STRAIN (06)

Main topics

A General deformation
B Tensor notations for strain
C Relationship between stress and strain

General deformation (changes of position of points in a body)

A Rigid body translation
1. All points displaced by an equal vector (equal magnitude and direction); no displacement of points relative to one another
2. \([X'] = [u] + [X]\) matrix addition

B Rigid body rotation
1. All points rotated by an equal amount about a common axis; no angular displacement of points relative to one another
2. \([X'] = [a][X]\) matrix multiplication; rows in \([a]\) are dir. cosines!

C Change in shape (distortional strain)
1. Change in linear dimension (normal strain or elongation)
   \[ \varepsilon = \frac{\Delta L}{L_o} = \frac{L_1 - L_o}{L_o} \] dimensionless!
2. Change in angles (shear strain)
   \[ \gamma = \theta_1 + \theta_2 = 2\varepsilon_{xy} = 2\varepsilon_{yx} \] dimensionless!
   For small angle changes, \(\theta_1 = \tan^{-1}\left(\frac{du_2}{dx_1}\right) \approx \frac{du_2}{dx_1}\)

D Change in volume (dilation)
\[ \Delta = \frac{\Delta V}{V_o} = \frac{V_1 - V_o}{V_o} \] dimensionless!

A positive shear strain corresponds to a decrease in the right angle.
III Tensor notations for infinitesimal strain

Displacement gradient matrix $J_u$ = Strain matrix $\varepsilon$ + rotation matrix $\Omega$

$$J_{ij} = \left( \frac{\partial u_i}{\partial x_j} \right)$$

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

In 2-D

$$\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2}
\end{bmatrix} =$$

$$\begin{bmatrix}
\frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \right) \\
\frac{1}{2} \left( \frac{\partial u_2}{\partial x_2} + \frac{\partial u_1}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right)
\end{bmatrix} +$$

$$\begin{bmatrix}
\frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_2} \right) \\
\frac{1}{2} \left( \frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_2}{\partial x_2} \right)
\end{bmatrix}$$

IV Relationship between stress and strain

A Uniaxial stress

$$\sigma_{xx} = E \varepsilon_{xx}$$

$E$ = Young's modulus; dimensions of stress

Analog:

$$F = kx$$

$k$ = spring constant; dimensions of stress*length

$$\varepsilon_{yy} = \varepsilon_{zz} = -\nu \varepsilon_{xx}$$

$\nu$ = Poisson's ratio (dimensionless)
B General conditions (for an isotropic medium)

\[
\varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu (\sigma_{yy} + \sigma_{zz}) \right] \\
\varepsilon_{yy} = \frac{1}{E} \left[ \sigma_{yy} - \nu (\sigma_{zz} + \sigma_{xx}) \right] \\
\varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - \nu (\sigma_{xx} + \sigma_{yy}) \right] \\
\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2G} \sigma_{xy} \\
\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2G} \sigma_{yz} \\
\varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2G} \sigma_{zx}
\]

(6.8)

(6.9)

(6.10)

(6.11)

(6.12)

(6.13)

For isotropic materials, the principal stresses parallel principal strains!

Relationships Among Elastic Constants for an Isotropic Solid
(from Mal and Singh, 1991, p. 16)

<table>
<thead>
<tr>
<th>Pair</th>
<th>(\lambda, G) ((\lambda, \mu))</th>
<th>(\lambda, \nu)</th>
<th>(G, k) ((\mu, k))</th>
<th>(\mu, E)</th>
<th>(k, \nu)</th>
<th>(E, \nu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda = Lame) parameter</td>
<td>(\frac{\lambda(1-2\nu)}{2\nu})</td>
<td>(k - \frac{2G}{3})</td>
<td>(\frac{G(E-2G)}{3G-E})</td>
<td>(\frac{3k\nu}{1+\nu})</td>
<td>(\frac{E}{(1+\nu)(1-2\nu)})</td>
<td></td>
</tr>
<tr>
<td>(G = Shear) modulus</td>
<td>(\frac{\lambda(1+\nu)}{3\nu})</td>
<td>(\frac{GE}{3(3G-E)})</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(k = Bulk) modulus</td>
<td>(\frac{G(3\lambda+2G)}{\lambda+G})</td>
<td>(\frac{\lambda(1+\nu)(1-2\nu)}{\nu})</td>
<td>(\frac{9kG}{3k+G})</td>
<td>(3k(1-2\nu))</td>
<td>(Young's) modulus</td>
<td></td>
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<tr>
<td>(E = )</td>
<td>(\frac{\lambda}{2(\lambda+G)})</td>
<td>(\frac{3k-2G}{2(3k+G)})</td>
<td>(\frac{E}{2G})</td>
<td>(Poisson's) ratio</td>
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<td></td>
</tr>
</tbody>
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References