Homework 2 Solution

3. We have no vertical load applied. Thus $w''' = 0$. We integrate 4 times to find

$$w''' = a$$
$$w'' = ax + b$$
$$w' = \frac{a}{2}x^2 + bx + c$$
$$w = \frac{a}{6}x^3 + \frac{b}{2}x^2 + cx + d$$

Boundary conditions are

$$w(0) = 0 \quad w'(0) = 0 \quad w(L) = -h \quad w'(L) = 0$$

The first two conditions gives $c = d = 0$. The two last conditions give

$$-h = \frac{a}{6}L^3 + \frac{b}{2}L^2$$
$$0 = \frac{a}{2}L^2 + bL$$

which yields:

$$a = \frac{12h}{L^3}$$
$$b = -\frac{6h}{L^2}$$

Thus, the solution becomes

$$w(x) = \frac{2h}{L'}x^3 - \frac{3h}{L'}x^2 = \frac{h}{L'}x^2\left(\frac{2x}{L} - 3\right)$$

a) Stresses are

$$\sigma(x, z) = \frac{E}{(1 - \nu^2)}zw'(x) = \frac{E}{(1 - \nu^2)}\left[\frac{12h}{L'^3}x - \frac{6h}{L'}\right] = \frac{6Eh}{(1 - \nu^2)L'^2}\left[\frac{2}{L}x - 1\right]$$

Stresses are zero at $x = L/2$ and are maximized for $x = 0, L$ where we expect fractures.

b) Maximum stresses are given by:

$$\sigma_{\text{max}} = \frac{6D}{t^2}w'(0) = \frac{6D}{t^2}[b] = \frac{36hD}{t^2L'}$$

Equating this stress to 25 MPa allows us to solve for $L$:

$$L = \frac{6}{t} \sqrt{\frac{hD}{\sigma_{\text{max}}}}$$

For maximum stress of 25 MPa, we find $L = 1.6$ km.

d) By modeling the single layer as a stack of thinner layers we can dramatically reduce the stresses.