ON THE STATISTICS OF FINE SCALE STRAIN IN THE THERMOCLINE

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INTRODUCTION

The interrelation between the large vertical-scale motion field, the fine scale (1-100 m) field, and the micro-scale is poorly understood. It is thought that motions become significantly non-Gaussian at vertical scales shorter than about 50 m (Holloway 1983). However, "non-Gaussian" is an extremely general description of a flow field. In this work we examine the fine-scale statistics of the vertical strain field in the sea. Strain is here defined as the vertical gradient of isopycnal vertical displacement. The objective is to identify a specific class of probability density functions (pdfs) which characterize the fine scale field during its transition from highly skewed (micro-scale) to Gaussian (large scale) behavior.

Theoretical studies of non-linear processes often assume quasi-Gaussian statistics. The non-linear condition is approximated through a perturbation expansion about a zeroth-order Gaussian state. If a joint-normal form is assumed for the pdf of vertical displacement of isopycnal pairs, $\eta(\rho_i),\eta(\rho_j)$, it is easily shown that the pdf of isopycnal separation Δz_{ij} is also Gaussian (Desaubies and Gregg 1981, henceforth DG81). There is always a finite probability that $\Delta z_{ij}(t)$ will vanish, resulting in singular values for vertical gradients of passive scalars $\frac{\partial\theta}{\partial z} = \frac{\theta(\rho_i) - \theta(\rho_j)}{\Delta z_{ij}(t)}$. From a mathematical viewpoint, a Gaussian zeroth order state is an awkward starting point for the description of the fine scale field.

Knowledge of the pdfs of isopycnal separations enables the statistical modeling of a number of phenomena of physical interest in addition to vertical gradient fluctuations. Measurements of variance can be used to infer skewness, kurtosis, etc., once a form for the separation pdf is established.

This work complements an introductory paper, Pinkel et al 1991, (Henceforth P91). A more complete discussion of statistical matters is presented in Pinkel and Anderson 1991 (PA91).

Isopycnal displacement data obtained in the 1986 experiment PATCHEX are used in this study. The data are derived from a series of nine thousand CTD profiles from the sea surface to 560 m, obtained over an 18.75 day interval. Three types of isopycnal separation statistics are accumulated. Probability density functions of isopycnals separation are formed at varying mean separations $\overline{\Delta z}$. These statistics are gathered in both fixed depth (Eulerian) and fixed density (semi-Lagrangian) reference frames. In addition, discrete probability functions are formed, describing the probability of occurrence of varying numbers of isopycnals in fixed vertical intervals.

The discrete probability distributions formed in the fixed depth intervals are found to be very nearly Poisson for vertical bins, H, of order 3 m and greater. The corresponding Eulerian and Lagrangian pdfs of isopycnal separation are very nearly gamma pdfs, as would be predicted from elementary Poisson theory.

This surprising finding enables the simple modeling of strain and gradient statistics in both reference frames. A single parameter, λ, specifies the entire description at all vertical scales $\overline{\Delta z} > 3$ m.

The measurements and data are described next. Statistical results are then presented, followed by discussion of these findings.

MEASUREMENTS

The data considered here are a set of 9,000 CTD profiles from the surface to 560 m. These were obtained during October 1986 from the Research Platform FLIP. FLIP was located at 34°N, 127°W, approximately 500 km west of Point Conception, CA. Position was maintained to within 300 m by a two point moor. Water depth at the site is 4 km.

The CTD's used are Seabird Instruments model SBE-9s. Two such instruments are profiled. The upper unit is cycled from the surface to 320 m. The lower system covers the depth range 250-560 m. Profiles are repeated at 3 min intervals. The drop rate of the sensors is approximately 3.5 m/s. It is not necessary to pump water through the conductivity cell to achieve adequate spatial resolution at this drop rate.

Following a time response correction to the temperature sensor, the vertical resolution of both the temperature and conductivity sensors is limited to 2 m by a low pass filter (Sherman 1989). Density profiles are then produced. A set of 560 isopycnals, of mean separation 1 m, is then followed for the duration of the data experiments (Fig. 1). The experimental approach is discussed in greater detail in P91.

The three hour record presented in Figure 1 represents a small portion of the 18.75 day data set. In it one sees a general trend toward decreasing isopycnal depth, associated with the baroclinic tide. Superimposed on this trend are higher frequency (1-2 cph) internal waves. These are extremely coherent with depth. Against this large scale background, the fine scale straining of the density field is seen. Isopycnals converge to form "sheets" of high vertical gradient and diverge, forming low gradient "layers". The typical time-scale for the fine scale variation appears to be from one-half to several hours, in this short record.

Protagonists in the present study are:

the isopycnal separation $\Delta z_{ij} \equiv z(\rho_i,t) - z(\rho_j,t)$

the normalized separation $\gamma_{ij}(t) = \Delta z_{ij}(t)/\overline{\Delta z}_{ij}$

and the finite difference strain $\hat{\gamma}_{ij} = \gamma_{ij}(t) - 1$.

The finite difference strain can be thought of as an approximation to the actual strain, $\partial\eta(\rho,t)/\partial z$, where $\eta \equiv z(\rho,t) - \overline{z}(\rho)$. Alternatively, separation variance statistics can be considered as precise estimates of the structure function

$$F(\overline{\Delta z}) = 1/\pi \int_{-\infty}^{\infty} (1 - \cos k\overline{\Delta z})S(k)dk = \langle \gamma^2 \mid (\overline{\Delta z}) \rangle - 1 \quad \text{(Tennekes and Lumley, 1972).} \qquad 1)$$

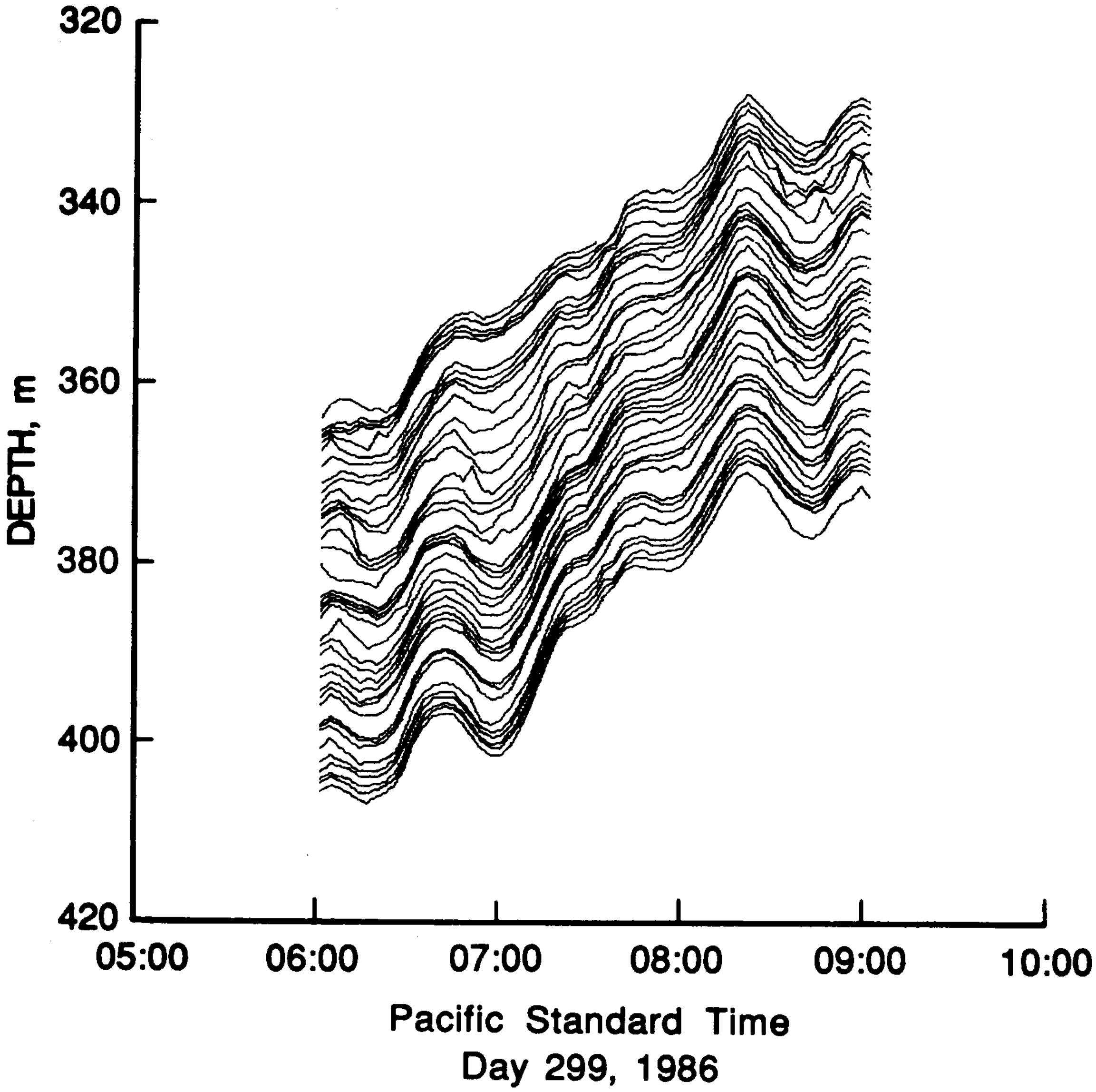

Fig. 1. An example of isopycnal depth fluctuations as seen in the PATCHEX data set. The statistics of isopycnal separation are the focus of the present study.

Here, S(k) is the vertical wavenumber spectrum of strain.

Obviously, no technique can produce quality estimates of the spectrum at high wavenumber in the absence of accurate measurements at small vertical scale Δz. Evenly spaced estimates of F at intervals of $\Delta z = n$ meters, $n = 1, N$, result in a wavenumber spectral estimate with Nyquist wavenumber of .5 cpm, and wavenumber resolution of N^{-1} m (McKean, 1974).

Three aspects of the measurements, sensor noise (ε), resolution, and statistical precision impact the present discussion. Sensor noise results in error in the estimates of density profiles. Noise has both correlated and uncorrelated aspects. At separations greater than a few meters, the noise which influences the estimate of one isopycnal is independent of that influencing another. Strain variance estimates are biased in the presence of noise.

$$<\gamma^2> = \frac{<\Delta z^2>}{\overline{\Delta z}^2} + \frac{2<\varepsilon^2>}{\overline{\Delta z}^2} \qquad 2)$$

The error in the estimate of the depth of a given isopycnal is approximately .25 m rms in the PATCHEX data set. This corresponds to a strain variance bias of .12 at $\overline{\Delta z}$ = 1 m, decreasing to .005 at $\overline{\Delta z}$ = 5 m. Thus the bias can be as large as 20 % of the total signal at $\overline{\Delta z}$ = 1 m (assuming the errors associated with two closely spaced isopycnals are indeed uncorrelated). Since the variance bias decreases as $\overline{\Delta z}^{-2}$ while the strain signal decreases more nearly as $\overline{\Delta z}^{-1}$, (Fig. 7) random error quickly becomes insignificant.

Noise has an effect on the pdfs of separation. The observed pdfs are a convolution of the true pdf of separation with the pdf, presumably Gaussian, of the noise. Thus, observed pdfs will be broader than the true, with their variance increased by the variance of the noise. This effect will be significant at small separation $\overline{\Delta z}$.

Resolution is a concern when considering closely spaced isopycnals. One would expect to observe fewer than the actual number of instances of close isopycnal spacing (small γ) in pdfs of separation. Conversely, fewer observations of "many isopycnals found in a fixed size bin", are expected in the discrete probability functions presented below. The finite resolution of the CTD is particularly damaging to the present study given that the spatial autocorrelation of strain transitions from positive (...if two adjacent isopycnals are squeezed into a sheet, it is likely that the immediate neighboring isopycnals are being drawn into the sheet...) to negative (...if five isopycnals are being drawn into a sheet, it is likely that there will be an absence of isopycnals, a layer, five m away) at about the resolution scale of the CTD, 2 m.

At sufficiently small scales the strain correlation is high and positive. A principal conjecture of P91 is that strain statistics approach log-normal form in this region. If the log-normal regime indeed exists, it occurs at scales unresolvable by the present CTD. Further exploration of the log-normal issue awaits improved instrumentation.

Statistical precision is often a concern when trying to compare pdfs estimated from data with classical functional forms. In this work, estimates of pdfs are formed in four 100 m depth ranges, 100-200 through 400-500 m. Thus, 9×10^5 measurements (9,000 profiles * 100 isopycnals) are available for each 100 m estimate. At issue is the fraction of these points that is statistically independent. This question is discussed by Briscoe (1977) in his investigation of the Gaussianity of the horizontal velocity and vertical displacement of the IWEX data set. From numerical simulations Briscoe finds that the effective decorrelation time for displacement is of order 1/2 day. A nineteen day displacement time series would consist of 38 independent samples, corresponding to 76 degrees of freedom.

The situation is more complicated for the strain field, given the significance of non-linear distortion at small scale. P91 demonstrate that the characteristic lifetime of "layer" events ($\gamma > 1$) is shorter than "sheet" events ($\gamma < 1$). The number of independent estimates of $\gamma < 1$ events per unit time is less than that of $\gamma > 1$ events. Monte Carlo simulations of the strain field will not model this effect appropriately unless the bi-spectrum of the field is properly specified.

In PA91 an attempt to estimate the statistical precision of the strain, as a function of γ itself, is presented. The variability of independent estimates of the strain pdf, from depth to depth at fixed γ, is used to determine the effective number of degrees of freedom. When large mean separations, $\overline{\Delta z} \geq 3$ m, are considered, the observations are consistent with an 100 degree of freedom process, more or less independent of γ. At very small mean separations, variations in statistical stability with γ is clearly seen. The effective number of degrees of freedom varies from 80 to 100 as γ increases from .2 to 2.

THE PDF OF STRAIN AND DISPLACEMENT

Joint pdfs of strain and displacement have been formed using the PATCHEX isopycnal displacement time series. The pdfs are formed for isopycnal pairs of mean separation $\overline{\Delta z} = 1 - 50$ m. For each mean separation, the pdfs are binned into 100 displacements ($\pm$ 50 m) by 100 strains ($\gamma = 0$ to 5) in four depths zones ($\overline{z} = 100$ - 200 through 400 - 500 m).

Two sets of joint pdfs are produced. Lagrangian pdfs are formed by tracking the evolution of specific isopycnals pairs (ρ_i,ρ_j) through the 19 day data set. Corresponding Eulerian pdfs are formed by tracking the separation between that pair of isopycnals, separated in the mean by $\overline{\Delta z}$, which is spanning a specific fixed depth, z_0, at each instant of time. The Eulerian study is repeated for fixed depths of 100-500 m, at 1 m increments. The resulting pdfs are averaged into 100-200 through 400-500 m bins, in correspondence with the averaging used in the Langrangian study.

A representative joint pdf of strain and displacement is presented in Fig. 2. The pdf is formed in a Lagrangian frame, using isopycnals of mean separation 3 m. Although not clearly apparent in the figure, it can be shown (PA91) that the data are consistent with the assumption that the joint pdf is separable, and that the pdf of average displacement $\eta_{ij} = (\eta(\rho_i) + \eta(\rho_j))/2$ is Gaussian. The pdf of strain is clearly non-Gaussian, with observations of close isopycnal spacing ($\gamma < 1$) more frequent than those of large separation ($\gamma > 1$).

THE PROBABILITY DENSITY FUNCTION OF STRAIN

The joint pdfs previously discussed can be integrated with respect to displacement to produce univariate pdfs of strain. Strain pdf are presented in Figs. 3 and 4 for the semi-Lagrangian and Eulerian frames respectively, for mean separations of Δz of 1-10 m. Sample pdfs have been formed for mean separation as great as 50 m. At scales greater than 10 m these appear very nearly Gaussian. Nevertheless, skewness and kurtosis estimates are significant to separations of order 30 m.

The observed pdfs have been fit to a variety of classical forms, including Rayleigh, Weibull, log normal and gamma. Significant discrepancies are subjectively apparent in all comparisons, with notable exception of the gamma pdf, which fits very well (Figs. 3, 4, light curves). The gamma pdf has the form

$$G(x) = \frac{\beta^{\alpha+1} x^{\alpha} e^{-\beta x}}{\Gamma(\alpha)} \qquad 3)$$

with mean $<x> = \alpha/\beta$ and variance $<x^2> - <x>^2 = \alpha/\beta^2$ (eg. Papoulis 1984)

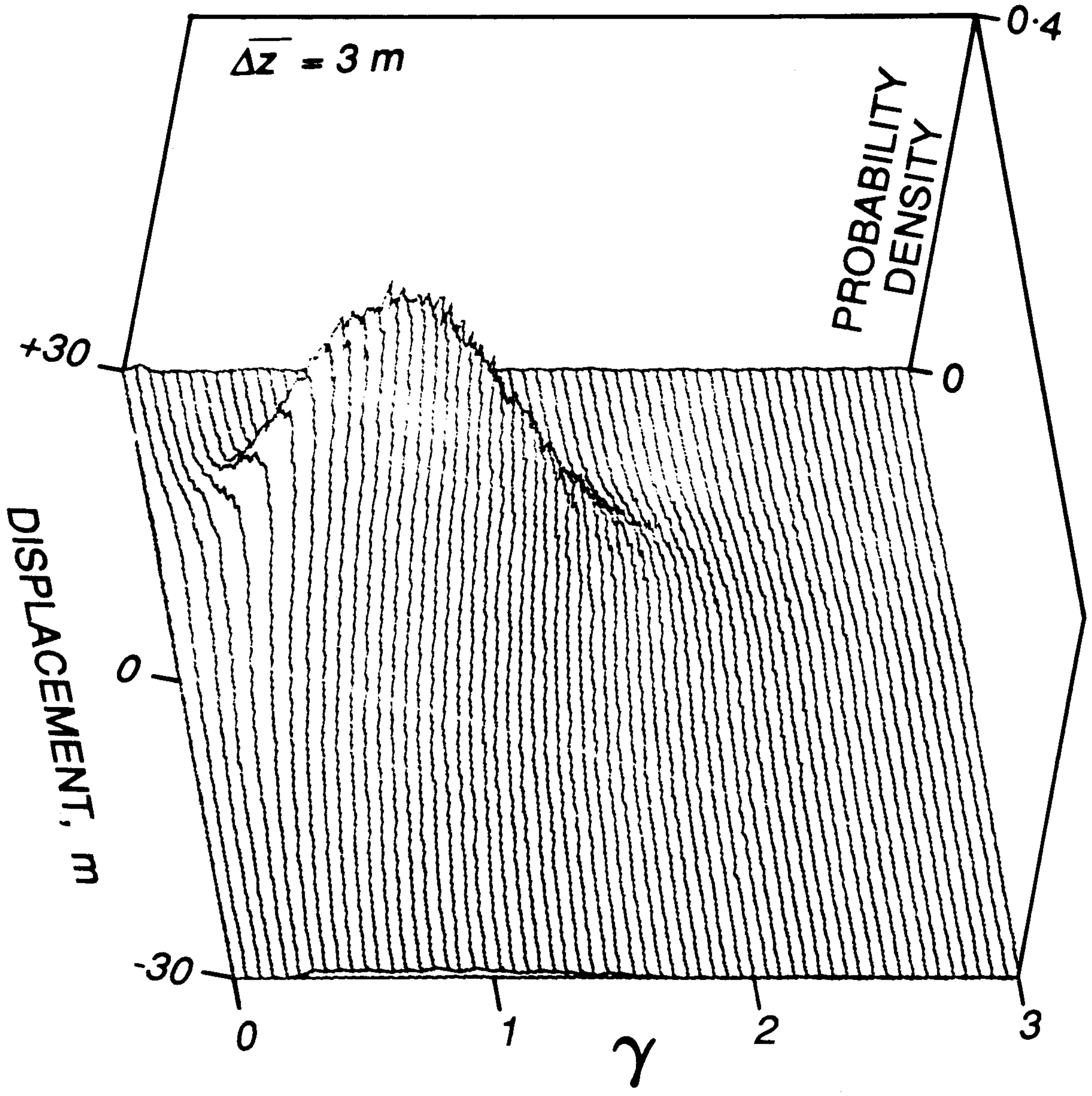

Fig. 2. An example of a joint probability denotes function of strain and displacement. This estimate was formed in a semi-Lagrangian frame, considering isopycnal pairs separated in the mean by 3 m. The PATCHEX data are consistent with the hypothesis that strain and displacement are independent quantities, with displacement obeying Gaussian statistics.

The semi- Lagrangian data are constrained to have $<\gamma> = 1$, $<\Delta z> = \overline{\Delta z}$, by initial choice of isopycnals. Hence $\alpha = \beta\Delta z$. The fits presented in Fig. 3 are thus one parameter fits, with sample variance matched to the model variance. The Eulerian observations are not constrained to unity mean. The fits are thus two parameter fits. The observed mean and variance are used to set model pdf parameters in Fig. 4.

The model gamma pdf is seen to fit the observations well in the 200-300 m depth range, except at separations less than 4 m. The fits are comparable in the other depth ranges, with the exception of the 300-400

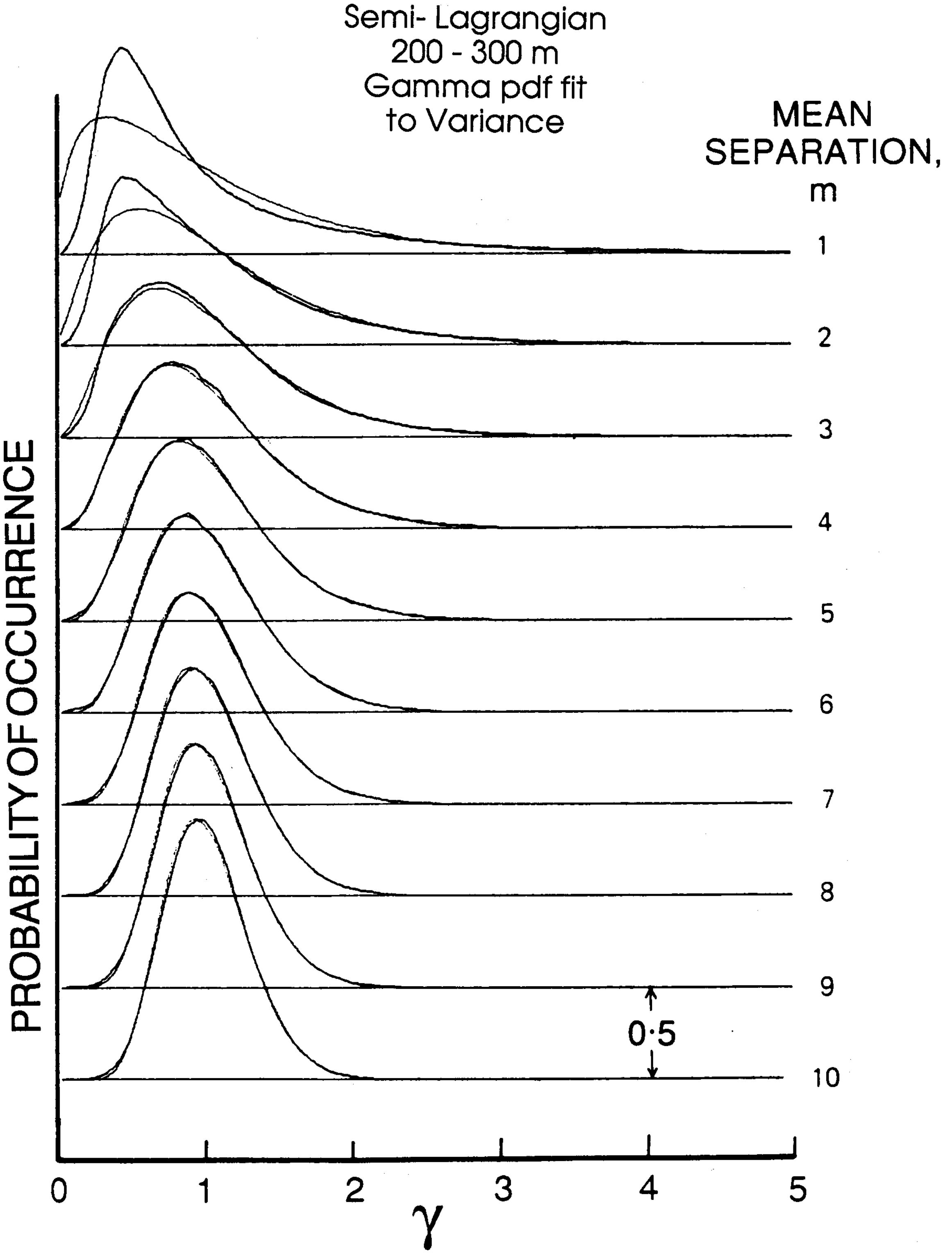

Fig. 3. Probability density functions of normalized separations, γ, formed in a semi-Lagrangian frame, for mean isopycnal separations 1-10 m. Light lines give model gamma pdfs, constrained to have unity mean and the observed variance. Data from 200-300 m depth zone are presented.

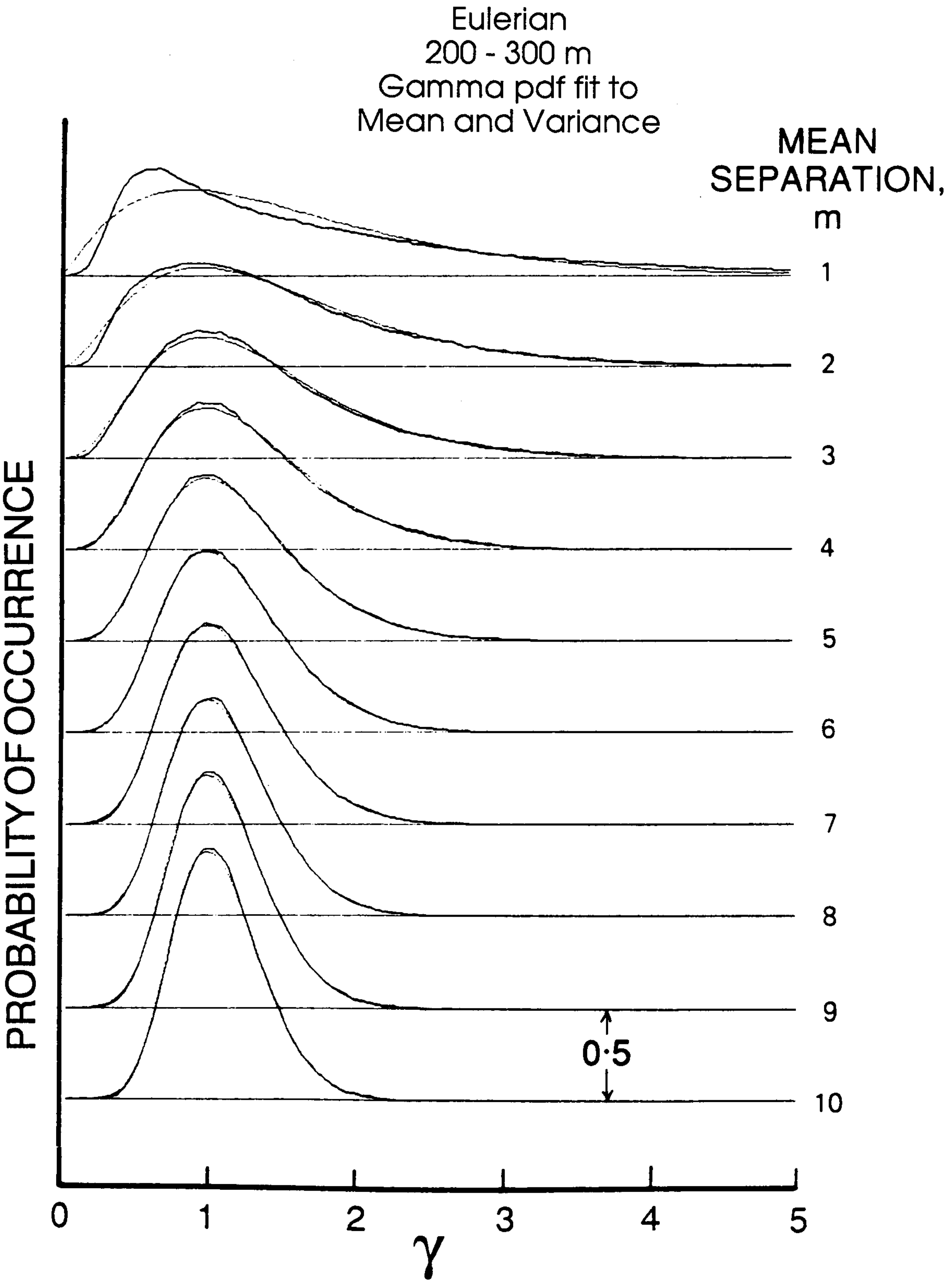

Fig. 4. Probability density functions of normalized separation, γ, as in Fig. 3 except formed in an Eulerian frame. Light lines give model gamma pdfs, constrained to have mean and variance identical to the observations. Data from 200-300 m depth zone are presented.

m interval, where the Lagrangian pdfs appear distorted at small γ over a range of $\overline{\Delta z} = 3 - 7$ m. This issue is addressed in PA91. The fits could be improved by employing a least squares fitting procedure. However, optimizing the fit is not the point of the present exercise.

Gamma pdfs are associated with the classical theory of Poisson processes. They describe the statistics of separation between the occurrence of Poisson "events" (Papoulis, 1984). If, indeed, a simple Poisson statistics describe the non-Gaussian behavior of the fine scale field, the problem of modeling the motion field in this regime can be significantly advanced.

POISSON STATISTICS

Poisson statistics describe the occurrence of discrete "events". If the probability of occurrence of these events is uniform and the occurrence of one event in no way influences the occurrence of any other events, Poisson statistics will apply (Papoulis, 1984). The Poisson probability function gives the probability that the number of events which occur in a dimensional interval of length H will equal any specified value, k.

$$P(n = k \mid H) = \frac{(\lambda H)^k e^{-\lambda H}}{k!} \qquad 4)$$

The Poisson probability function has the interesting property that the mean number of "events" occurring in an interval H, λH, is equal to the variance in the number of events.

It is not clear exactly what constitutes a "Poisson event" in the context of the fine scale variability in the thermocline. We have tracked a set of isopycnal surfaces with mean spacing arbitrarily chosen to be 1 m. A Poisson-like investigation can be conducted, tracking the number of isopycnals which are found to occur in fixed depth bins of varying size H. This is done for the four 100-m depth ranges used in the previous studies. The results are presented for the 200-300 m range in Fig. 5. Not surprisingly, the mean number of isopycnals found in a depth bin of thickness H is H, given our initial choice of isopycnals mean separation. The oceanic signal is seen in the variation of the higher moments with H. To facilitate comparison with the classical Poisson distribution, which is constrained to have mean equal to variance, the observed probability functions are re-scaled in terms of equivalent "Poisson events", $\hat{n} = \lambda n$. Here $\hat{n}$ is the number of "events" alleged to occur in the fixed depth bin, n is the observed number of isopycnals, and λ is a scale factor chosen such that the mean number of "events" is equal to the event variance. The fitting of model Poisson probability functions (Fig. 5, light line) to the data is accomplished by setting the model mean equal to the rescaled observed mean, λH. The fit is impressive, with significant discrepancies apparent only in 1 and 2 bin sizes.

THE THERMOCLINE AS A POISSON PROCESS

The excellent fits of the Gamma and Poisson pdfs to the observations at vertical scales greater than 3 m encourages the adoption of Poisson statistics as a zeroth order model for the thermocline. In this section we review relevant aspects of Poisson theory. Subsequently we discuss observed departures from the predictions of the Poisson model, and why these departures are to be expected in a real fluid.

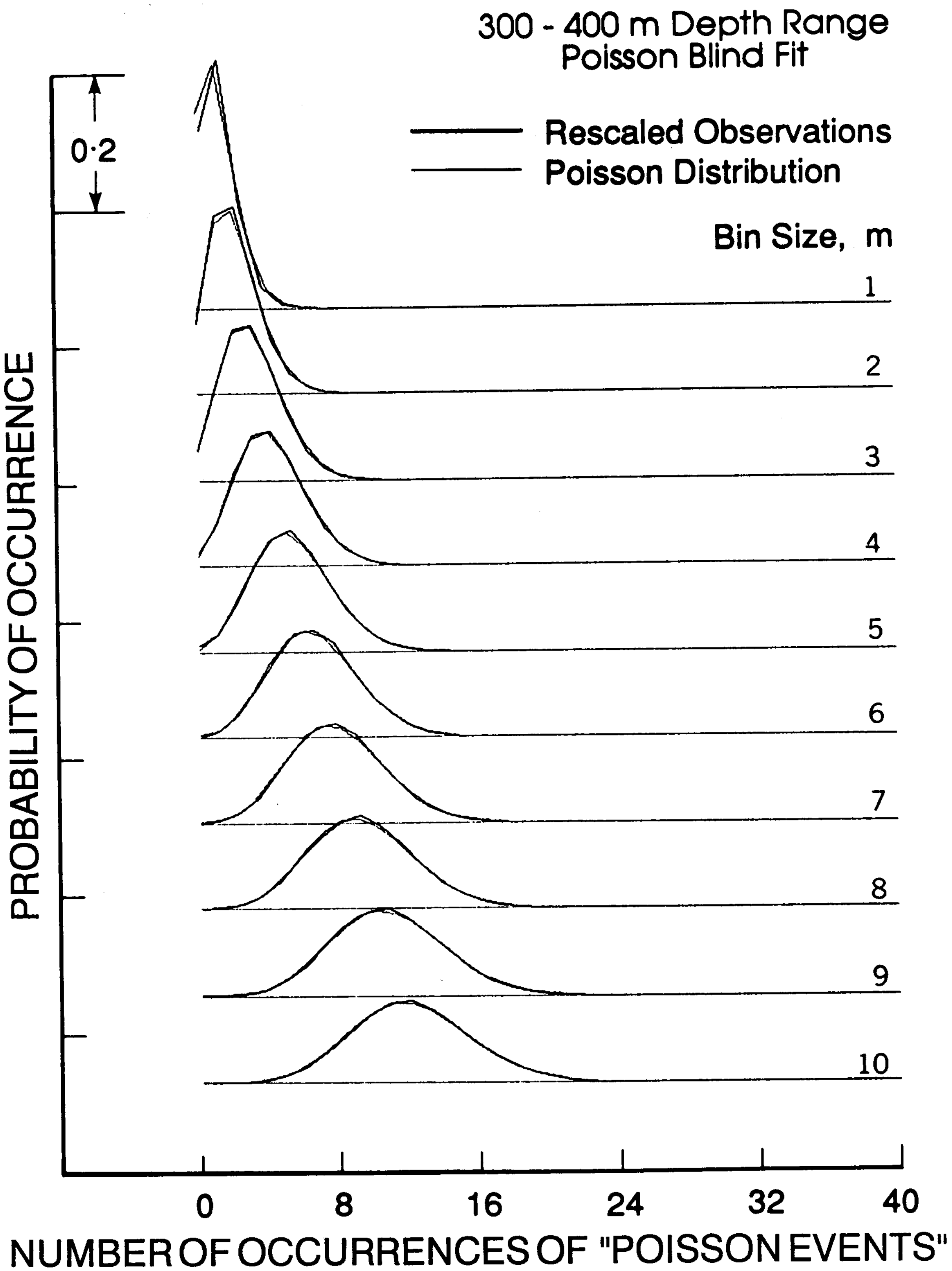

Fig. 5. Probability function of the number of isopycnals found in fixed depth bins of size H = 1-10 m. The observed number of isopycnals is re-scaled by the factor λ, such that the sample mean equals the variance. The light curve gives a theoretical Poisson distribution. Data from the depth range 300-400 m are presented here.

Papoulis (1984) notes that the statistics of separation of Poisson events take two distinct forms. If two adjacent events are considered at random, the gamma pdf

$$G(\Delta z \mid 1, \lambda) = \lambda e^{-\lambda \Delta z}/\Gamma(1) \tag{5}$$

describes the separation statistics. However if the two adjacent random points are constrained to bracket some fixed point, z_0, the pdf of separation is given by

$$G(\Delta z \mid 2, \lambda) = \frac{\lambda^2 \Delta z e^{-\lambda \Delta z}}{\Gamma(2)}. \tag{6}$$

The added constraint of requiring the points to bracket a fixed point alters the statistics.

We identify (6) as the appropriate pdf for isopycnal separation in an Eulerian frame, at the fixed depth z_0. Expression (5) is the appropriate semi-Lagrangian separation pdf, for a mean separation $\overline{\Delta z} = \lambda^{-1}$. When considering the pdf of distance between an arbitrary random point and one removed by (n – 1) intermediate points, the order of the appropriate gamma pdf is simply increased, e.g.,

$$G(\Delta z \mid n, \lambda) = \frac{\lambda^n \Delta z^{n-1} e^{\lambda \Delta z}}{\Gamma(n)}. \tag{7}$$

The continuous observations presented here are consistent with the interpretation that Poisson "events"" occur every λ^{-1} meters. Thus the discrete parameter n can be replaced by its continuous equivalent $\lambda^* = \lambda \overline{\Delta z}$.

$$G^L(\Delta z \mid \overline{\Delta z}, \lambda) = \lambda \cdot \frac{(\lambda \Delta z)^{\lambda^*} e^{-\lambda \Delta z}}{\Gamma(\lambda^*)} \tag{8}$$

$$G^E(\Delta z \mid \overline{\Delta z}, \lambda) = \frac{\lambda (\lambda \Delta z)^{\lambda^* + 1}}{\Gamma(\lambda^* + 1)} e^{-\lambda \Delta z} = \gamma\, G^L(\Delta z \mid \overline{\Delta z}, \lambda) \tag{9}$$

The skewness seen in the semi-Lagrangian frame reflects the relatively passive advection of the density field by the velocity field, on the fine scale. Isopycnals which find themselves close together experience nearly identical advecting velocity fields. Hence they remain together for a relatively long time. The Eulerian pdf is less skewed than its semi-Lagrangian counterpart, for a given mean separation $\overline{\Delta z}$. This reflects the fact that when isopycnal pairs are traveling closer together than average, they are less likely to span a specific reference depth than when they are farther apart. Not surprisingly, the chance of spanning a given point while simultaneously having separation Δz increases linearly with Δz. This result is reflected in equations 5,6 and 8,9, and is demonstrated for arbitrary strain pdfs in PA91, under the assumption that strain and displacement are independent quantities.

The immense power in the Poisson model results from the fact that the single dimensional parameter λ (m^{-1}) describes the variability of strain not just at a particular vertical scale $\overline{\Delta z}$, but at all scales.

Thus, mean isopycnal separation is given by

$$<\Delta z_L> = \frac{\lambda^*}{\lambda} \equiv \overline{\Delta z} \qquad 10)$$

$$<\Delta z_E> = \frac{\lambda^* + 1}{\lambda} = \overline{\Delta z} + \lambda^{-1}$$

Separation variance is:

$$<\Delta z^2>_L - \overline{\Delta z}^2 = <\Delta\eta^2>_L = \lambda^* / \lambda = \overline{\Delta z} / \lambda$$

$$<\Delta z^2>_E - <\Delta z>_E^2 = <\Delta\eta^2>_E = \lambda^* + \frac{1}{\lambda^2} = \overline{\Delta z} / \lambda + 1 / \lambda^2 \qquad 11)$$

The growth of variance with mean separation is linear. If this pattern were seen in the data down to arbitrarily small separation, it would correspond to a white vertical wavenumber spectrum of strain,

$$S_\gamma(k) = \lambda^{-1}\,m \qquad k > 0$$

$$S_\gamma(k) = 0 \qquad k = 0 \qquad 12)$$

The corresponding isopycnal vertical displacement spectrum is of k^{-2} form.

THE DOMAIN OF VALIDITY OF THE POISSON MODEL

One anticipates departures from the Poisson model at both large and small vertical scales. This can be seen by considering the variance of isopycnal separation, Eq. 11, which is predicted to grow linearly without bound as mean separation increases. This behavior is a consequence of the fact that the model k^{-2} vertical wavenumber spectrum of displacement extends to arbitrarily low wavenumber. In a finite depth ocean, the governing physics will surely change as the scale of the waveguide thickness is approached, resulting in a departure from Poisson behavior.

At small vertical scales, of order λ^{-1}, the model itself becomes inconsistent. The Eulerian separation variance fails to vanish at $\overline{\Delta z} = 0$. This corresponds to a limit of infinite strain variance in the Eulerian frame as the vertical scale tends to zero. The Poisson model Cox number presented in Table 1 also becomes singular at small scales.

There is a simple interpretation for these a-physical aspects of the Poisson model. The lack of predictability is central to the concept of a Poisson process. The occurrence of a specific Poisson "event" in no way influences the occurrence of subsequent events. However, at sufficiently small scales in the ocean one expects the strain field to be correlated. The oceanic strain spectrum cannot be white. It presumably is band limited, with a bandwidth which is the inverse of this correlation scale. To the extent that a "sheet and layer" model of the strain field is valid, one expects the autocorrelation function of strain to assume negative as well as positive values. (If a large number of isopycnals is found concentrated in a given region, a "sheet", it is likely that there will be a relative absence of isopycnals, a "layer" nearby.) A purely Poisson model is not appropriate at small vertical scales, where the strain field is correlated.

Discrepancies between the observations and the idealized Poisson model can be seen in plots of observed variance vs. mean separation. These are presented in Figs. 6 and 7 for both the Eulerian and semi-Langrangian studies. Data are presented to vertical separations of 50 m.

Focusing attention on the semi-Lagrangian separation variance, Fig. 7 c,d, disagreement with the Poisson model is found at both smaller and larger scale. At small scales, the variance first increases more rapidly than $\overline{\Delta z}^{+1}$ ($\overline{\Delta z} < 4$ m), then more gradually ($\overline{\Delta z} > 4$ m). This corresponds to the positive and negative regions of strain correlation, as sensed by the low resolution CTDs used in the experiment.

At large separations, the variance increase never does approach the $\overline{\Delta z}^{+1}$ Poisson form, except perhaps in the 100-200 m depth range. Simple simulations suggest that this discrepancy is related to the depth variability of the wavefield. Isopycnal displacement variance changes significantly with depth, as a result of the changing profile of Vaisala frequency. WKB theory suggests that at depths removed from the sea surface or sea floor,

$$<\eta^2>(z) = <\eta^2>_0 \frac{N_0}{N(z)} \qquad 13)$$

(Garrett and Munk 1972).

While this classical expression is only approximately correct for the PATCHEX data, it could be used to rescale (WKB stretch) the displacement field to produce a data set more nearly homogeneous in depth. This has not been done here, in the interest of presenting the basic results in a model independent format.

A related check on the applicability of the Poisson model is to examine the variability in the observed estimate of λ, the Poisson "constant", as a function of bin size H or mean separation $\overline{\Delta z}$. This is presented in Fig. 8, for the fixed depth Poisson study and the semi-Langrangian isopycnal separation study. At 10 m scales λ has a minimum value of approximately 1.1 m^{-1}. Slight increases in this constant are apparent at both small and larger scales. Again, the finite correlation of the strain field is thought to cause the small scale departure from ideal Poisson behavior. The non-homogeneity of the wavefield with depth is a suggested source of the discrepancy at large scale.

THE MODELING OF VERTICAL GRADIENTS

If one abandons the assumption of Gaussianity, it simplifies the modeling of vertical gradients in the thermocline. There is no need to linearize about small departures from the mean gradient. Recalling the expression for the instantaneous gradient of a passive scalar,

$$\frac{\partial\theta}{\partial z} = \frac{\overline{\Delta\theta}}{\overline{\Delta z}} \cdot \gamma^{-1}(t),$$

interest is focused on the statistics of $\mu(t) \equiv \gamma^{-1}(t)$. If $p(\gamma)$ is an arbitrary pdf of γ ($p(\gamma) \equiv 0$ for $\gamma \leq 0$), then $g(\mu) = p(\mu^{-1})/\mu^2$ gives the corresponding pdf of the scalar gradient.

Using this approach, one can derive general expressions for quantities of interest.

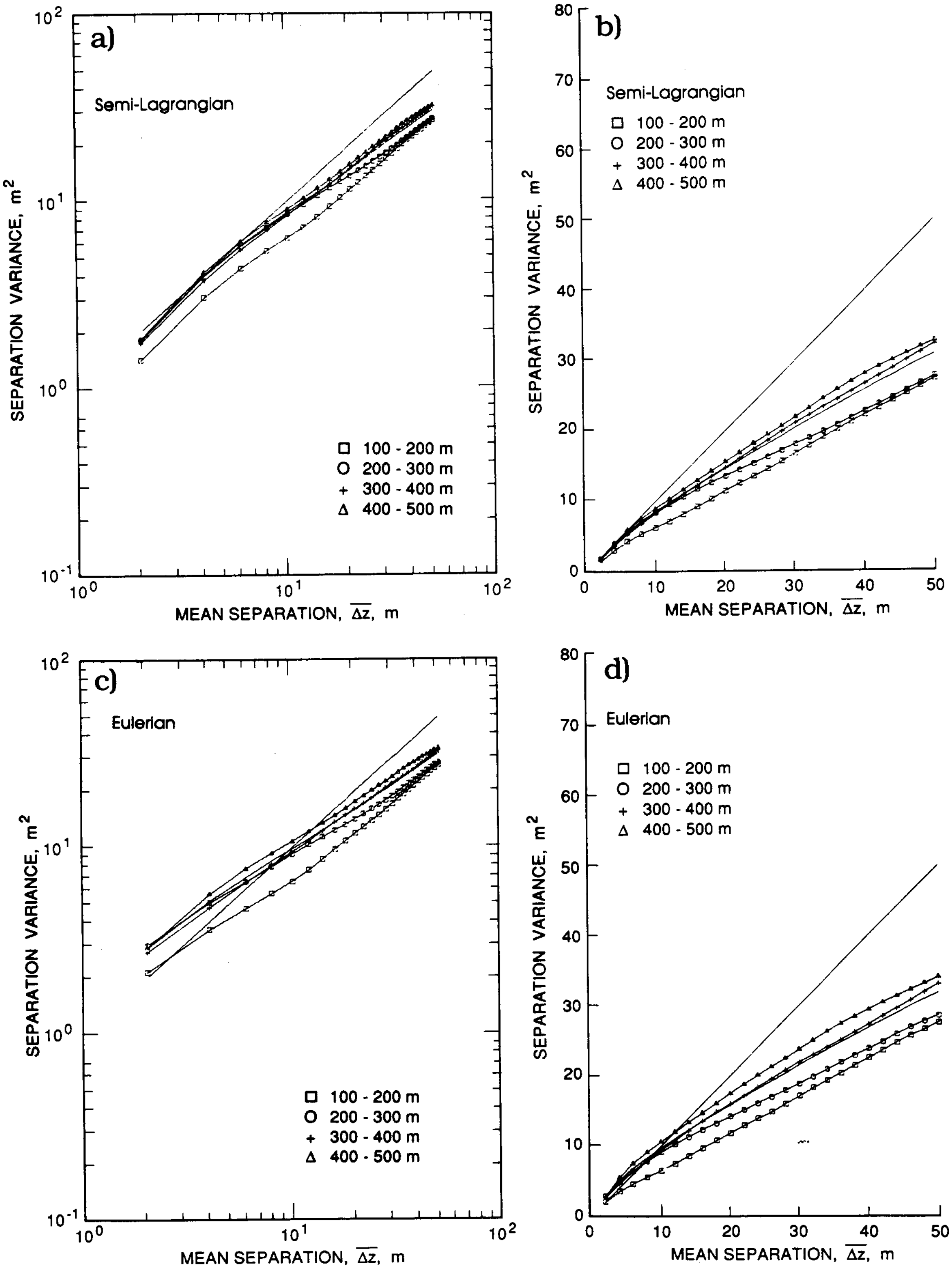

Fig. 6. Isopycnal separation variance as observed in semi-Lagrangian (a,b) and Eulerian (c,d) frames. A Poisson model would indicate a linear increase of variance with range.

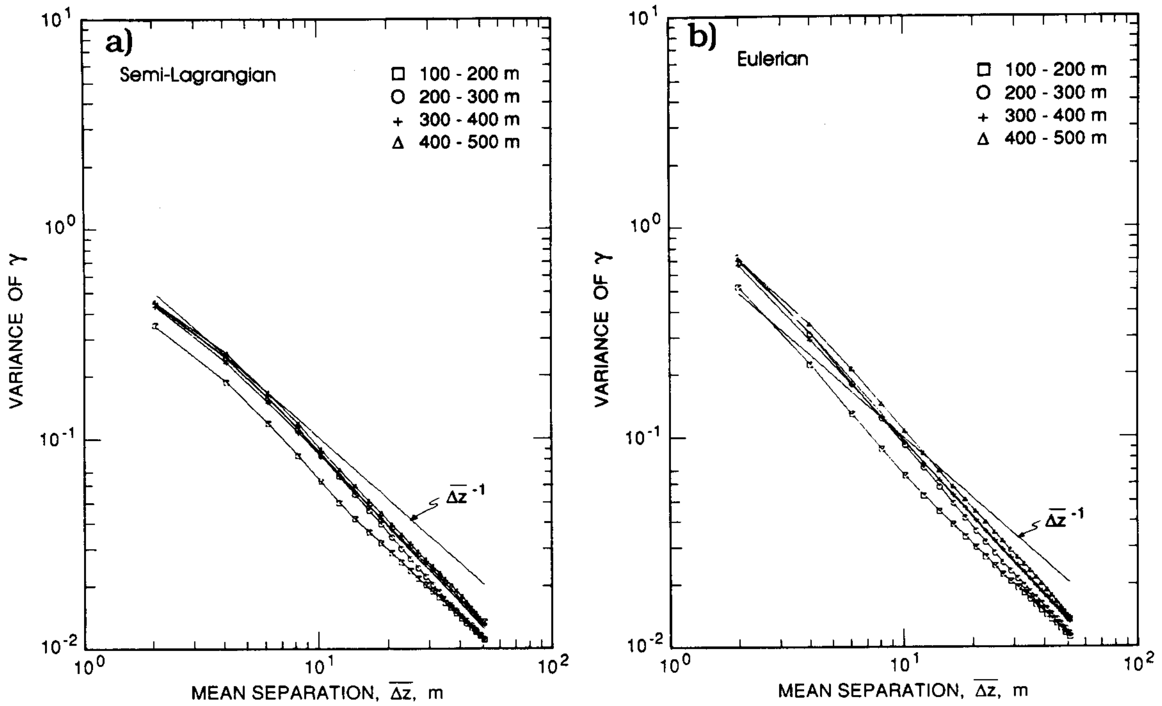

Fig 7. Normalized separation variance,l $<\gamma^2>$ - 1, as observed in semi-Lagrangian (a) and Eulerian (b) frames.

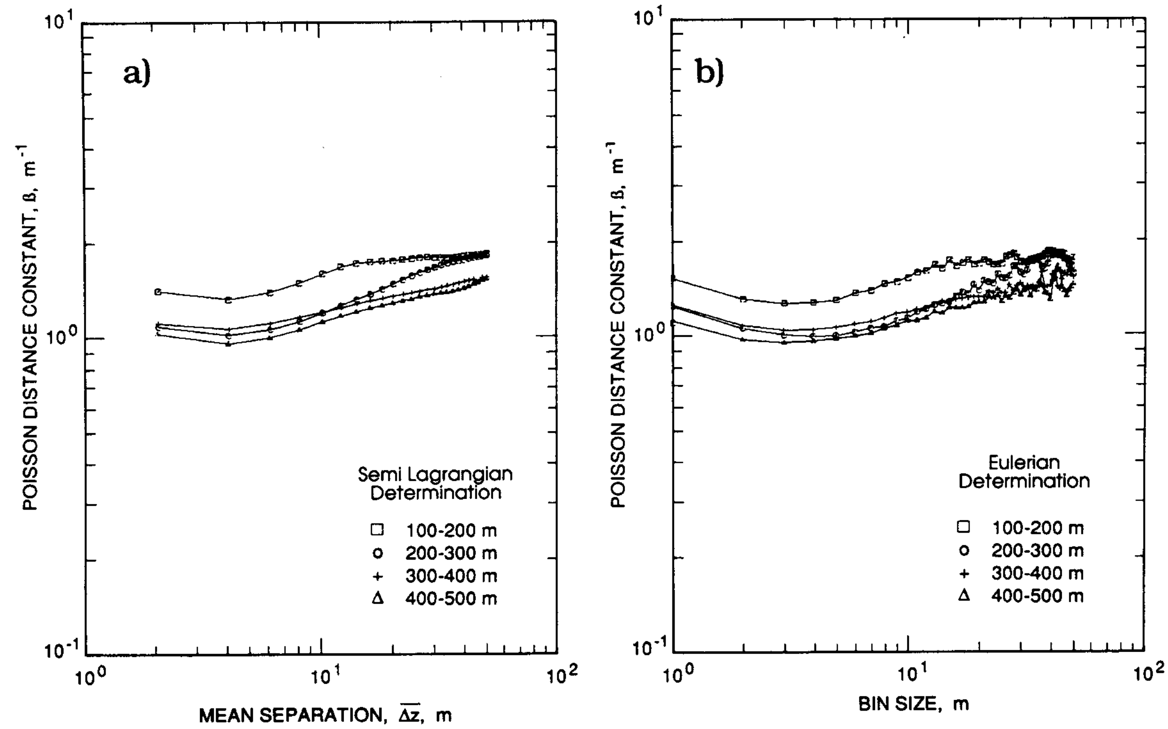

Fig. 8. a. The Poisson scale factor λ as determined by the mean to variance ratio of the semi-Lagrangian pdfs. b. The scale factor as determined by the mean to variance ratio of the number of isopycnals in fixed depth bins of varying size, H.

The Eulerian mean gradient is given by:

$$<\frac{\Delta\theta}{\Delta z}>_E = \frac{\Delta\theta}{\overline{\Delta z}}<\mu>_E = \frac{\Delta\theta}{\overline{\Delta z}}\int_0^\infty \mu g^E(\mu\,|\,\Delta\overline{z})d\mu$$

$$= \frac{\Delta\theta}{\Delta z}\int_0^\infty \gamma^{-1}p^E(\gamma\,|\,\overline{\Delta z})d\gamma$$

$$= \frac{\Delta\theta}{\Delta\overline{z}}\int_0^\infty p^L(\gamma\,|\,\overline{\Delta z})d\gamma = \frac{\Delta\theta}{\overline{\Delta z}} \qquad 14)$$

Here, we make use of the fact that $p^E(\gamma) = \gamma p^L(\gamma)$ is a general result, applying not only to the Poisson process (PA91). Also, the Eulerian Cox fine scale number

$$C_E \equiv <(\frac{\Delta\theta}{\Delta z})^2>_E / <\frac{\Delta\theta}{\Delta z}>_E^2 = \qquad 15)$$

$$\int_0^\infty \gamma^{-2}p^E(\gamma\,|\,\overline{\Delta z})d\gamma = \int_0^\infty \gamma^{-1}p^L(\gamma\,|\,\overline{\Delta z})\,d\gamma = <\mu>_L$$

In a semi-Langrangian frame, the mean gradient is seen to be

$$<\frac{\Delta\theta}{\Delta z}>_L = \frac{\Delta\theta}{\overline{\Delta z}}\int_0^\infty \gamma^{-1}p^L(\gamma\,|\,\overline{\Delta z})d\gamma \qquad 16)$$

Similarly,

$$C_L \equiv <(\frac{\Delta\theta}{\Delta z})^2>_L / \frac{\Delta\theta}{\Delta z}>_L^2 = \int_0^\infty \gamma^{-2}p^L(\gamma\,|\,\overline{\Delta z})d\gamma/(\int_0^\infty \gamma^{-1}p^L(\gamma\,|\,\overline{\Delta z})d\gamma)^2 \qquad 17)$$

These results are independent of the specific form of the separation pdf $p(\gamma\,|\,\overline{\Delta z})$.

Here we have demonstrated that the pdf of separation are very nearly gamma pdfs at vertical scales greater than 3 m. A summary of the corresponding model pdfs for strain and vertical gradients, as well as expressions for the Cox numbers are presented in Table 1. Note that, over the range of validity of this model, the fine scale Cox number is of the order two or less, far below values typically encountered at microstructure scales.

TABLE 1. Poisson Model Results

$\lambda = 1.1$ $\lambda^* = \overline{\Delta z}\,\lambda$ $\gamma = \Delta z/\overline{\Delta z}$ $\mu = 1/\lambda$

	Semi Lagrangian	Eulerian
Strain, γ	$p^L(\gamma\,\vert\,\overline{\Delta z}) = \dfrac{\lambda^*(\lambda^*\gamma)^{\lambda^*-1}e^{-\lambda^*\gamma}}{\Gamma(\lambda^*)}$	$p^E(\gamma\,\vert\,\overline{\Delta z}) = \dfrac{(\lambda^*\gamma)^{\lambda^*}e^{-\lambda^*\gamma}}{\Gamma(\lambda^*)}$
Vertical Gradients, μ	$g^L(\mu\,\vert\,\overline{\Delta z}) = \dfrac{(\lambda^*/\mu)^{\lambda^*}e^{-\lambda^*/\mu}}{\mu\Gamma(\lambda^*)}$	$g^E(\mu\,\vert\,\overline{\Delta z}) = \dfrac{(\lambda^*/\mu)^{\lambda^*+1}e^{-\lambda^*/\mu}}{\mu\Gamma(\lambda^*+1)}$
Cox Number	$C_L = (\lambda^*-1)/(\lambda^*-2)$	$C_E = \lambda^*/(\lambda^*-1)$

The Poisson model pdfs can be tested against existing observations of fine scale gradients. The observations of Gregg 1977, DG81 are presented in Fig. 9. Using a single value $\Delta\theta/\overline{\Delta z}$ for Gregg's local mean gradient, and the parameter $\lambda_0 = 1.1\ \mathrm{m}^{-1}$ set by our Eastern Pacific 1986 observations, it is seen that Gregg's central Pacific observations, taken nearly a decade earlier, are well fit by the model. Significant discrepancies are seen only at small differencing intervals $\overline{\Delta z}$. Here, separation statistics deviate from the Poisson-gamma model, perhaps approaching a log-normal form. In addition, negative temperature gradients are sometimes seen in Gregg's observations. These intrusive/overturn events are outside the scope of the present model.

SUMMARY

A series of 9000 CTD profiles from the surface to 560 m has been used to study the statistics of the fine scale strain field in the thermocline. A set of isopycnals, of 1 m mean spacing, is tracked for the 18.75 day duration of the observations. Three statistical studies are performed. Isopycnal separation statistics are formed in both isopycnal following and fixed-depth reference frames. A related investigation simply counts the number of isopycnals found in fixed depth intervals of varying scale. At vertical scales greater than 2-3 m, the statistics of the isopycnal counting study are Poisson. The corresponding isopycnal separation statistics are described by gamma pdfs. The relationship between these three types of information is very nearly as predicted by classical Poisson theory. (e.g., Papoulis 1984).

In this study, the instantaneous separation between isopycnals is simply the sum of the separations of the intervening isopycnals. For example:

$$z_{220} - z_{200} = (z_{210} - z_{200}) + (z_{220} - z_{210})$$
$$= (z_{201} - z_{200}) + (z_{202} - z_{201}) + \ldots + (z_{219} - z_{218}) + (z_{220} - z_{219}) \quad \text{etc.}$$

Here z_{220} indicates the instantaneous depth of the isopycnal whose mean depth is 220 m. It is easily shown that the semi Lagrangian gamma pdf associated with 20 m (say) mean separation can be given as the convolution of the gamma pdf of 10 m separation with itself, or the convolution of the pdf of 5 m separation, repeated 4 times, etc. This key mathematical property of the gamma pdf is only applicable to the present problem if the various "sub-separations" which are added together to form the larger separation are themselves statistically independent. Statistical independence implies that the separation (strain)

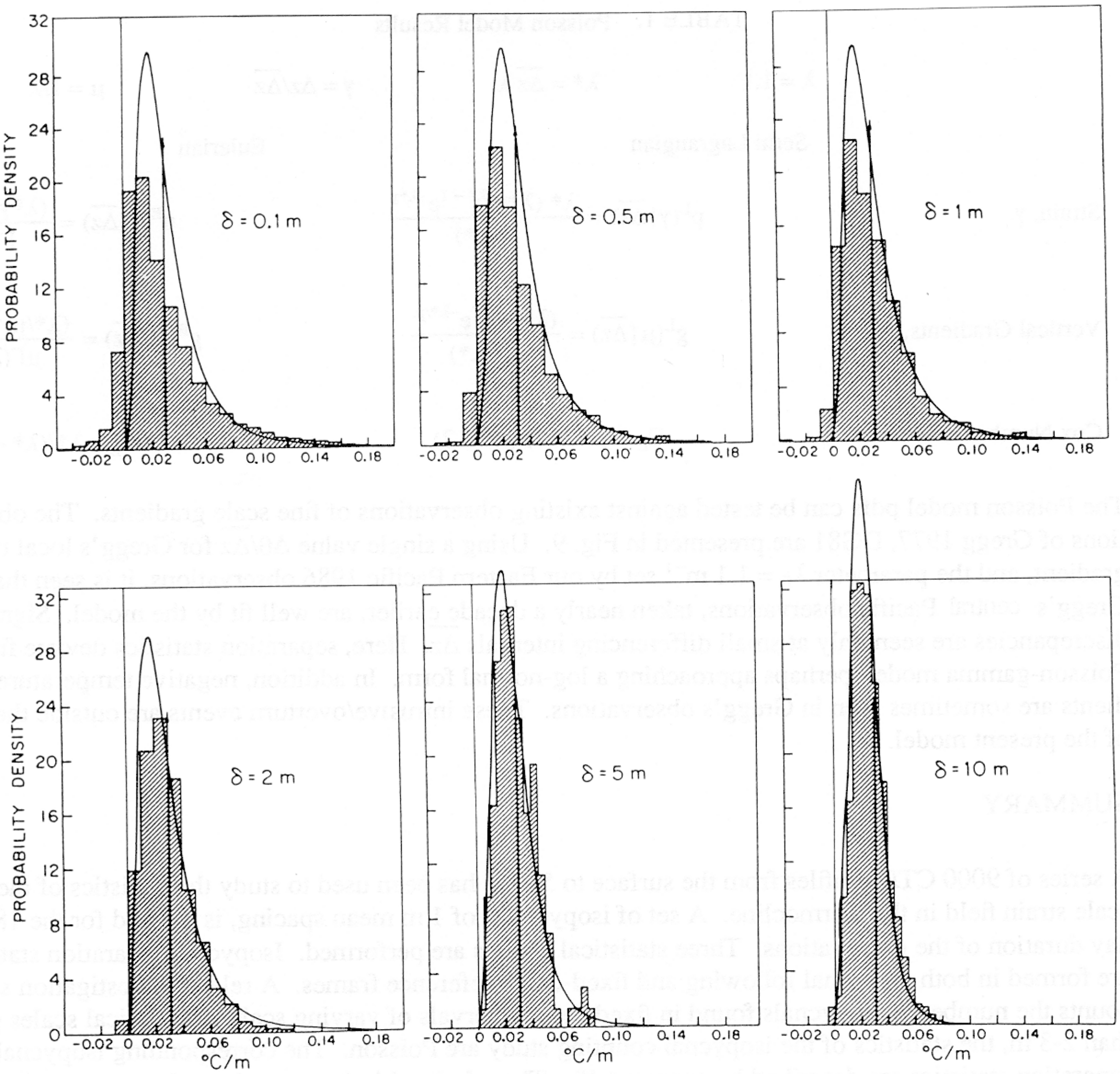

Fig. 9. Histograms of temperature gradient obtained at a variety of fixed separations $\Delta\bar{z}$ by Gregg in the Central Pacific in 1977 from Desaubies and Gregg (1981). The solid curves give the predictions of the Poisson model using the single constant λ = 1.1 m from the East Pacific 1986 PATCHEX data set.

statistics are uncorrelated. At sufficient small scale where the strain field is correlated we expect the Poisson model to deviate from observation.

A single constant λ (~1.1 m^{-1} here) should describe the mean, variance, skewness, etc. at all vertical scales where the model is applicable. The associated vertical wavenumber spectrum of strain should be white, with spectral level λ^{-1}. In fact, modest variation in λ is seen in the observations. A realistic description of (the second moment of) the strain field requires specification of high and low wavenumber bounds on the spectrum. These bounds are necessary to insure finite variance of the strain field and to acknowledge the finite depth of the ocean. Additional physical principles, beyond the scope of the Poisson model, need to be invoked to determine these bounds. An investigation of the non-Poisson aspects of the present data, with a focus on the small-scale/high wavenumber region of the spectrum, will be presented in a subsequent work.

In spite of the weak but apparent discrepancy between the observations and the Poisson model at small vertical scale, the Poisson-gamma pdfs represent a powerful starting point for a description of the fine scale fields in the sea.

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