Eigenvectors and eigenvalues of real symmetric matrices

Eigenvectors can reveal planes of symmetry and together with their associated eigenvalues provide ways to visualize and describe many phenomena simply and understandably. The solutions involve finding special reference frames.

Applications of eigenvectors and eigenvalues in structural geology

* Strain
* Stress
* Curvature (Shapes of surfaces)

Equations for eigenvalue problems

(1) $[A][X] = \lambda[X]$

Since $\lambda[X] = \lambda[IX]$, subtracting $\lambda[IX]$ from both sides of (1) yields an alternate form:

(2) $[A-I\lambda][X] = 0$

Meaning

If a certain transformation or process, given by [A], acts on a particular non-zero vector X, such that $[A][X] = \lambda[X]$, where λ is a constant, then λ is called an eigenvalue and the corresponding vector X is called an eigenvector. The transformed vector $\lambda[X]$ has the same direction as X, although its length might differ.

Trivial solution for $[A][X] = \lambda[X]$

X = 0 always solves equation (1) or (2). The solution is unique if and only if $|A-I\lambda| \neq 0$.

Example 1

If two lines intersect at a unique point (the origin), then their slopes must differ, so the determinant |A| must not equal zero. The following example with the lines y = 2x and y = -x illustrates this.

The equations of the lines are:

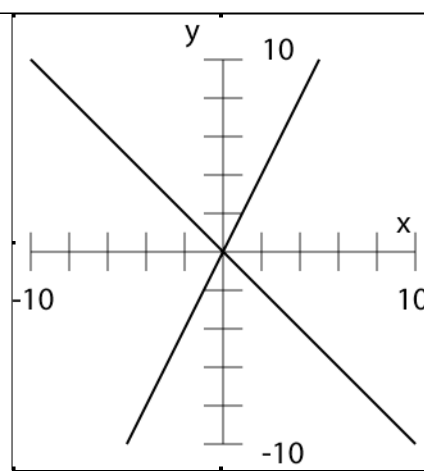
(a) $$\begin{bmatrix} 2x - y = 0 \\ x + y = 0 \end{bmatrix}$$

In matrix form these become:

(b) $$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix} = 0\begin{bmatrix} x \\ y \end{bmatrix}$$

Since $\lambda = 0$, $|A-I\lambda| = |A| = 3$:

(c) $$\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = (2)(1) - (-1)(1) = 3 \neq 0$$



The first equation in (a) stems from y = 2x. Replacing y by 2x in the second equation in (a) yields 3x = 0, so x = 0. Since y = 2x, y = 0. Thus x = 0, y = 0 is the solution, as the graph above confirms, and this solution is unique. Note the determinant of A, in (c), is indeed not zero.

Eigenvector solution for [A][X] = λ[X]

Eigenvectors, in contrast to trivial solutions, are *required* to be non-zero solutions to (1) or (2). If an eigenvector solution exists in addition to X = 0, then the solution is not unique, hence |A-Iλ|=0; this requirement also means that the rows in |A-Iλ| are not linearly independent.

Example 2

Consider the lines 2y = -2x and y = -x. These lines plot on top of each other.

(d) $$\begin{matrix} 2x+2y=0 \\ x+y=0 \end{matrix}$$

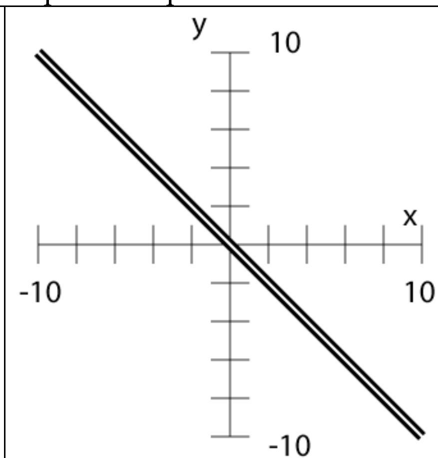
In matrix form these become:

(e) $$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix} = 0\begin{bmatrix} x \\ y \end{bmatrix}$$

Here again λ = 0, so |A-Iλ| equals |A|:

(f) $$\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = (2)(1)-(2)(1)=0$$

The determinant |A-Iλ| = 0, and the slopes of the lines are equal. Any points along the direction y/x = -1 provide a solution. So here λ = 0 is the eigenvalue and the corresponding eigenvector is given by the direction y/x = -1.



Characteristic equation: |A-Iλ|=0

The polynomial equation derived from |A-Iλ|=0 yields eigenvalues as its roots and is called the characteristic equation.

Example 3 (General 2-D Example)

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

(3) $$[A-I\lambda]=\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

(4) $$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 = (a-\lambda)(d-\lambda)-(b)(c) = \lambda^2-(a+d)\lambda+ad-bc = \lambda^2 - tr(A)\lambda + |A|,$$

where tr(A) is the trace of A (i.e., the sum of the terms along the main diagonal of A).

By solving the quadratic equation we obtain the eigenvalues

(5) $$\lambda_1,\lambda_2 = \frac{tr(A) \pm \sqrt{(tr(A))^2 - 4|A|}}{2}$$

In general, an n x n matrix has n eigenvalues, but some of the eigenvalues might be identical. Note that eigenvalues can be zero even though eigenvectors can not be (see example 2).

Eigenvalues and eigenvectors for a real symmetric 2 x 2 matrix

Eigenvalues (scalars)

If A is a real symmetric 2x2 matrix such that b = c, then $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, and from eq. (5)

(6) $$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - b^2)}}{2} = \frac{(a+d) \pm \sqrt{(a^2 - 2ad + d^2) + 4b^2}}{2}$$

(7) $$\lambda_1, \lambda_2 = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4b^2}}{2}$$

The squared terms under the radical sign can not be negative, so <u>λ_1 and λ_2 must be real numbers</u>.

Eigenvectors (directions)

Start with equation (2)

(8a) $(a-\lambda)x + by = 0$ or (8b) $bx + (d-\lambda)y = 0$ (Note: 9a and 9b are not linearly independent)

(9a) $by = (\lambda - a)x$ or (9b) $(\lambda - d)y = bx$

(10a) $\dfrac{y}{x} = \dfrac{(\lambda - a)}{b}$ or (10b) $\dfrac{y}{x} = \dfrac{b}{(\lambda - d)}$

Equation (10) gives the slope of the line for a particular eigenvalue. Now consider the numerator and denominator of the slope as the y- and x-components of the eigenvector, respectively. The square root of the sum of the squares of the component lengths gives the length of this eigenvector. To find the direction cosines of the line with respect to the x- and y-axes (i.e., to find the components of a unit eigenvector), divide the x- and y- components by that length.

(11a) $n_x = \dfrac{b}{\sqrt{b^2 - (\lambda - a)^2}} = \dfrac{(\lambda - d)}{\sqrt{b^2 - (\lambda - d)^2}}$ (11b) $n_y = \dfrac{(\lambda - a)}{\sqrt{b^2 - (\lambda - a)^2}} = \dfrac{b}{\sqrt{b^2 - (\lambda - d)^2}}$

<u>**The eigenvectors X_1 and X_2 of a symmetric 2x2 matrix are orthogonal**</u>

Proof: The product of the slopes of eigenvectors X_1 and X_2 is -1

Start with equations (10a) and (10b). For λ_1 $\left(\dfrac{y}{x}\right)_1 = \dfrac{(\lambda - a_1)}{b}$, and for λ_2: $\left(\dfrac{y}{x}\right)_2 = \dfrac{b}{(\lambda_2 - d)}$.

Now multiply the slopes and use equation (7) for the values of the eigenvalues:

$$\left(\frac{y}{x}\right)_1 \left(\frac{y}{x}\right)_2 = \frac{(\lambda_1 - a)}{b} \frac{b}{(\lambda_2 - d)} = \frac{(\lambda_1 - a)}{(\lambda_2 - d)} = \frac{\left(\lambda_1 - \frac{2a}{2}\right)}{\left(\lambda_2 - \frac{2d}{2}\right)} = \frac{\dfrac{(d-a) - \sqrt{(a-d)^2 + 4b^2}}{2}}{\dfrac{(a-d) + \sqrt{(a-d)^2 + 4b^2}}{2}} = -1$$

Diagonalization of a real symmetric 2x2 matrix

A symmetric matrix [A] can be expressed in terms of matrices containing its eigenvalues and its eigenvector components by manipulating the equation AX = λX a bit. This permits matrix [A] to be re-expressed in a form that has more geometric or physical meaning. Start with the general eigenvalue equation:

$$(12)\,[A][X]=\lambda[X]=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}a_{11}x_1+a_{12}x_2\\ a_{21}x_1+a_{22}x_2\end{bmatrix}=\begin{bmatrix}\lambda x_1\\ \lambda x_2\end{bmatrix}$$

Here [X] contains just the components for <u>one</u> eigenvector and λ is just <u>one</u> eigenvalue. If [A] is a 2x2 matrix, then [X] is a 2x1 matrix, and λ is a constant. One can easily build on this equation by stacking <u>all</u> the eigenvectors (represented as column vectors below partitioned by dots) side-by-side in a 2x2 matrix, and by putting all the eigenvalues in a 2x2 matrix:

$$(13)\,[A]\begin{bmatrix}x_1^{(1)} \vdots\, x_1^{(2)}\\ x_2^{(1)} \vdots\, x_2^{(2)}\end{bmatrix}=\begin{bmatrix}\lambda_1 x_1^{(1)} \vdots\, \lambda_2 x_1^{(2)}\\ \lambda_1 x_2^{(1)} \vdots\, \lambda_2 x_2^{(2)}\end{bmatrix}=\begin{bmatrix}x_1^{(1)} \vdots\, x_1^{(2)}\\ x_2^{(1)} \vdots\, x_2^{(2)}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

In abbreviated form, equation (13) becomes

$$(14)\,[A][X_1 \vdots X_2]=[\lambda_1 X_1 \vdots \lambda_2 X_2]=[X_1 \vdots X_2]\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda 2\end{bmatrix}$$

Now let [S] be the matrix of eigenvectors and [Λ] be the diagonal matrix of eigenvalues:

$$(15)\quad [S]=[X_1 \vdots X_2]$$

$$(16)\quad [\Lambda]=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$$

Substituting (5) and (6) into (4)

(17) [A][S] = [S][Λ]

By pre-multiplying both sides of (17) by $[S]^{-1}$ the eigenvalue matrix [Λ] can be obtained from the original matrix [A] and the eigenvector matrix [S]:

(18) $[S]^{-1}[A][S] = [S]^{-1}\,[S][\Lambda] = [\Lambda]$

Alternatively, by post-multiplying both sides of (17) by $[S]^{-1}$ the original matrix [A] can be obtained from its eigenvalue matrix and its eigenvector matrix:

(19) $[A][S]\,[S]^{-1} = [A] = [S][\Lambda]\,[S]^{-1}$

Application to the equation of an ellipse (Principal Axes Thereom)

Consider the equation of an ellipse

(20) $ax^2 + 2bxy + cy^2 = 1$

The equation can be re-written using matrices as:

(21) $$ax^2 + 2bxy + dy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + by \\ bx + dy \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} [A] \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

Using equation (18) the matrix can be rewritten

(22) $$ax^2 + 2bxy + dy^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} n_x^{(1)} & n_x^{(2)} \\ n_y^{(1)} & n_y^{(2)} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} n_x^{(1)} & n_y^{(1)} \\ n_x^{(2)} & n_y^{(2)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1,$$

where the n-terms are the components of the unit eigenvectors of symmetric matrix [A].

Since the unit eigenvectors of a real symmetric matrix are orthogonal, we can let the direction of λ_1 parallel one Cartesian axis (the x'-axis) and the direction of λ_2 parallel a second Cartesian axis (the y'-axis). In light of this, we rewrite the rightmost matrix of the eigenvectors in the equation above:

(23) $$\begin{bmatrix} n_x^{(1)} & n_y^{(1)} \\ n_x^{(2)} & n_y^{(2)} \end{bmatrix} = \begin{bmatrix} a_{x'x} & a_{x'y} \\ a_{y'y} & a_{y'x} \end{bmatrix}$$

<u>This means that the matrix of unit eigenvectors for a symmetric 2x2 matrix can be interpreted as a rotation matrix that relates coordinates in one orthogonal reference frame (here the x, y reference frame) to coordinates in an orthogonal reference frame along axes defined by the eigenvectors (here the x', y' reference frame)</u>. This is a rather important result. For the example here:

(24) $$\begin{bmatrix} n_x^{(1)} & n_y^{(1)} \\ n_x^{(2)} & n_y^{(2)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{x'x} & a_{x'y} \\ a_{y'y} & a_{y'x} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

This allows the equation of the ellipse to be expressed in the x', y' reference frame as

(25) $$ax^2 + 2bxy + dy^2 = \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'\lambda_1 & y'\lambda_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \lambda_1 x'^2 + \lambda_2 y'^2 = 1$$

or

(26) $$ax^2 + 2bxy + dy^2 = \left(\frac{x'}{\sqrt{\lambda_1}} \right)^2 + \left(\frac{y'}{\sqrt{\lambda_2}} \right)^2 = 1$$

The square roots of the eigenvalues for unit eigenvectors are the lengths of the semi-major and semi-minor axes of the ellipse. This result is much harder to derive by other methods.

Examples of [A][X] = λ[X]

Co-axial finite strain

(27) $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b=c \\ c=b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad or \quad [X'] = [F][X]$$

The F-matrix for co-axial finite strain is symmetric. In light of the material on the previous page, the unit eigenvectors give the directions in which the points given by X' are the greatest and least distance from the origin. The square roots of the eigenvalues give the magnitudes of those distances.

Second partial derivatives in general

(28) $$\begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x} & \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y} \\ \frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial x} & \frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} d\frac{\partial z}{\partial x} \\ d\frac{\partial z}{\partial y} \end{bmatrix}$$

A matrix of mixed partial second derivatives (a Hessian matrix) is symmetric because the second derivative does not depend on the order of differentiation, so the off diagonal terms are equal. The eigenvectors give the directions in which the first partial derivatives increase or decrease the most. The eigenvalues give the magnitudes of those changes in the first partial derivatives. The second partial derivatives along the main diagonal can differ from those off-diagonal if z locally has a quadratic form (e.g., $z = ax^2 + bxy + cy^2$). If z represents elevation, the principal values can be used to identify points that are at the top of domes or ridges, at the base of bowls or valleys, or at saddles.

Infinitesimal strains in a plane

(29) $$\begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial\left(\frac{\partial u}{\partial x}\right)}{\partial x} & \frac{\partial\left(\frac{\partial u}{\partial x}\right)}{\partial y} \\ \frac{\partial\left(\frac{\partial u}{\partial y}\right)}{\partial x} & \frac{\partial\left(\frac{\partial u}{\partial y}\right)}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} d\frac{\partial u}{\partial x} \\ d\frac{\partial u}{\partial y} \end{bmatrix}$$

A matrix of mixed partial second derivatives of displacements is symmetric because the second derivative does not depend on the order of differentiation. The eigenvectors give the directions in which the changes in elongation (or extension) increase or decrease the most. The eigenvalues give the magnitudes of those changes in the elongations (or extensions).

Stresses

(30) $$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \end{bmatrix} = T \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

The stress matrix is symmetric in order for a body to be at equilibrium. The eigenvectors give the directions in which the tractions on a plane through a point are greatest or least. The eigenvalues give the magnitudes of the greatest and the least tractions. These are also known as the principal stress magnitudes.

References

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