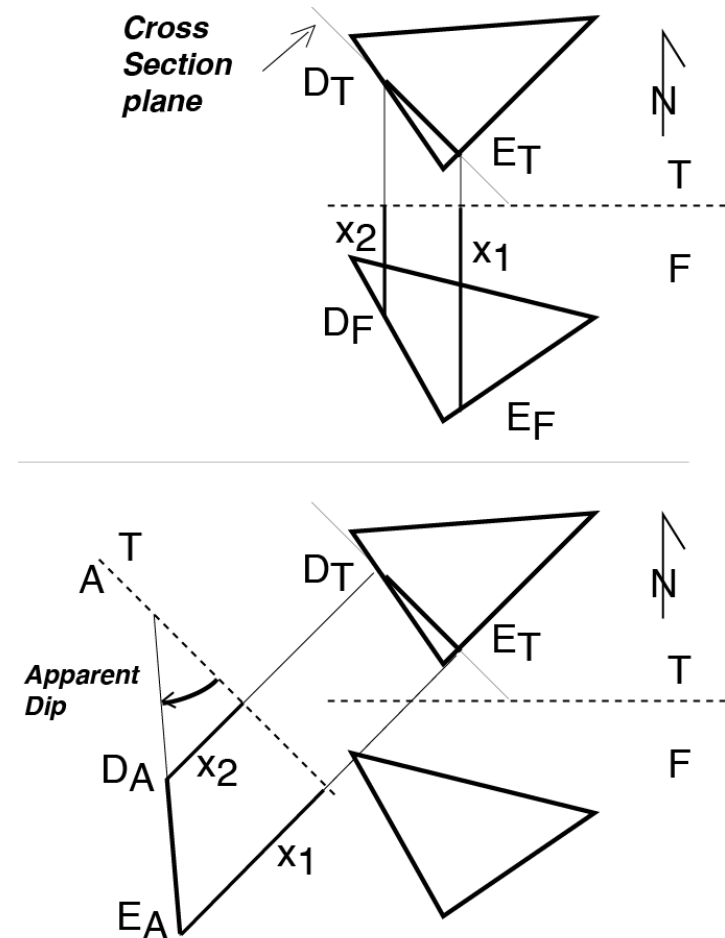
# 4. INTERSECTIONS OF PLANES

C Apparent dip and orthographic projection example

(1) In the top view, intersect the cross section plane with plane ABC. Call the line of intersection DE. Point D is on line AC. Point E is on line BC.

Next we find the plunge of line DE.

(2) Project the line of intersection DE onto an auxiliary view that is taken parallel to the line of cross section. The plunge of line DE is the apparent dip of plane ABC as seen in the cross section.



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V Intersection of features in a point

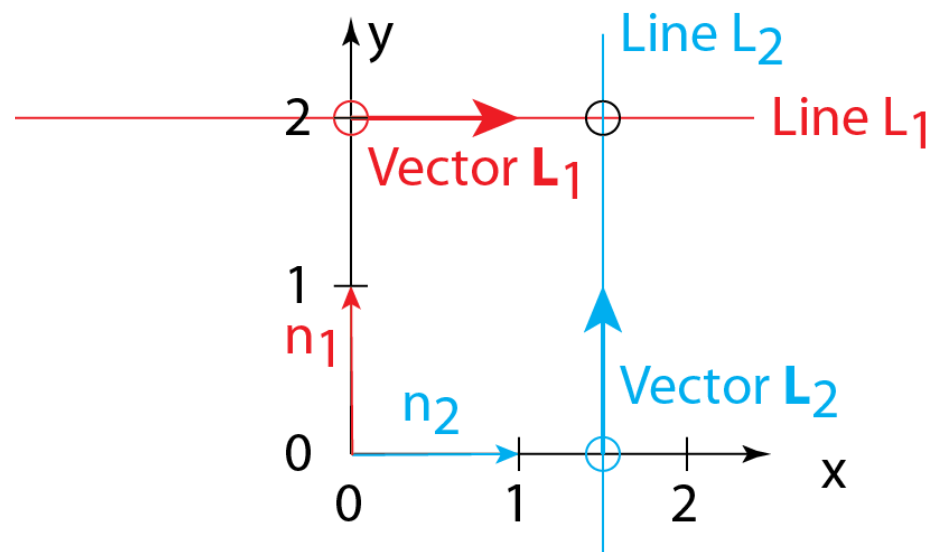
A Intersection of two lines ( $L_1$  and  $L_2$ ) at a point

1 Intersection criteria

a Vectors along the lines are not parallel  
(i.e.,  $\vec{L}_1 \times \vec{L}_2 \neq 0$ )

b Vectors along the lines (and normals to the lines) lie in the same plane:

$$(\vec{n}_1 \times \vec{L}_1) \times (\vec{n}_2 \times \vec{L}_2) = 0$$



## 4. INTERSECTIONS OF PLANES

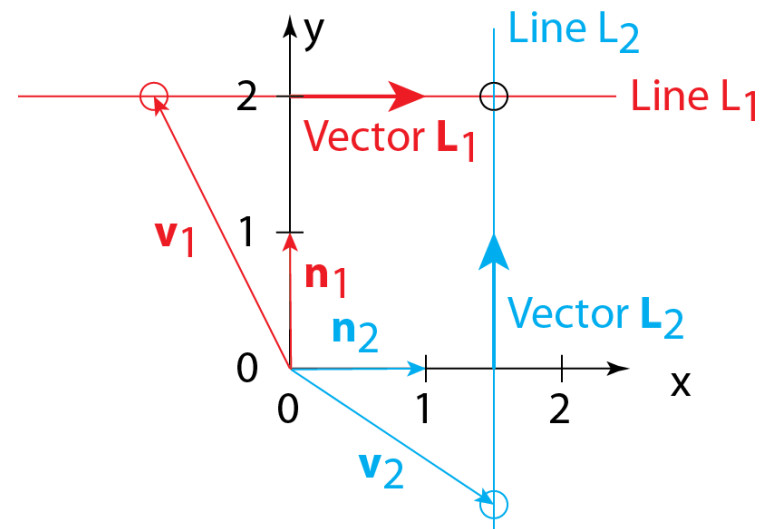
### A Intersection of two lines (cont.)

- 2 Write equations for two lines ( $n \bullet v = d$ )

$$\vec{n}_1 \bullet \vec{v}_1 = a_{n_1x}x + a_{n_1y}y = d_1$$

$$\vec{n}_2 \bullet \vec{v}_2 = a_{n_2x}x + a_{n_2y}y = d_2$$

- 3 Find  $n_1, n_2, d_1, d_2$



Example

$$(0)(-1) + (1)(2) = 2 = d_1$$

$$(1)(1.5) + (0)(-1) = 1.5 = d_2$$

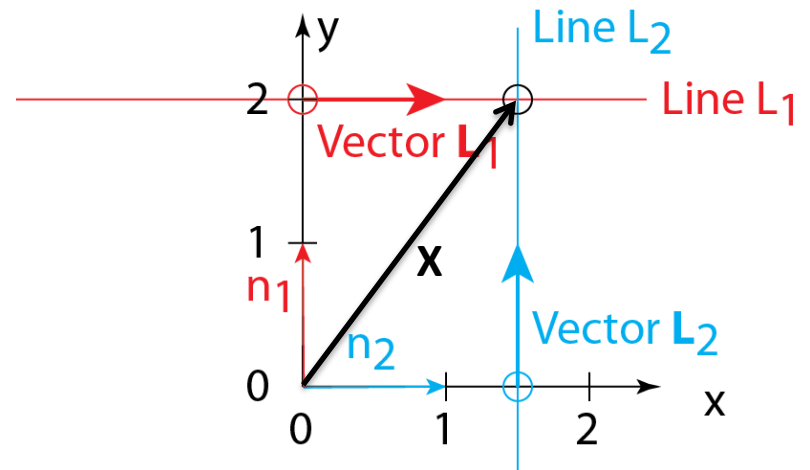
## 4. INTERSECTIONS OF PLANES

### A Intersection of two lines (cont.)

- 3 Write equations for lines in matrix form

$$\begin{bmatrix} a_{n_1x} & a_{n_1y} \\ a_{n_2x} & a_{n_2y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$
$$[A] \quad [X] = [B]$$

We seek the vector  $[X]$  that satisfies both equations (i.e., a point  $(x,y)$  on both lines)



### Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$$
$$[A] \quad [X] = [B]$$

# 4. INTERSECTIONS OF PLANES

## A Intersection of two lines (cont.)

4 Solve for  $[X]$

$$[A][X]=[B]$$

$$[A]^{-1}[A][X]=[A]^{-1}[B]$$

$$[X]=[A]^{-1}[B]$$

### a Matlab

```
>> A = [0 1;1 0]
```

```
A =
```

```
0 1
```

```
1 0
```

```
>> B = [2;1.5]
```

```
B =
```

```
2.0000
```

```
1.5000
```

```
>> X = A\B
```

```
X =
```

```
1.5000
```

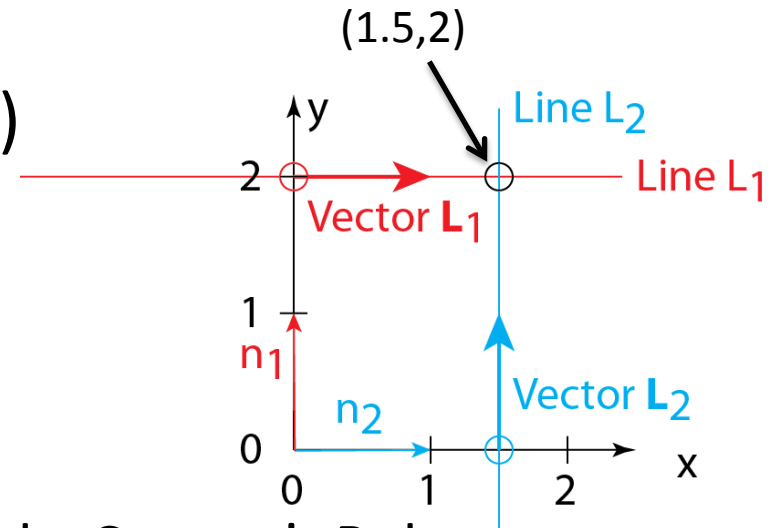
```
2.0000
```

```
>> X=inv(A)*B
```

```
X =
```

```
1.5000
```

```
2.0000
```



### b Cramer's Rule

$$x = \frac{\begin{vmatrix} B_1 & A_{12} \\ B_2 & A_{22} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}} = \frac{\begin{vmatrix} 1.5 & 1 \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{-2}{-1} = 2$$

$$y = \frac{\begin{vmatrix} A_{11} & B_1 \\ A_{21} & B_2 \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 1.5 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}} = \frac{-1.5}{-1} = 1.5$$

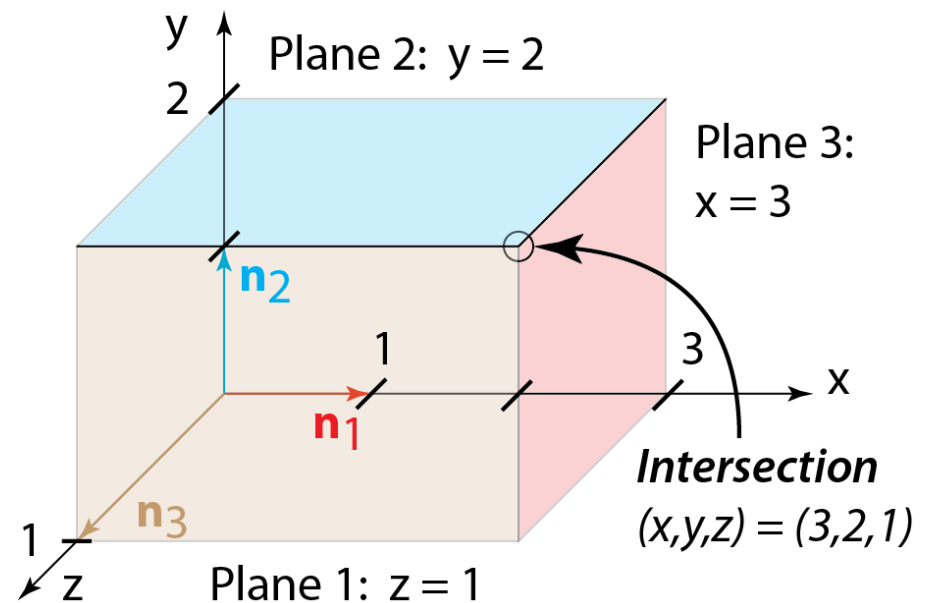
# 4. INTERSECTIONS OF PLANES

B Intersection of three plane ( $P_1$ ,  $P_2$ , and  $P_3$ ) at a point

## 1 Intersection criteria

Normals of any planes are not parallel; scalar triple product of normals to planes  $\neq 0$

(i.e.,  $n_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0$ )



# 4. INTERSECTIONS OF PLANES

## B Intersection of three planes (cont.)

2 Write equations for three planes  
( $\mathbf{n} \cdot \mathbf{v} = d$ )

$$\vec{n}_1 \cdot \vec{v}_1 = a_{n_1x}x + a_{n_1y}y + a_{n_1z}z = d_1$$

$$\vec{n}_2 \cdot \vec{v}_2 = a_{n_2x}x + a_{n_2y}y + a_{n_2z}z = d_2$$

$$\vec{n}_3 \cdot \vec{v}_3 = a_{n_3x}x + a_{n_3y}y + a_{n_3z}z = d_3$$

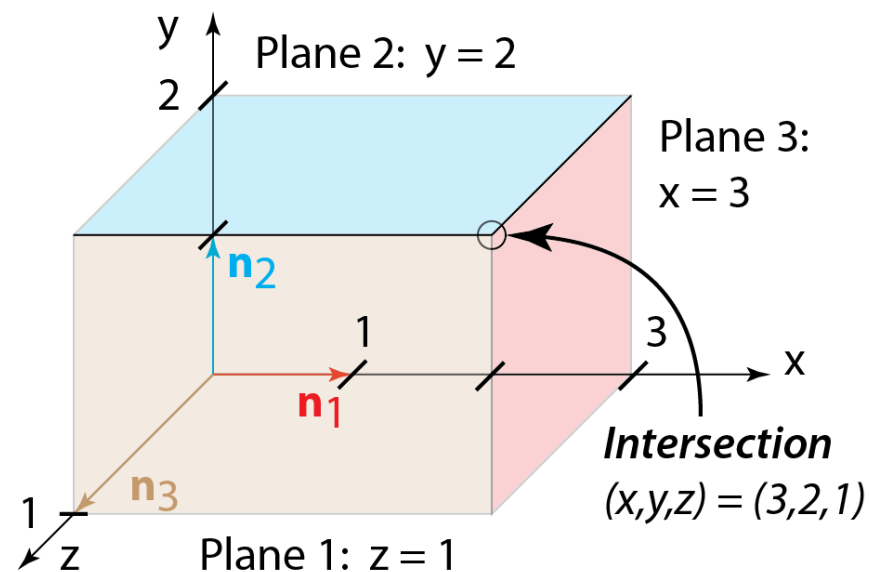
3 Find  $n_1, n_2, n_3, d_1, d_2, d_3$

Example

$$n_1 = 1\vec{i} + 0\vec{j} + 0\vec{k}$$

$$n_2 = 0\vec{i} + 1\vec{j} + 0\vec{k}$$

$$n_3 = 0\vec{i} + 0\vec{j} + 1\vec{k}$$



$$(1)(3) + (0)(y) + (0)(z) = 3 = d_1$$

$$(0)(x) + (1)(2) + (0)(z) = 2 = d_2$$

$$(0)(x) + (0)(y) + (1)(1) = 1 = d_3$$



# 4. INTERSECTIONS OF PLANES

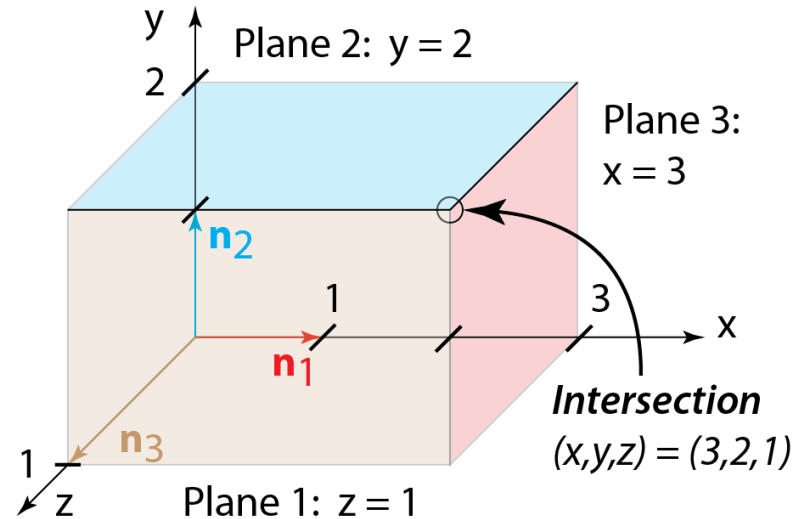
## B Intersection of three planes (cont.)

3 Write equations for lines in matrix form

$$\begin{bmatrix} a_{n_1x} & a_{n_1y} & a_{n_1z} \\ a_{n_2x} & a_{n_2y} & a_{n_2z} \\ a_{n_3x} & a_{n_3y} & a_{n_3z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$[A] \quad [X] = [B]$$

We seek the vector  $[X]$  that satisfies each equation (i.e., one point  $(x,y,z)$  on all three planes)



Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$[A] \quad [X] = [B]$$

# 4. INTERSECTIONS OF PLANES

## B Intersection of three planes (cont.)

### 4 Solve for $[X]$

$$[A][X]=[B]$$

$$[A]^{-1}[A][X]=[A]^{-1}[B]$$

$$[X]=[A]^{-1}[B]$$

#### a Matlab

```
>> A = [0 1;1 0]
```

```
A =  
 1  0  1  
 0  1  0  
 0  0  1
```

```
>> B = [3;2;1]
```

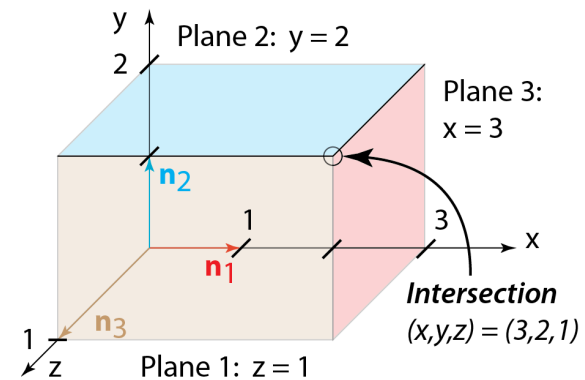
```
B =  
 3.0000  
 2.0000  
 1.0000
```

```
>> X = A\B
```

```
X =  
 3.0000  
 2.0000  
 1.0000
```

```
>> X=inv(A)*B
```

```
X =  
 3.0000  
 2.0000  
 1.0000
```



b Cramer's Rule  
Could use Cramer's rule, but it is cumbersome.

# 4. INTERSECTIONS OF PLANES

## C Intersection of a line and a plane

- 1 A line ( $L_1$ ) and a plane ( $P_1$ ) intersect at a point
- 2 Point of intersection can be viewed as the intersection of 3 planes

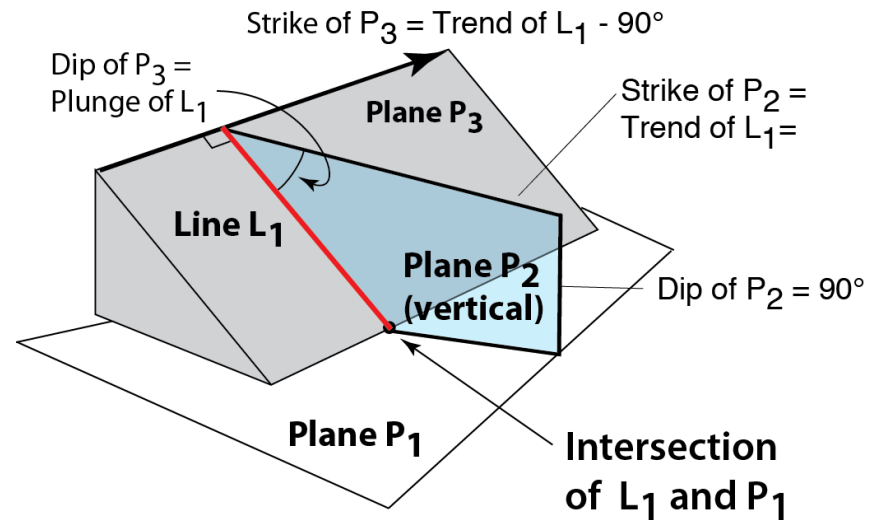
a Plane  $P_1$  (white)

b Plane  $P_2$  (blue)

$P_2$  intersects plane  $P_3$  at line  $L_1$ . Plane  $P_2$  is a vertical plane containing  $L_1$ , so  $P_2$  strikes parallel to the trend of  $L_1$ .

c Plane  $P_3$  (gray)

$P_3$  intersects plane  $P_2$  at line  $L_1$ . Plane  $P_3$  is an inclined plane that contains  $L_1$ . The dip of  $P_3$  equals the plunge of  $L_1$ , and the strike of  $P_3$  is  $90^\circ$  from the trend of  $L_1$ .



## 4. INTERSECTIONS OF PLANES

### Two Math Handbook References

Gellert, W., Küstner, H., Hellwich, M., and Kästner, H., 1977, The VNR concise encyclopedia of mathematics: Van Nostrand Reinhold, New York, 760 p.

Tuma, J.J., 1979, Engineering mathematics handbook: McGraw-Hill, New ork, 394 p.