Solution of Simultaneous Linear Equations (AX=B)

- Preliminary: matrix multiplication
- Defining the problem
- Setting up the equations
- Arranging the equations in matrix form
- Solving the equations
- Meaning of the solution
- Examples
  - Geometry
  - Balancing chemical equations
  - Dimensional analysis

Matrix Multiplication (*)

$$A * B$$

$$Let \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Operate across rows of A and down columns of B

$$A * B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

If A*B = C, then

A is nxm, B is mxn, and C is nxn

Matrix Multiplication (*)

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

[2x3] [3x1] = [2x1]

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

[2x2] [2x1] + [2x1][1x1]=[2x1]

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

[2x2] [2x1] + [2x1] = [2x1]

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

[2x1] + [2x1] = [2x1]

Matrix Multiplication (.*)

$$A.*B$$

$$Let \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Multiply elements of A with counterparts in B

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} .* \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix}$$

If A.*B = C, then

A is nxm, B is nxm, and C is nxm

Matrix Multiplication (.*)

$$A.*B$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} .* \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 9 \\ 16 & 25 & 36 \end{bmatrix}$$

Defining the Problem (Two intersecting lines)

- What is the point where two lines in the same plane intersect

- *Alternative1*: What point that lies on one line also lies on the other line?

- *Alternative 2*: What point with coordinates (x,y) satisfies the equation for line 1 and simultaneously satisfies the equation for line 2?

Setting up the Equations

Equation for line 1

$y = m_1 x + b_1$

$-m_1 x + y = b_1$

Now multiply both sides by a constant c_1

$c_1(-m_1 x + y) \quad = (c_1)b_1$

$\underline{-c_1 m_1} x + \underline{c_1} y = \underline{(c_1)b_1}$

$\underline{a_{11}} x + \underline{a_{12}} y \quad = \underline{b^*_1}$

Equation for line 2

$y = m_2 x + b_2$

$-m_2 x + y = b_2$

Now multiply both sides by a constant c_2

$c_2(-m_2 x + y) \quad = (c_2)b_2$

$\underline{-c_2 m_2} x + \underline{c_2} y = \underline{(c_2)b_2}$

$\underline{a_{21}} x + \underline{a_{22}} y \quad = \underline{b^*_2}$

Setting up the Equations

Equation for line 1

$a_{11}x + a_{12}y \quad = b^*_1$

Equation for line 2

$a_{21}x + a_{22}y \quad = b^*_2$

The variables are on the left sides of the equations.
Only constants are on the right sides of the equations.
The left-side coefficients have slope information.
The right-side constants have y-intercept information.

We have two equations and two unknowns here.
This means the equation can have a solution.

Arranging the Equations in Matrix Form (AX = B)

Form from prior page

$a_{11}x + a_{12}y \quad = b^*_1$

$a_{21}x + a_{22}y \quad = b^*_2$

Matrix form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b^*_1 \\ b^*_2 \end{bmatrix}$$

Matrix A of known coefficients

Matrix X of unknown variables

Matrix B of known constants

We want to find values of x and y (i.e., X) that simultaneously satisfy both equations.

Solving the equations

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b*_1 \\ b*_2 \end{bmatrix}$$

AX = B

(1) $a_{11}x + a_{12}y = b^*_1$

(2) $a_{21}x + a_{22}y = b^*_2$

We use eq. 2 to eliminate x from eq. (1)

$$a_{11}x + a_{12}y = b^*_1$$

$$-(a_{11/}a_{21})(a_{21}x + a_{22}y) = -(a_{11/}a_{21})(b^*_2)$$

$$[a_{12} -(a_{11/}a_{21})(a_{22})](y) = b^*_1 -(a_{11/}a_{21})(b^*_2)$$

Solving the equations

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b^*{}_1 \\ b^*{}_2 \end{bmatrix}$$

AX = B

The equation of the previous slide

$[a_{12} - (a_{11/}a_{21})(a_{22})]\,(y) \qquad = b^*{}_1 - (a_{11/}a_{21})(b^*{}_2)$

has one equation with one unknown (y). This can be solved for y.

$y = [b^*{}_1 - (a_{11/}a_{21})(b^*{}_2)]/\,[a_{12} - (a_{11/}a_{21})(a_{22})]$

Similarly, we could solve for x:

$x = [b^*{}_2 - (a_{22/}a_{12})(b^*{}_1)]/\,[a_{21} - (a_{22/}a_{12})(a_{11})]$

Solving the Equations (Cramer's Rule)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b^*_1 \\ b^*_2 \end{bmatrix}$$

AX = B

$$x = \frac{\begin{vmatrix} b^*_1 & a_{12} \\ b^*_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b^*_1 a_{22} - a_{12} b^*_2}{a_{11} a_{22} - a_{12} a_{21}}$$

$$y = \frac{\begin{vmatrix} a_{11} & b^*_1 \\ a_{21} & b^*_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11} b^*_2 - b^*_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}$$

Note: if the denominators equal zero, the equations have no unique simultaneous solution (e.g., lines are parallel)

Solving the equations

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b*_1 \\ b*_2 \end{bmatrix}$$

AX = B

Many equations for many problems can be set up in this form (see examples):

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b*_1 \\ \vdots \\ b*_n \end{bmatrix}$$

AX = B

Matlab allows these to be solved like so:

X = A\B

Meaning of the Solution

$$AX = B$$

The solution X is the collection of variables that simultaneously satisfy the conditions described by the equations.

Example 1
Intersection of Two Lines

1x + 1y = 2

0x + 1y = 1

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

By inspection, the intersection is at y=1, x=1.

In Matlab:

```
A = [1 1;0 1]
B = [2;1]
X = A\B
```

Example 2
Intersection of Two Lines

$1x + 1y = 2$

$2x + 2y = 2$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Doubling the first equation yields the left side of the second equation, but not the right side of the second equation - what does this mean?

In Matlab:

```
A = [1 1;2 2]
B = [2;2]
X = A\B
```

Example 3
Intersection of Two Lines

1x + 1y = 1

2x + 2y = 2

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Doubling the first equation yields the second equation - what does this mean?

In Matlab:

```
A = [1 1;2 2]
B = [1;2]
X = A\B
```

Example 4
Intersection of Two Lines

1x + 2y = 0

2x + 2y = 0

$$\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Equations where the right sides equal zero are called homogeneous. They can have a “trivial” solution (x=0,y=0) or an infinite number of solutions. Which is the case here?

In Matlab:

```
A = [1 2;2 2]
B = [0;0]
X = A\B
```

Example 5
Intersection of Two Lines

$1x + 1y = 0$

$2x + 2y = 0$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is the case here?

In Matlab:

```
A = [1 1;2 2]
B = [0;0]
X = A\B
```

Example 6
Intersection of Three Planes

$1x + 1y + 0z = 2$

$0x + 1y + 0z = 1$

$0x + 0y + 1z = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

By inspection, the intersection is at z=0, y=1, x=1.

In Matlab:

```
A = [1 1 0;0 1 0; 0 0 1]
B = [2;1;0]
X = A\B
```

Example 7
Solution of a Chemical Equation

Hydrogen + Oxygen = Water

What are the unknowns? *H*, *O*, and *W* (the # of hydrogens, oxygens, and waters)

How many unknowns are there? 3

What are the chemical formulas?

$$H * H_2 + O * O_2 = W * H_2O$$

$$H * H_2 + O * O_2 - W * H_2O = 0$$

Example 7 (cont.)

What are the basic chemical components? H_2, O_2

How many components are there? 2

How many equations are there? 2 (see next page)

Example 7(cont.)

$H * H_2 + O * O_2 - W * H_2O = 0$

	Hydrogen H_2	Oxygen O_2	Water H_2O
H_2	1	0	-1
O_2	0	1	-0.5

Matrix equation

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -0.5 \end{bmatrix} \boxed{\begin{bmatrix} H \\ O \\ W \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2 equations and three unknowns…

Example 7(cont.)

More unknowns than equations.

Need to reduce the # of unknowns.

Let the # of waters (W) = 1.

Initial Eqn.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -0.5 \end{bmatrix}\begin{bmatrix} H \\ O \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} H \\ O \end{bmatrix} + \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}[1] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[2x3] [3x1] = [2x1] [2x2] [2x1] + [2x1] [1x1] =[2x1]

Revised Eqn.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} H \\ O \end{bmatrix} + \begin{bmatrix} -1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} H \\ O \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

[2x2] [2x1] + [2x1] = [2x1] [2x2] [2x1] = [2x1]

Example 7(cont.)

$H = 1$; $O = 0.5$; ($W = 1$)

Balanced chemical equation

$1{*}H_2 + 0.5{*}O_2 = 1{*}\ H_2O$

Now we can see why the solution need not be unique: the coefficients on each side of the equation can be scaled to yield other valid solutions.

Example 8
Dimensional Analysis

The dimensions of a physical equation must be the same on opposing sides of the equal sign

Example 8 (cont.)

<u>Fundamental physical quantities and their SI units</u>

M: mass (e.g., kg)

L: length (e.g., m)

T: time (e.g., sec)

θ: Temperature (e.g., K)

<u>Derived physical quantities</u>

Gravitational acceleration (g) = LT^{-2} (e.g., m/sec^2)

Energy = (Force)(Distance) = (MLT^{-2}) (L) = ML^2T^{-2}

Pressure = Force/area = $(MLT^{-2})/L^2$ = $ML^{-1}T^{-2}$

Example 8 (cont.)

Suppose the kinetic energy (E) of a body depends on it mass (M) and its velocity (v), such that E= f (M,v). Find the function f.

$E = M^a v^b$

$ML^2T^{-2} = M^a (L/T)^b$

Dimensioned starting equation

$M^1L^2T^{-2} M^{-a} (L/T)^{-b} = 1$

$M^{1-a}L^{2-b}T^{-2+b} = 1^0$

Dimensionless equation

Focus on the dimensions. Since L, M, and T are independent terms, the exponent for each term must be zero. Hence

$1 - a = 0 \longrightarrow 1-a = 0$

$\left.\begin{array}{r} 2-b = 0 \\ -2+b = 0 \end{array}\right\} \longrightarrow 2-b = 0$

Example 8 (cont.)

$$\begin{matrix} 1-a = 0 \\ 2-b = 0 \end{matrix} \rightarrow \begin{matrix} -a = -1 \\ -b = -2 \end{matrix} \rightarrow \begin{matrix} -1a = -1 \\ -1b = -2 \end{matrix} \rightarrow \begin{matrix} -1a + 0b = -1 \\ 0a - 1b = -2 \end{matrix}$$

Matrix equation

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Example 8 (cont.)

$$E = M^a v^b$$

$$a = 1$$

$$b = 2$$

So the form of the equation is: $E = M^1 v^2$

This solution generally will need to be multiplied by a dimensionless constant k. Here the dimensionless constant is 1/2.

$$E = kM^1 v^2 = (1/2)M^1 v^2$$

This is the form of the final dimensioned equation.

Appendix

- Re-arranging elements in a matrix in a matrix equation

Re-arranging elements in a matrix in a matrix equation

Consider the following equations:

$a_{11}\ x_1 + a_{12}\ x_2 + a_{13}\ x_3 = b_1$

$a_{21}\ x_1 + a_{22}\ x_2 + a_{23}\ x_3 = b_2$

$a_{31}\ x_1 + a_{32}\ x_2 + a_{33}\ x_3 = b_3$

These yield the following matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Coefficient a_{ij} acts on element x_j to contribute to element b_i

Re-arranging elements in a matrix in a matrix equation

The equations can be re-arranged:

$a_{12}\, x_2 + a_{11}\, x_1 + a_{13}\, x_3 = b_1$

$a_{22}\, x_2 + a_{21}\, x_1 + a_{23}\, x_3 = b_2$

$a_{32}\, x_2 + a_{31}\, x_1 + a_{33}\, x_3 = b_3$

These yield the following matrix equation:

$$\begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Coefficient a_{ij} acts on element x_j to contribute to element b_i