4. INTERSECTIONS OF PLANES

I Main Topics

A Equation of a plane

B Pole to a plane using cross-products

C Intersection of two planes in a line
(apparent dip problem)

D Intersection of three planes in a point
(solution of simultaneous linear equations)

4. INTERSECTIONS OF PLANES

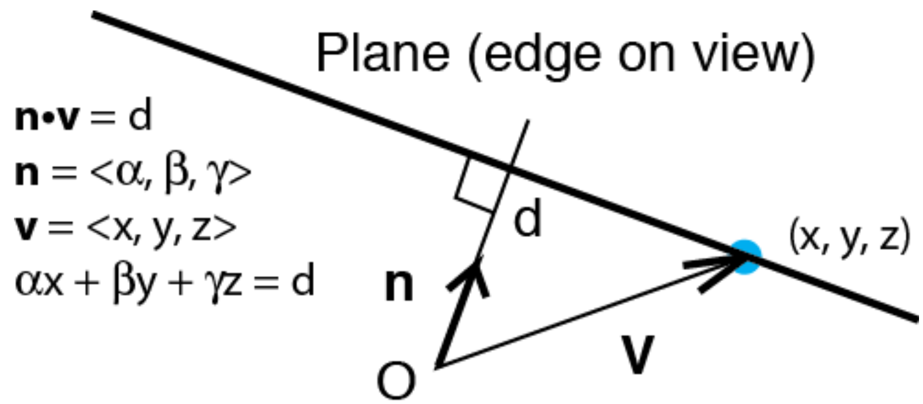
II Equation for plane: **n•v** = d

A Meaning

1 The distance from a known reference point to a plane, as measured in the direction of a unit vector **n** normal to the plane, is d.

2 d > 0 if **n** points from the reference point towards the plane;

3 d < 0 if **n** points from the plane towards the reference point.



4. INTERSECTIONS OF PLANES

II Equation for plane: **n•v** = d (cont.)

B Solution for **n**=α**i**+β**j**+γ**k** = <α,β,γ>

1 If x=north, y=east, z=down

$\alpha = \cos\phi\cos\theta$, $\beta = \cos\phi\sin\theta$, $\gamma = \sin\phi$

2 If x=east, y=north, z=up

$\alpha = \cos\phi\sin\theta$, $\beta = \cos\phi\cos\theta$, $\gamma = -\sin\phi$

where θ is the pole trend and ϕ is the pole plunge

3 Matlab:[alpha,beta,gamma]=sph2cart(trend,plunge,1)

Trend and plunge must be in radians.

4 Matlab: [trend,plunge,R]=cart2sph(alpha,beta,gamma)

5 **<u>Good for z = down</u>**; trend and plunge must be in radians.

4. INTERSECTIONS OF PLANES

II Equation for plane: **n•v** = d (cont.)

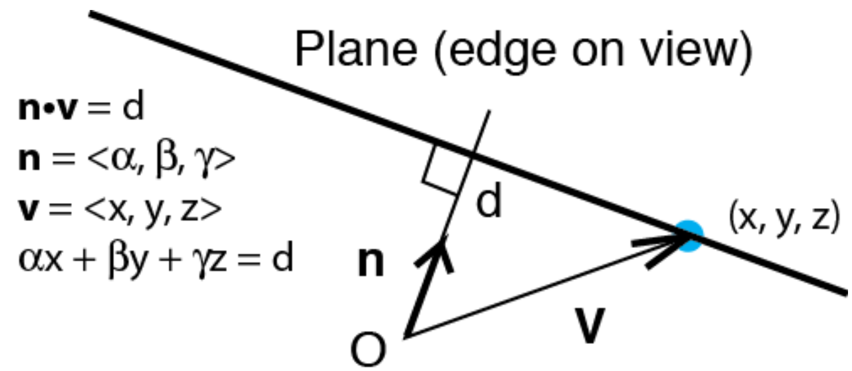
C **v** = x**i**+y**j**+z**k** = <x,y,z>

D Solution for d

1 $d = \alpha x + \beta y + \gamma z$

(Normal form)

2 d=dot(n,v)



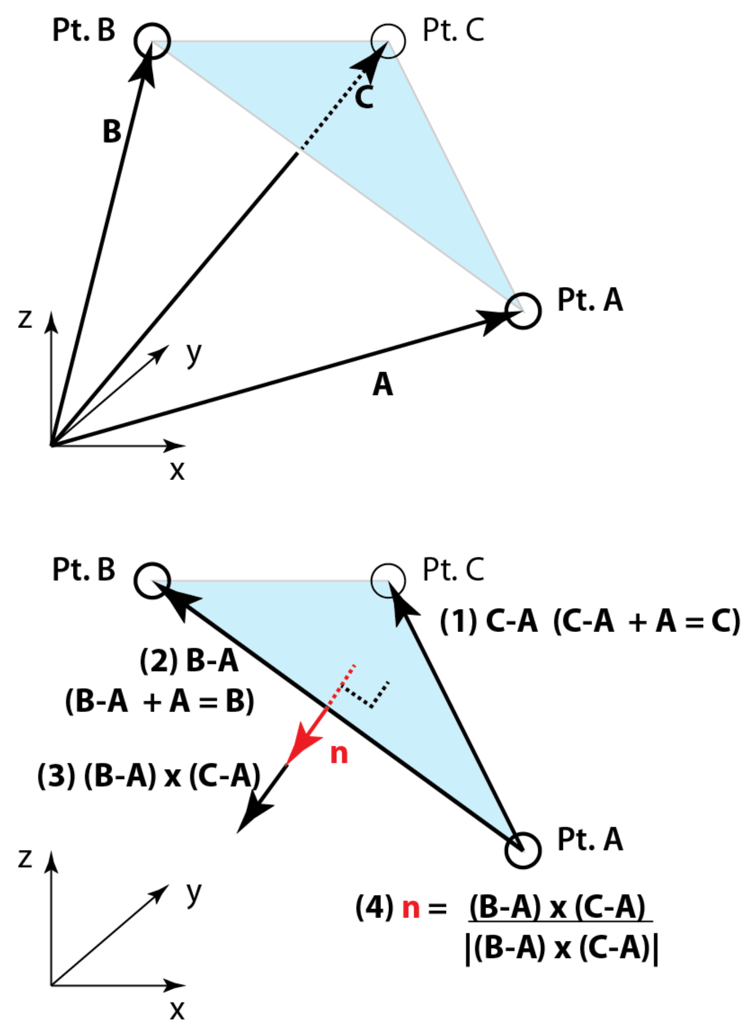
4. INTERSECTIONS OF PLANES

III Pole to a plane using cross-products

A Consider three non-colinear points A,B,C that form a <u>clockwise</u> circuit as viewed from above

B (**B-A**) x (**C-A**) is normal to the plane

C $\color{red}{\mathbf{n}}$ = $\dfrac{(\mathbf{B\text{-}A}) \times (\mathbf{C\text{-}A})}{|(\mathbf{B\text{-}A}) \times (\mathbf{C\text{-}A})|}$, <u>unit</u> normal to the plane.



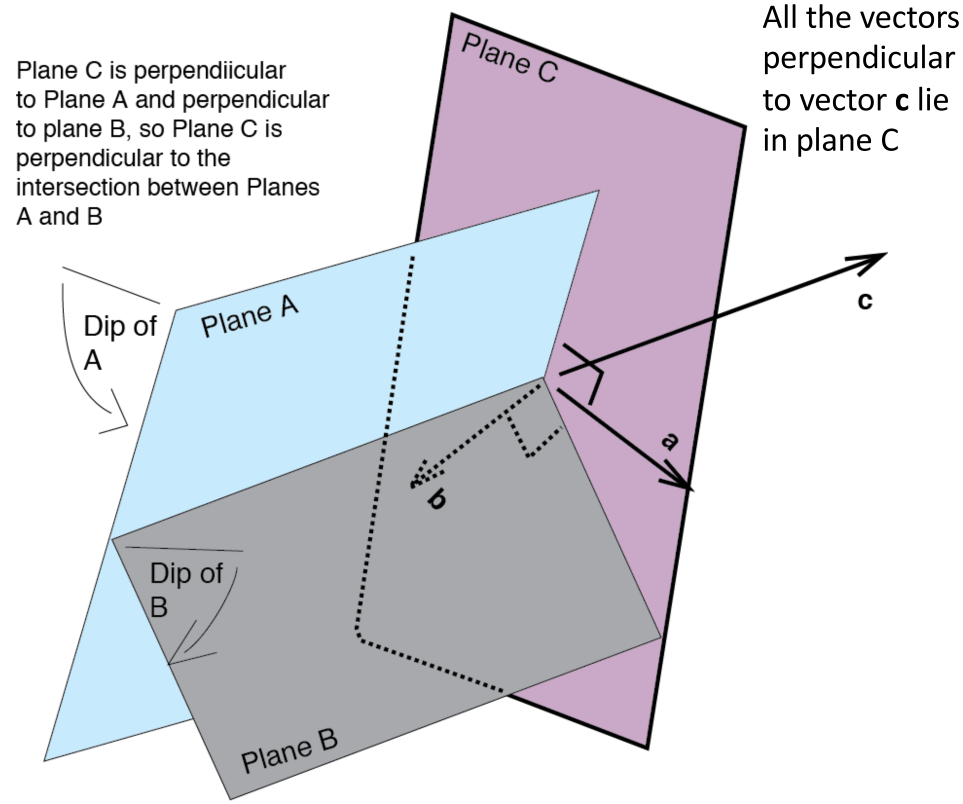
4. INTERSECTIONS OF PLANES

IV Intersection of two planes in a line

A Two planes A and B intersect in a line. The equations of those two planes define the line.

B The direction of intersection is along the vector **a x b**, where **a** is normal to plane A, and **b** is normal to plane B

Direction of Intersection of Two Planes

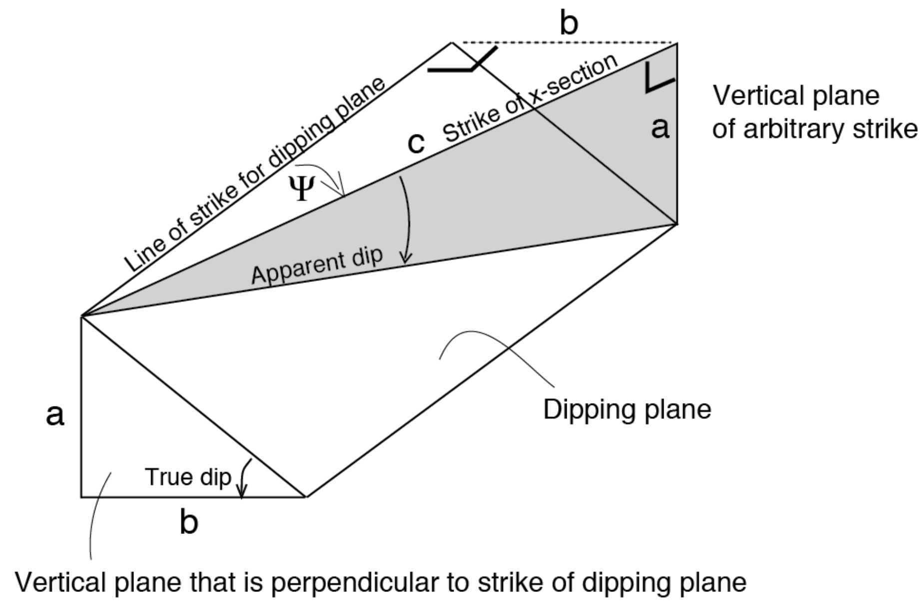


a is unit normal to Plane A and lies in Plane C
b is unit normal to Plane B and lies in Plane C
c is unit normal to Plane C
c= axb/laxbl

4. INTERSECTIONS OF PLANES

IV Intersection of two planes in a line (cont.)

- C Apparent dip
  - 1 How a geologic plane appears to dip in a vertical cross section of <u>arbitrary</u> strike
  - 2 Plunge of the line of intersection between a geologic plane and a vertical cross section plane of <u>arbitrary</u> strike
  - 3 Plunge of the cross product of two vectors
    - a Vector normal to a geologic plane
    - b Vector normal to vertical cross section plane of <u>arbitrary</u> strike
    - c <u>Make sure cross product points down to get a pole</u>



Apparent Dip

1) $a/b = \tan(\text{true dip})$.
2) $a = b \tan(\text{true dip}) = c \tan(\text{app. dip})$. So
3) $b \tan(\text{true dip}) = c \tan(\text{app. dip})$

Now consider the angle Ψ between the line of strike of the cross section and the line of strike for the dipping plane.

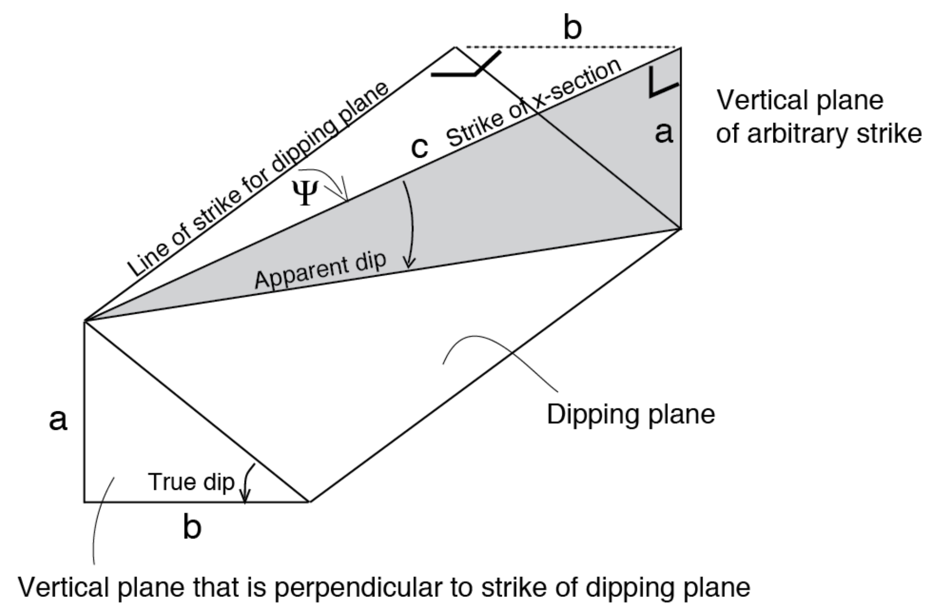
4) $b/c = \sin\Psi$, so
5) $c = b/\sin\Psi$.

Now insert (5) into the right side of (3)

6) $b \tan(\text{true dip}) = (b/\sin\Psi) \tan(\text{app. dip})$
7) $\text{app. dip} = \tan^{-1}\{[\tan(\text{true dip})][\sin\Psi]\}$

Since $\sin\Psi \leq 1$, app. dip ≤ true dip.

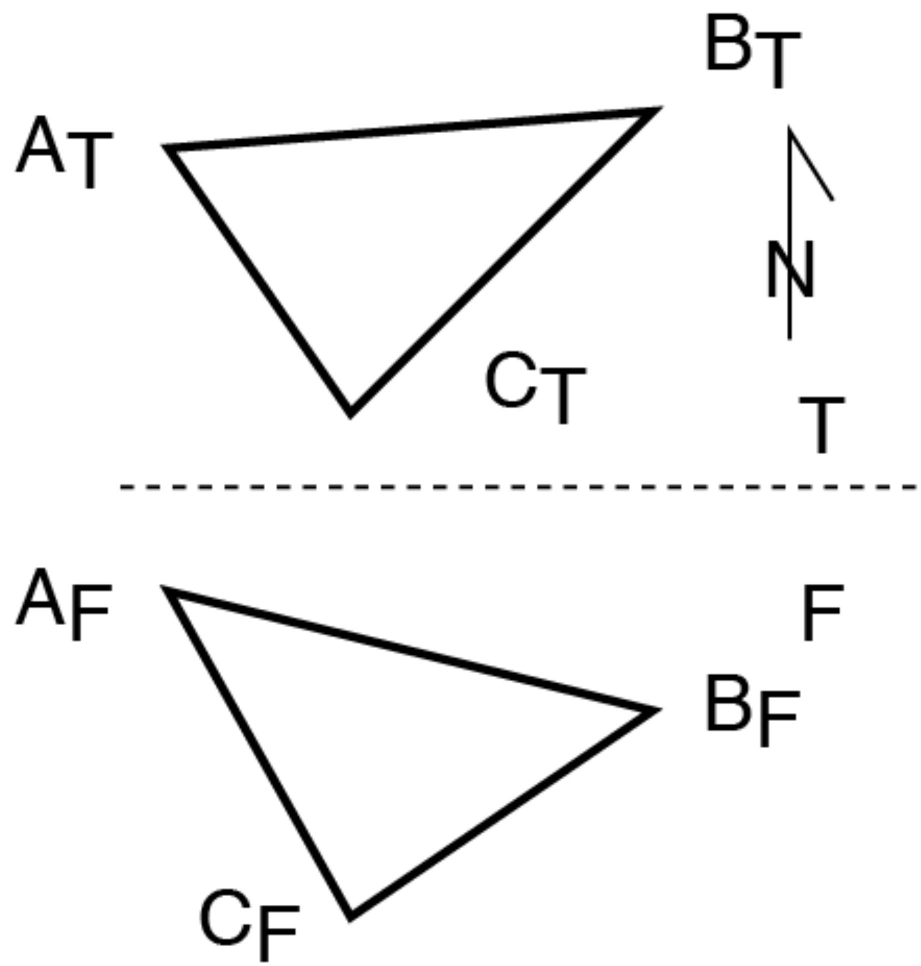
If $\Psi = 90°$, then app. dip = true dip.



4. INTERSECTIONS OF PLANES

- IV Intersection of two planes in a line (cont.)
  - D Apparent dip and orthographic projection example

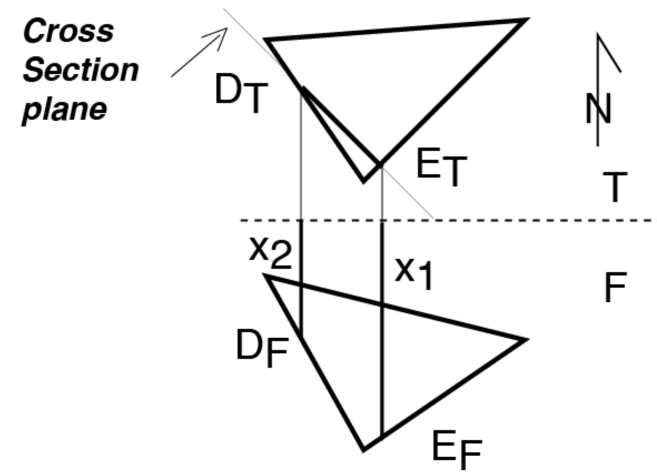
Find the apparent dip of plane ABC in a vertical cross section that strikes 315°.



4. INTERSECTIONS OF PLANES

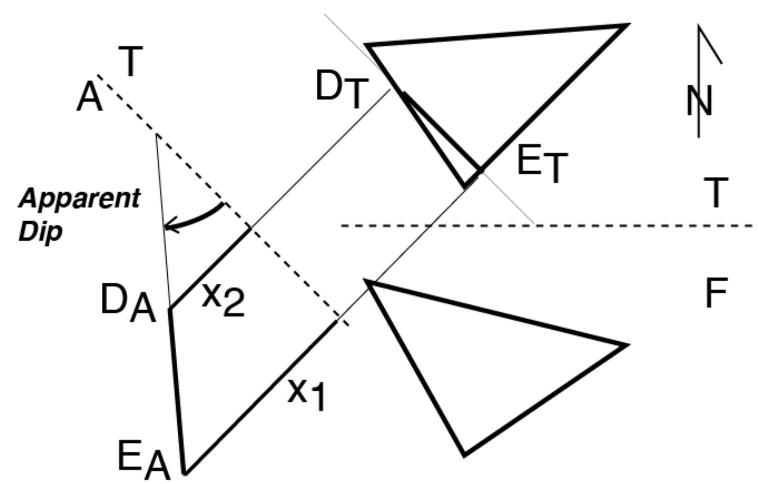
C Apparent dip and orthographic projection example

(1) In the top view, intersect the cross section plane with plane ABC. Call the line of intersection DE. Point D is on line AC. Point E is on line BC.



Next we find the plunge of line DE.

(2) Project the line of intersection DE onto an auxiliary view that is taken parallel to the line of cross section. The plunge of line DE is the apparent dip of plane ABC as seen in the cross section.



4. INTERSECTIONS OF PLANES

V Intersection of features in a point (*in 3D*)

A Intersection of two lines (L_1 and L_2)

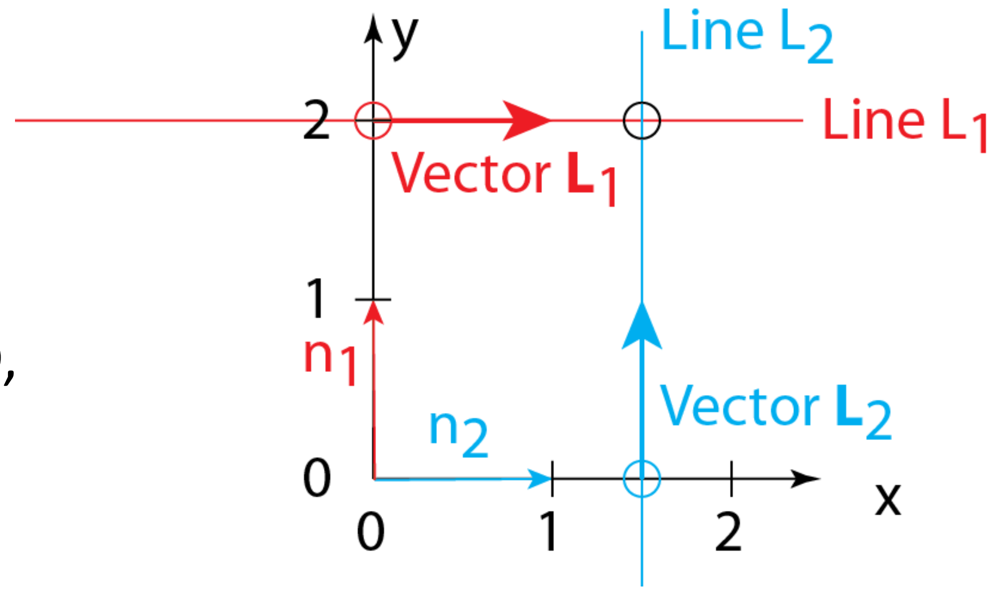
1 Intersection criteria

a Vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ are *not* parallel:

$$\vec{\mathbf{L}}_1 \times \vec{\mathbf{L}}_2 \neq 0$$

b The lines must lie in the same plane in 3D, so the normal to the plane containing $\mathbf{L}_1$ and $\mathbf{n}_1$ *in 3D* must parallel the normal to the plane containing $\mathbf{L}_2$ and $\mathbf{n}_2$:

$$(\vec{\mathbf{n}}_1 \times \vec{\mathbf{L}}_1) \times (\vec{\mathbf{n}}_2 \times \vec{\mathbf{L}}_2) = 0$$



Vector $\mathbf{L}_1$ is along line L_1

Vector $\mathbf{L}_2$ is along line L_2

Vector $\mathbf{n}_1$ is from the origin to L_1

Vector $\mathbf{n}_2$ is from the origin to L_2

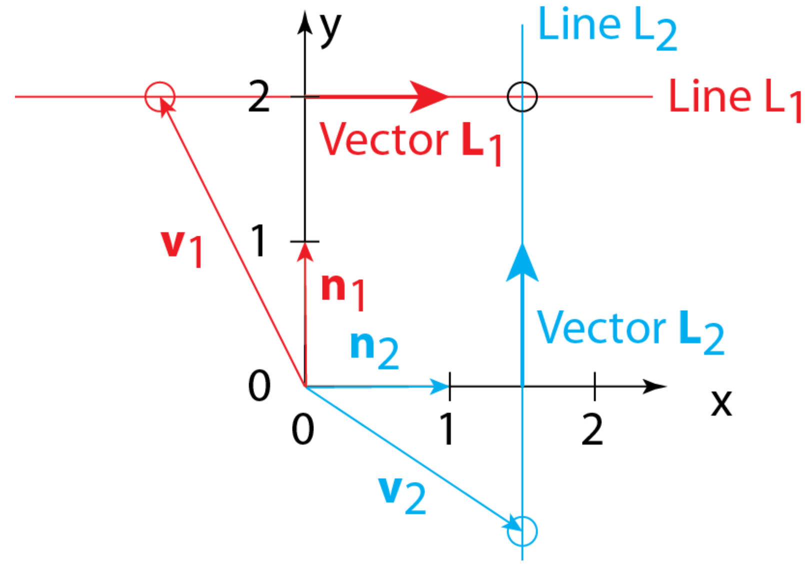
4. INTERSECTIONS OF PLANES

A Intersection of two lines (cont.)

2 Write equations for two lines ($n \bullet v = d$)

$$\vec{n}_1 \bullet \vec{v}_1 = a_{n_1x}x_1 + a_{n_1y}y_1 = d_1$$
$$\vec{n}_2 \bullet \vec{v}_2 = a_{n_2x}x_2 + a_{n_2y}y_2 = d_2$$

3 Find n_1, n_2, d_1, d_2



Example

$$(0)(-1)+(1)(2)=2=d_1$$
$$(1)(1.5)+(0)(-1)=1.5=d_2$$

4. INTERSECTIONS OF PLANES

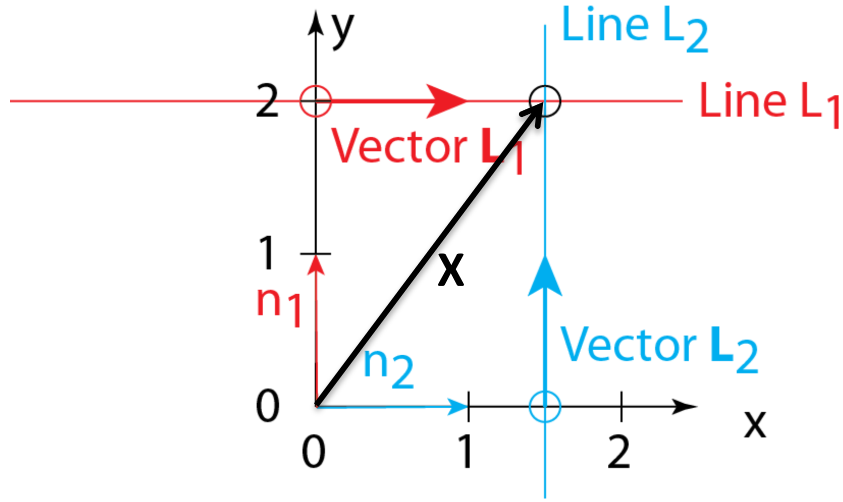
A Intersection of two lines (cont.)

3 Write equations for lines in matrix form

$$\begin{bmatrix} a_{n_1x} & a_{n_1y} \\ a_{n_2x} & a_{n_2y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$[A] \quad [X] = [B]$$

We seek the vector [X] that satisfies both equations (i.e., a point (x,y) on both lines)



Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$$

$$[A] \quad [X] = [B]$$

4. INTERSECTIONS OF PLANES

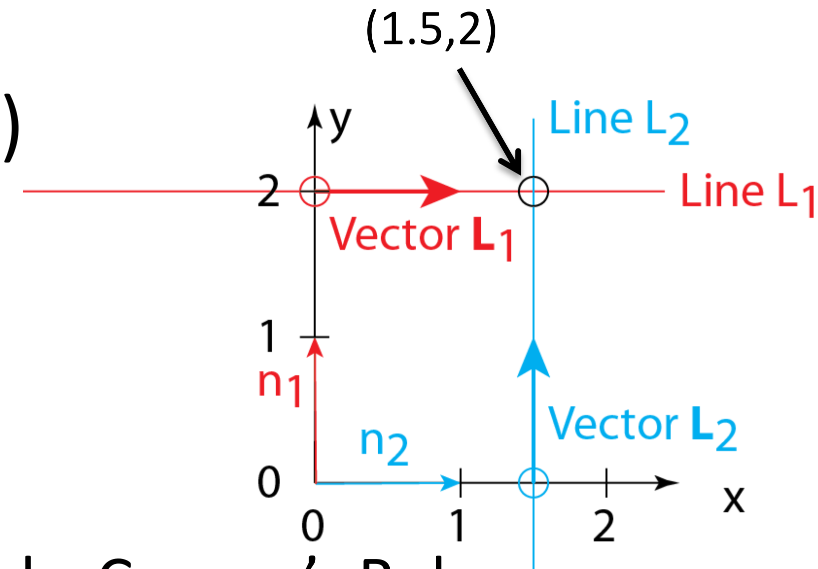
A Intersection of two lines (cont.)

4 Solve for [X]

$$[A][X]=[B]$$

$$[A]^{-1}[A][X]=[A]^{-1}[B]$$

$$[X]=[A]^{-1}[B]$$



a Matlab

```
>> A = [0 1;1 0]
A =
    0    1
    1    0
>> B = [2;1.5]
B =
    2.0000
    1.5000
```

```
>> X = A\B
X =
    1.5000
    2.0000
>> X=inv(A)*B
X =
    1.5000
    2.0000
```

b Cramer's Rule

$$x=\frac{\begin{vmatrix} B_1 & A_{12} \\ B_2 & A_{22} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}}=\frac{\begin{vmatrix} 1.5 & 1 \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}=\frac{-2}{-1}=2$$

$$y=\frac{\begin{vmatrix} A_{11} & B_1 \\ A_{21} & B_2 \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}}=\frac{\begin{vmatrix} 0 & 1.5 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}=\frac{-1.5}{-1}=1.5$$

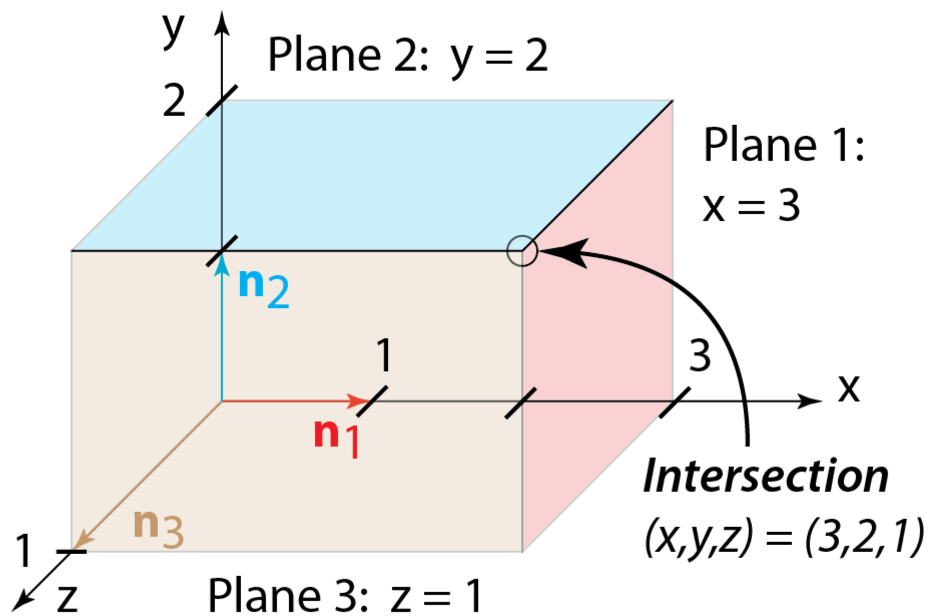
4. INTERSECTIONS OF PLANES

B Intersection of three plane (P_1, P_2, and P_3) at a point

1 Intersection criteria

Normals of any two planes are not parallel; scalar triple product of normals to planes ≠ 0:

$$\vec{\mathbf{n}}_1 \bullet (\vec{\mathbf{n}}_2 \times \vec{\mathbf{n}}_3) = 0$$



4. INTERSECTIONS OF PLANES

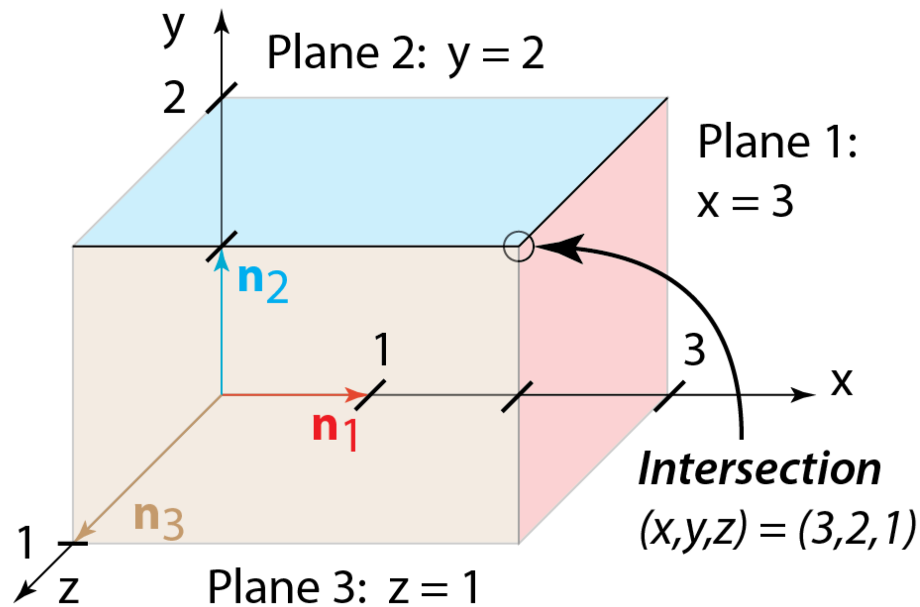
B Intersection of three planes (cont.)

2 Write equations for three planes (n•v=d)

$$\vec{n}_1 \bullet \vec{v}_1 = a_{n_1x}x_1 + a_{n_1y}y_1 + a_{n_1z}z_1 = d_1$$

$$\vec{n}_2 \bullet \vec{v}_2 = a_{n_2x}x_2 + a_{n_2y}y_2 + a_{n_2z}z_2 = d_2$$

$$\vec{n}_3 \bullet \vec{v}_3 = a_{n_3x}x_3 + a_{n_3y}y_3 + a_{n_3z}z_3 = d_3$$



3 Find n_1, n_2, n_3, d_1, d_2, d_3

Example

$$\vec{n}_1 = 1\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\vec{n}_2 = 0\vec{i} + 1\vec{j} + 0\vec{k}$$

$$\vec{n}_3 = 0\vec{i} + 0\vec{j} + 1\vec{k}$$

$$(1)(3) + (0)(y) + (0)(z) = 3 = d_1$$

$$(0)(x) + (1)(2) + (0)(z) = 2 = d_2$$

$$(0)(x) + (0)(y) + (1)(1) = 1 = d_3$$

4. INTERSECTIONS OF PLANES

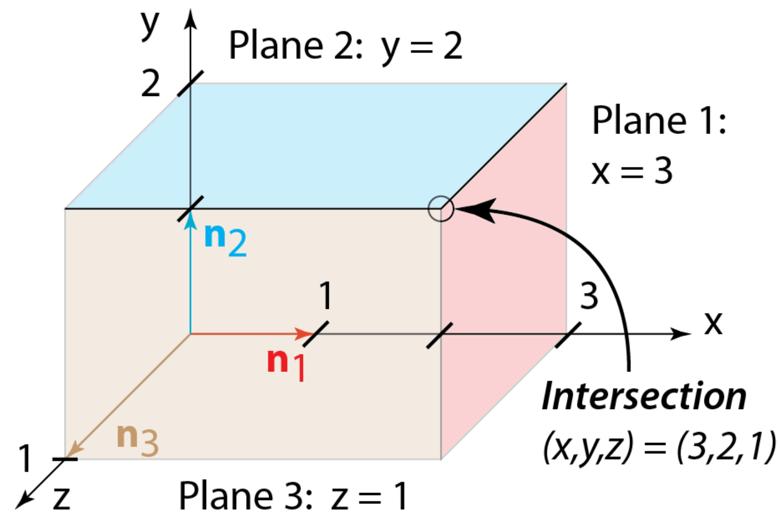
B Intersection of three planes (cont.)

3 Write equations for lines in matrix form

$$\begin{bmatrix} a_{n_1x} & a_{n_1y} & a_{n_1z} \\ a_{n_2x} & a_{n_2y} & a_{n_2z} \\ a_{n_3x} & a_{n_3y} & a_{n_3z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$[A] \qquad [X] = [B]$$

We seek the vector [X] that satisfies each equation (i.e., one point (x,y,z) on all three planes)



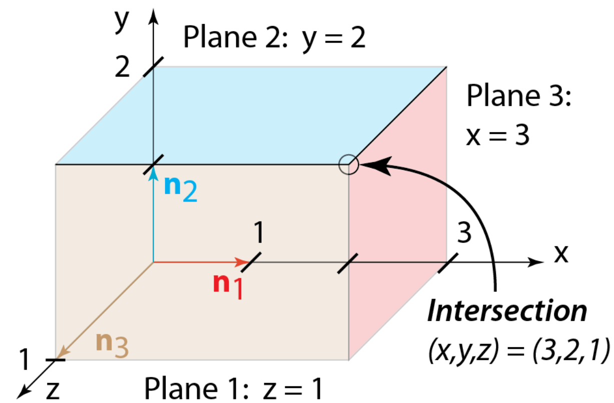
Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$[A] \qquad [X] = [B]$$

4. INTERSECTIONS OF PLANES

B Intersection of three planes (cont.)



4 Solve for [X]

$$[A][X]=[B]$$

$$[A]^{-1}[A][X]=[A]^{-1}[B]$$

$$[X]=[A]^{-1}[B]$$

a Matlab

```
>> A = [0 1;1 0]
A =
    1    0    1
    0    1    0
    0    0    1
>> B = [3;2;1]
B =
    3.0000
    2.0000
    1.0000
```

```
>> X = A\B
X =
    3.0000
    2.0000
    1.0000
>> X=inv(A)*B
X =
    3.0000
    2.0000
    1.0000
```

b Cramer's Rule

Could use Cramer's rule, but it is cumbersome.

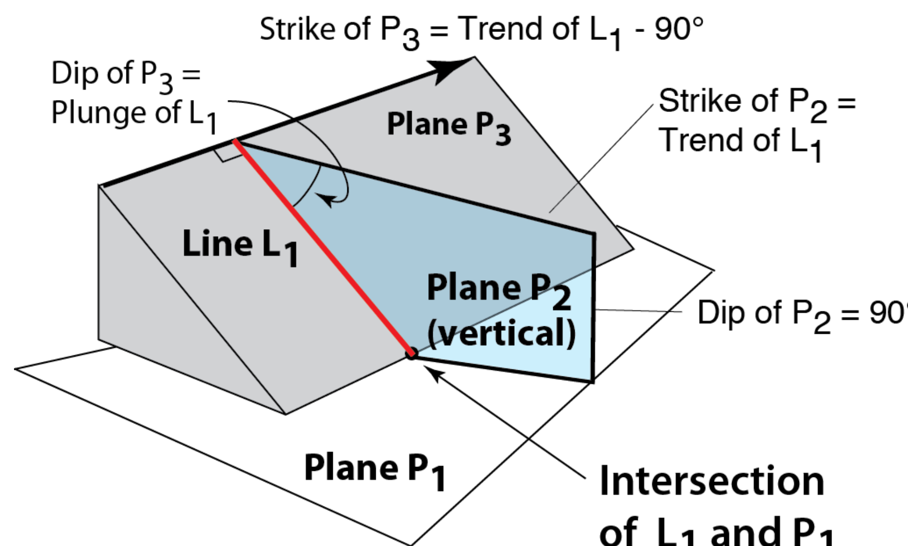
4. INTERSECTIONS OF PLANES

C Intersection of a line and a plane

1. A line (L_1) and a plane (P_1) intersect at a point
2. Point of intersection can be viewed as the intersection of 3 planes
   - a Plane P_1 (white)
   - b Plane P_2 (blue)

     P_2 intersects plane P_3 at line L_1. Plane P_2 is a vertical plane containing L_1, so P_2 strikes parallel to the trend of L_1.
   - c Plane P_3 (gray)

     P_3 intersects plane P_2 at line L_1. Plane P_3 is an inclined plane that contains L_1. The dip of P_3 equals the plunge of L_1, and the strike of P_3 is 90° from the trend of L_1.



4. INTERSECTIONS OF PLANES

Two Math Handbook References

Gellert, W., Küstner, H., Hellwich, M,. and Kästner, H., 1977, The VNR concise encyclopedia of mathematics: Van Nostrand Reinhold, New York, 760 p.

Tuma, J.J., 1979, Engineering mathematics handbook: McGraw-Hill, New ork,394 p.