GG612
Lecture 3

Strain and Stress
Should complete infinitesimal strain by adding rotation.

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Main Theme

• Representation of complicated quantities describing strain and stress at a point in a clear manner

Vector Conventions

• $X = \text{initial position}$
• $X' = \text{final position}$
• $U = \text{displacement}$
Matrix Inverses

- \( AA^{-1} = A^{-1}A = [I] \)
- \( [AB]^{-1} = [B^{-1}][A^{-1}] \)
- \( ABB^{-1}A^{-1} = A[I][A^{-1}] = [I] \)
- \( [AB][AB]^{-1} = [I] \)
- \( [B^{-1}A^{-1}] = [AB]^{-1} \)

Matrix Inverses and Transposes

- \( a\cdot b = [a^T][b] \)
- \( [AB]^T = [B^T][A^T] \)
- \( A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} \)
- \( B = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix} \)
- \( AB = \begin{bmatrix} a_1\cdot b_1 & a_2\cdot b_1 & \cdots & a_n\cdot b_1 \\ a_1\cdot b_2 & a_2\cdot b_2 & \cdots & a_n\cdot b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1\cdot b_m & a_2\cdot b_m & \cdots & a_n\cdot b_m \end{bmatrix} \)

\( [AB]^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_m \end{bmatrix} \)

\( B^T A' = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \cdots & b_m \end{bmatrix} \)

\( [AB]^T = B^T A' \)
Rotation Matrix \([\mathbf{R}]\)

- Rotations change the orientations of vectors but not their lengths
- \(\mathbf{X} \cdot \mathbf{X} = |\mathbf{X}| |\mathbf{X}| \cos \theta_{xx}\)
- \(\mathbf{X} \cdot \mathbf{X} = X' \cdot X'\)
  - \(X' = RX\)
- \(\mathbf{X} \cdot \mathbf{X} = [\mathbf{RX}] \cdot [\mathbf{RX}]\)
- \(\mathbf{X} \cdot \mathbf{X} = [\mathbf{X}^T \mathbf{R}^T] [\mathbf{RX}]\)
- \([\mathbf{X}^T] [\mathbf{X}] = [\mathbf{X}^T \mathbf{R}^T] [\mathbf{RX}]\)
- \([\mathbf{X}^T] [\mathbf{I}] [\mathbf{X}] = [\mathbf{X}^T] [\mathbf{R}^T] [\mathbf{R}] [\mathbf{X}]\)
- \([\mathbf{I}] = [\mathbf{R}^T] [\mathbf{R}]\)
- But \([\mathbf{I}] = [\mathbf{R}^{-1}] [\mathbf{R}], \) so
- \([\mathbf{R}^T] = [\mathbf{R}^{-1}]\)

Rotation Matrix \([\mathbf{R}]\)

2D Example

\[
\mathbf{R} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-sin \theta & \cos \theta
\end{bmatrix} : [\mathbf{X}'] = [\mathbf{R}] [\mathbf{X}]
\]

\[
\begin{bmatrix}
X' \\
Y'
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} \rightarrow 
\begin{bmatrix}
X' = \cos \theta x + \sin \theta y \\
Y' = -\sin \theta x + \cos \theta y
\end{bmatrix} \rightarrow \begin{bmatrix}
x'^2 + y'^2 = x^2 + y^2
\end{bmatrix}
\]

\[
\mathbf{R}^T = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

\[
\mathbf{R} \mathbf{R}^T = \begin{bmatrix}
\cos \theta & \sin \theta \\
-sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
\mathbf{R}^T = \mathbf{R}^{-1}
\]
General Concepts

Deformation = Rigid body motion + Strain

Rigid body motion

Rigid body translation
- Treated by matrix addition
  \[ [X'] = [X] + [U] \]

Rigid body rotation
- Changes orientation of lines, but not their length
- Axis of rotation does not rotate; it is an eigenvector
- Treated by matrix multiplication
  \[ [X'] = [R] [X] \]

General Concepts

- Normal strains
  Change in line length
  - Extension (elongation) = \( \Delta s/s_0 \)
  - Stretch = \( S = s'/s_0 \)
  - Quadratic elongation = \( Q = (s'/s_0)^2 \)

- Shear strains
  Change in right angles

- Dimensions: Dimensionless
Homogeneous strain

• Parallel lines to parallel lines (2D and 3D)
• Circle to ellipse (2D)
• Sphere to ellipsoid (3D)

\[
\begin{bmatrix}
X'
\end{bmatrix} = \begin{bmatrix} F \end{bmatrix} \begin{bmatrix} X \end{bmatrix}
\]
\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\
y\end{bmatrix}
\]
\[
\begin{bmatrix}
X
\end{bmatrix} = \begin{bmatrix} F \end{bmatrix}^{-1} \begin{bmatrix} X' \end{bmatrix}
\]
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} x' \\
y'
\end{bmatrix}
\]
Matrix Representations: Positions (2D)

dx' = \frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy

dy' = \frac{\partial y'}{\partial x} dx + \frac{\partial y'}{\partial y} dy

\begin{bmatrix}
  dx' \\
  dy'
\end{bmatrix} =
\begin{bmatrix}
  \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\
  \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y}
\end{bmatrix}
\begin{bmatrix}
  dx \\
  dy
\end{bmatrix}

\begin{bmatrix}
  dX'
\end{bmatrix} = [F][dX]

If derivatives are constant (e.g., at a point)

\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} =
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}

\begin{bmatrix}
  X'
\end{bmatrix} = [F][X]
Matrix Representations
Displacements (2D)

\[
du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy
\]

\[
dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy
\]

\[
\begin{bmatrix}
  du \\
  dv
\end{bmatrix}
= \begin{bmatrix}
  \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
  \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}
\begin{bmatrix}
  dx \\
  dy
\end{bmatrix}
\]

\[
[dU] = [J_u][dX]
\]

Matrix Representations
Displacements (2D)

\[
u = \frac{\partial u}{\partial x} x + \frac{\partial u}{\partial y} y
\]

\[
v = \frac{\partial v}{\partial x} x + \frac{\partial v}{\partial y} y
\]

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix}
= \begin{bmatrix}
  \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
  \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

\[
[U] = [J_u][X]
\]

If derivatives are constant (e.g., at a point)
Matrix Representations
Positions and Displacements (2D)

\[ U = X' - X \]
\[ U = FX - X = FX - IX \]
\[ U = [F-I]X \]
\[ [F-I] = J_u \]
\[ J_u = \begin{bmatrix} a - 1 & b \\ c & d - 1 \end{bmatrix} \]

Matrix Representations
Positions and Displacements

Lagrangian: f(X)
\[ [X'] = [F][X] \]
\[ U = X' - X = FX - X \]
\[ U = FX - IX = [F - I]X \]

Eulerian: g(X')
\[ [X] = [F^{-1}][X'] \]
\[ U = X' - X = X' - F^{-1}X' \]
\[ U = [I - F^{-1}]X' \]
Squares of Line Lengths

\[ s^2 = |\vec{X}| \cdot |\vec{X}| \cos(\theta_{\vec{X}\vec{X}}) \]
\[ s^2 = \vec{X} \cdot \vec{X} = X^T X \]
\[ s^2 = X^T X \]

\[ s'^2 = \vec{X}' \cdot \vec{X}' \]
\[ s'^2 = [FX]^T [FX] \]
\[ s'^2 = X^T F^T FX \]

E (strain matrix)

\[ \frac{s'^2 - s^2}{2} = \frac{dX^T [F^T F - I] dX}{2} \]
\[ \frac{s'^2 - s^2}{2} = \frac{dX^T [E] dX}{2} \]
\[ E \equiv \frac{[F^T F - I]}{2} \]
\[ \varepsilon \text{ (Infinitesimal Strain Matrix, 2D)} \]

\[
E \equiv \left[ F^T F - I \right] = \frac{1}{2} \left[ \left[ J_u + I \right]^T \left[ J_u + I \right] - I \right]
\]

\[
E = \frac{1}{2} \begin{bmatrix}
\frac{\partial u}{\partial x} + 1 & \frac{\partial v}{\partial x} \\
\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + 1
\end{bmatrix}
\begin{bmatrix}
\frac{\partial u}{\partial x} + 1 & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} + 1
\end{bmatrix}
- \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

If partial derivatives \(<1\), their squares can be dropped to obtain the infinitesimal strain matrix \(\varepsilon\)

\[
\varepsilon = \frac{1}{2} \begin{bmatrix}
\left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \right) & \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) & \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{bmatrix}
\]

\[ J_u = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix} \quad \varepsilon = \frac{1}{2} \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}
= \frac{1}{2} \left[ J_u + J_u^T \right]
\]

\[
J_u = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
\left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \right) & \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) & \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{bmatrix}
+ \frac{1}{2} \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix}
= \frac{1}{2} \left[ J_u + J_u^T \right]
\]

\[ \varepsilon \text{ is symmetric} \quad \omega \text{ is anti-symmetric} \]

Linear superposition
ε (Infinitesimal Strain Matrix, 2D)  
Meaning of components

\[
\varepsilon = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{du}{ds} \\
\frac{dv}{ds}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} \\
\varepsilon_{yx} & \varepsilon_{yy}
\end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy}
\end{bmatrix}
\]

First column in \( \varepsilon \): relative displacement vector for unit element in x-direction  
\( \varepsilon_{yx} \) is displacement in the y-direction of right end of unit element in x-direction

Pure strain without rotation

\[
\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}
\]

ε (Infinitesimal Strain Matrix, 2D)  
Meaning of components

\[
\varepsilon = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{du}{ds} \\
\frac{dv}{ds}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} \\
\varepsilon_{yx} & \varepsilon_{yy}
\end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix}
\varepsilon_{xy} \\
\varepsilon_{yy}
\end{bmatrix}
\]

Second column in \( \varepsilon \): relative displacement vector for unit element in y-direction  
\( \varepsilon_{yx} \) is displacement in the x-direction of upper end of unit element in y-direction

Pure strain without rotation

\[
\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}
\]
\[ \varepsilon \text{ (Infinitesimal Strain Matrix, 2D)} \]

**Meaning of components**

\[ \varepsilon = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y} \end{bmatrix} \]

- \( \varepsilon_{11} = \varepsilon_{xx} \) = elongation of line parallel to \( x \)-axis
- \( \varepsilon_{12} = \varepsilon_{xy} \) = \( (\Delta \theta)/2 \)
- \( \varepsilon_{21} = \varepsilon_{yx} \) = \( (\Delta \theta)/2 \)
- \( \varepsilon_{22} = \varepsilon_{yy} \) = elongation of line parallel to \( y \)-axis

\[ \frac{\Delta \theta}{2} = \frac{(\psi_2 - \psi_1)}{2} = \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \]

Shear strain > 0 if angle between +\( x \) and +\( y \) axes decreases

---

\[ \omega \text{ (Infinitesimal Strain Matrix, 2D)} \]

**Meaning of components**

\[ \omega = \begin{bmatrix} 0 & \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \\ \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) & 0 & \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \\ \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) & 0 \end{bmatrix} \]

- \( \omega_{xy} \) = displacement in the \( y \)-direction of right end of unit element in \( x \)-direction

\[ \frac{du}{ds} \]

\[ \frac{dv}{ds} \]

First column in \( \omega \): relative displacement vector for unit element in \( x \)-direction

\( \omega_z \ll 1 \) radian
\( \omega \) (Infinitesimal Strain Matrix, 2D)
Meaning of components

\[
\omega = \begin{bmatrix}
0 & \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{du}{ds} \\
\frac{dv}{ds}
\end{bmatrix} = \begin{bmatrix}
0 & -\omega_z \\
\omega_z & 0
\end{bmatrix} \begin{bmatrix}
0 \\
\omega_z
\end{bmatrix}
\]

Second column in \( \omega \): relative displacement vector for unit element in y-direction
\( \omega_y \) is displacement in the negative x-direction of upper end of unit element in y-direction

\( \varepsilon \) (Infinitesimal Strain Matrix, 2D)
Meaning of components

\[
\varepsilon = \begin{bmatrix}
\left( \frac{\partial u}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \left( \frac{\partial v}{\partial y} \right)
\end{bmatrix}
\]

\( \varepsilon_{11} = \varepsilon_{xx} = \text{elongation of line parallel to x-axis} \)
\( \varepsilon_{12} = \varepsilon_{yx} = (\Delta \theta)/2 \)
\( \varepsilon_{21} = \varepsilon_{xy} = (\Delta \theta)/2 \)
\( \varepsilon_{22} = \varepsilon_{yy} = \text{elongation of line parallel to y-axis} \)

\[
\frac{\Delta \theta}{2} = \frac{(\psi_2 - \psi_1)}{2} = \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\]

Shear strain > 0 if angle between +x and +y axes decreases
Coaxial Finite Strain

\[ F = \begin{bmatrix} a & b \\ b & d \end{bmatrix}; \quad [F][X] = \lambda [X] \]

- \( F = F^T \)
- All values of \( X' \cdot X' \) are positive if \( X' \neq 0 \)
- \( F \) is positive definite
  - \( F \) has an inverse
  - Eigenvalues > 0
  - \( F \) has a square root

1. Eigenvectors (\( X \)) of \( F \) are perpendicular because \( F \) is symmetric (\( X_1 \cdot X_2 = 0 \))
2. \( X_1, X_2 \) solve \( d(X' \cdot X')/d\theta = 0 \)
3. \( X_1, X_2 \) along major axes of strain ellipse
4. \( X_1 = X_1'; \quad X_2 = X_2' \)
5. Principal strain axes do not rotate
Non-coaxial Finite Strain

- The vectors that transform \textit{from} the axes of the reciprocal strain ellipse \textit{to} the principal axes of the strain ellipse rotate.
- The rotation is given by the matrix that rotates the principal axes of the reciprocal strain ellipse to those of the strain ellipse.

\[
[X'] = [F][X]
\]
\[
X' \cdot X' = [X][F^T F][X]
\]
\[
[F^T F] \text{ is symmetric}
\]
\[
\text{Eigenvectors of } [F^T F] \text{ give principal strain directions}
\]
\[
\text{Square roots of eigenvalues of } [F^T F] \text{ give principal stretches}
\]
\[
[X] = [F^{-1}][X']
\]
\[
X \cdot X = [X'][F^{-1}]^T [F^{-1}][X']
\]
\[
[F^{-1}]^T [F^{-1}] \text{ is symmetric}
\]
\[
\text{Eigenvectors of } [F^{-1}]^T [F^{-1}] \text{ give principal strain directions}
\]
\[
\text{Square roots of eigenvalues of } [F^{-1}]^T [F^{-1}] \text{ give (reciprocal) principal stretches}
\]
Non-coaxial Finite Strain

1. The strain ellipse and the reciprocal strain ellipse have the same eigenvalues but different eigenvectors.

2. \([F^T F] = ([F^{-1}]^T [F^{-1}])^{-1}\)

3. \([([F^{-1}]^T [F^{-1}])^{-1} = ([F^{-1}]^{-1} [F^{-1}]^T)^{-1}] = FF^T.\)
Coaxial vs. Non-coaxial Strain

Coaxial
- $F = F^T$ (F is symmetric)
- $FF^T = F^T F = F^2$ (F$^2$ is symmetric)
- $FX = \lambda X$
- $[F^T]^2 X = \lambda^2 X$
- $F = U = V$

Non-coaxial
- $F \neq F^T$ (F is not symmetric)
- $F^T F \neq F^2$ (but both symmetric)
- $FX = \lambda X$
- $[F^T]^2 X_1 = \lambda_1^2 X_1 ; \lambda_1 = \lambda \neq \lambda$
- $[F^T] X_2 = \lambda_2^2 X_2 ; X \neq X_1 \neq X_2$
- $F = RU = R[F^T]^{1/2} = VR = [F^T F]^{1/2} R$

\[ F = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad F^T = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \]

\[ FF^T = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \]

\[ F = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad F^T = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \]

\[ FF^T = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \]
Polar Decomposition Theorem

Suppose
(1) \([F] = [R][U]\),
where \(R\) is a rotation matrix and \(U\) is a symmetric stretch matrix. Then
(2) \(FF^T = [RU]^T[RU] = U^T R^T RU = U^T U\)

However, \(U\) is postulated to be positive definite, so
(3) \(U^T U = U^2 = FF^T\)

Since \(FF^T\) gives squares of line lengths, if \(U\) gives strains without rotations, it too should give the same squares of line lengths. Hence
(4) \(U = (FF^T)^{1/2}\)

From equation (1):
(5) \(R = FU^{-1}\)

Polar Decomposition Theorem

Suppose
(1) \([F] = [V][R^*]\),
where \(R^*\) is a rotation matrix and \(V\) is a symmetric stretch matrix. Then
(2) \(FF^T = [VR^*]^T[VR^*] = VR^*R^{-1}V^T = VR^*R^{-1}V^T = VV^T\)

However, \(V\) is postulated to be positive definite, so
(3) \(VV^T = V^2 = FF^T\)

Since \(FF^T\) gives squares of line lengths, if \(V\) gives strains without rotations, it too should give the same squares of line lengths. Hence
(4) \(V = (FF^T)^{1/2}\)

From equation (1):
(5) \(R^* = V^{-1}F\)
Polar Decomposition Theorem

Proof that the polar decompositions are unique.

Suppose different decompositions exist

\[ F = R_1U_1 = R_2U_2 \]

\[ X' \cdot X' = [FX] \cdot [FX] = [FX]^T [FX] = X'^T F^T FX \]

\[ F^T F = \begin{bmatrix} R_1U_1 \end{bmatrix}^T \begin{bmatrix} R_1U_1 \end{bmatrix} = U_1^T R_1^T R_1 U_1 = U_1^T R_1^T R_1 U_1 \]

\[ = U_1^T U_1 = U_1 \]

\[ F^T F = \begin{bmatrix} R_2U_2 \end{bmatrix}^T \begin{bmatrix} R_2U_2 \end{bmatrix} = U_2^T R_2^T R_2 U_2 = U_2^T R_2^T R_2 U_2 \]

\[ = U_2^T U_2 = U_2 \]

\[ U_1^2 = U_2^2 \]

\[ U_1 = U_2 = U \]

\[ F = R_1U_1 = R_2U_1 \]

\[ R_1 = R_2 = R \]

Polar Decomposition Theorem

• The same procedure can be followed to show that the decomposition \( F = VR^* \) is unique.

These results are very important: \( F \) can be decomposed into only one symmetric matrix that is pre-multiplied by a unique rotation matrix, and \( F \) can be decomposed into only one symmetric matrix that is post-multiplied by a unique rotation matrix.
Polar Decomposition Theorem

Proof that \( F = RU = VR \)

Intuitively, we might expect that \( R = R^* \). This is straightforward to show.

\[
\begin{align*}
F &= VR^* = RV = \left[R R^T\right]VR = R\left[VR^T\right] = R\left[R^T VR^T\right]
\end{align*}
\]

Now consider the character of \( \left[R R^T\right] \) by taking its transpose

\[
\begin{align*}
\left[VR^T\right]^T &= \left[R R^T\right]^T \left[VR^T\right]^T = \left[R^T VR^T\right] = \left[R^T\right]^T \left[VR^T\right]
\end{align*}
\]

The transpose of \( \left[R R^T\right] \) equals \( \left[R^T\right]^T \left[VR^T\right] \), so \( \left[R R^T\right] \) is symmetric (definite-positive) matrix. It also is pre-multiplied by a rotation matrix. That means equation (11) can be re-written as

\[
F = R^* U \quad \text{if}
\]

Equating the two right sides above

\[
F = RU = R^* U \quad \text{if}
\]

The results of (ii) show that the rotation matrix and U-matrix are uniquely defined, so \( R = R^* \), hence

\[
F = RU = VR
\]

Polar Decomposition Theorem

Comparison of eigenvectors and eigenvalues

Now compare the eigenvectors and eigenvalues of \( U \) and \( V \) (see example 3.2.1 of Lai et al.). Suppose \( \hat{X} \) is an eigenvector of \( U \) and \( \lambda \) is an eigenvalue of \( U \).

\[
U\hat{X} = \lambda \hat{X}
\]

\[
RU\hat{X} = \lambda R\hat{X}
\]

\[
\left[RU \right] \hat{X} = \lambda R\hat{X}
\]

\[
\left[RU \right] \|VR\| = F
\]

\[
\|VR\| \hat{X} = \lambda R\hat{X}
\]

\[
V\left[R\hat{X}\right] = \lambda \left[R\hat{X}\right]
\]

So \( RX \) is an eigenvector of \( V \), and \( \lambda \) is an eigenvalue of \( V \). Since \( \lambda \) is also an eigenvalue of \( U \) (see the first step), that means the eigenvalues of \( U \) and \( V \) are the same, even though the eigenvectors are not.

The rotation matrix \( R \) rotates the eigenvectors of \( U \) to the orientation of the eigenvectors of \( V \). This means that the matrix \( U \) describes the principal axes of the reciprocal strain ellipse, and the matrix \( V \) describes the principal axes of the strain ellipse.
Stress

1. Stress vector
2. Stress state at a point
3. Stress transformations
4. Principal stresses

16. STRESS AT A POINT

16. STRESS AT A POINT

I Stress vector (traction) on a plane
A \( \tau = \lim_{A \to 0} \frac{\vec{F}}{A} \)
B Traction vectors can be added as vectors
C A traction vector can be resolved into normal (\( \tau_n \)) and shear (\( \tau_s \)) components
   1 A normal traction (\( \tau_n \)) acts perpendicular to a plane
   2 A shear traction (\( \tau_s \)) acts parallel to a plane
D Local reference frame
   1 The n-axis is normal to the plane
   2 The s-axis is parallel to the plane

III Stress at a point (cont.)
A Stresses refer to balanced internal "forces (per unit area)". They differ from force vectors, which, if unbalanced, cause accelerations
B "On-in convention": The stress component \( \sigma_{ij} \) acts on the plane normal to the i-direction and acts in the j-direction
   1 Normal stresses: \( i=j \)
   2 Shear stresses: \( i \neq j \)
16. STRESS AT A POINT

III Stress at a point

C Dimensions of stress: force/unit area

D Convention for stresses
   1 Tension is positive
   2 Compression is negative
   3 Follows from on-in convention
   4 Consistent with most mechanics books
   5 Counter to most geology books

\[ \sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \quad \text{2-D} \]

\[ \sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad \text{3-D} \]

E In nature, the state of stress can (and usually does) vary from point to point

F For rotational equilibrium,
\[ \sigma_{xy} = \sigma_{yx}, \sigma_{xz} = \sigma_{zx}, \sigma_{yz} = \sigma_{zy} \]
16. STRESS AT A POINT

IV Principal Stresses (these have magnitudes and orientations)
A Principal stresses act on planes which feel no shear stress
B The principal stresses are normal stresses.
C Principal stresses act on perpendicular planes
D The maximum, intermediate, and minimum principal stresses are usually designated $\sigma_1$, $\sigma_2$, and $\sigma_3$, respectively.
E Principal stresses have a single subscript.

F Principal stresses represent the stress state most simply

\[
\sigma_\theta = \begin{bmatrix}
\sigma_1 & 0 \\
0 & \sigma_2
\end{bmatrix} \quad \text{2-D, 2 components}
\]

\[
\sigma_\theta = \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix} \quad \text{3-D, 3 components}
\]
19. Principal Stresses


17. Mohr Circle for Trazions

- From King et al., 1994 (Fig. 11)
- Coulomb stress change caused by the Landers rupture. The left-lateral ML=6.5 Big Bear rupture occurred along dotted line 3 hr 26 min after the Landers main shock. The Coulomb stress increase at the future Big Bear epicenter is 2.2-2.9 bars.

19. Principal Stresses

II Cauchy’s formula
   A Relates traction (stress vector) components to stress tensor components in the same reference frame
   B 2D and 3D treatments analogous
   C $\tau_i = \sigma_{ij} n_j = n_j \sigma_{ij}$

Note: all stress components shown are positive

19. Principal Stresses

II Cauchy’s formula (cont.)
   C $\tau_i = n_j \sigma_{ij}$
      1 Meaning of terms
         a $\tau_i =$ traction component
         b $n_i =$ direction cosine of angle between n-direction and j-direction
         c $\sigma_{ij} =$ traction component
         d $\tau_i$ and $\sigma_{ij}$ act in the same direction
19. Principal Stresses

II Cauchy’s formula (cont.)

D Expansion (2D) of \( \tau_i = n_j \sigma_{ji} \)

1. \( \tau_x = n_x \sigma_{xx} + n_y \sigma_{yx} \)
2. \( \tau_y = n_x \sigma_{xy} + n_y \sigma_{yy} \)

\( n_j = \cos \theta_{nj} = a_{nj} \)

E Derivation:
Contributions to \( \tau_x \)

1. \( \tau_x = w^{(1)} \sigma_{xx} + w^{(2)} \sigma_{yx} \)
2. \( \frac{F_x}{A_x} = \left( \frac{A_x}{A_w} \right) F^{(1)}_x + \left( \frac{A_x}{A_y} \right) F^{(2)}_x \)
3. \( \tau_x = n_x \sigma_{xx} + n_y \sigma_{yx} \)

Note that all contributions must act in \( x \)-direction.

\( n_x = \cos \theta_{nx} = a_{nx} \)
\( n_y = \cos \theta_{ny} = a_{ny} \)
19. Principal Stresses

II Cauchy's formula (cont.)
E Derivation:

Contributions to $\tau_y$

1 $\tau_y = w^{(3)}\sigma_{xy} + w^{(4)}\sigma_{yy}$

2 $\frac{F_y}{A_y} = \left(\frac{A_x}{A_y}\right) F^{(3)} + \left(\frac{A_y}{A_x}\right) F^{(4)}$

3 $\tau_y = n_x\sigma_{xy} + n_y\sigma_{yy}$

Note that all contributions must act in $y$-direction.

$n_x = \cos \theta_{nx} = a_{nx}$

$n_y = \cos \theta_{ny} = a_{ny}$

19. Principal Stresses

II Cauchy's formula (cont.)
F Alternative forms

1 $\tau_i = n_i\sigma_{ji}$

2 $\tau_i = \sigma_{ij}n_j$

3 $\tau_i = \sigma_{ij}n_j$

4 $\begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$

5 Matlab

a $t = s'*n$

b $t = s*n$

$\tau_x = n_x\sigma_{xx} + n_y\sigma_{yx}$

$\tau_y = n_x\sigma_{xy} + n_y\sigma_{yy}$

$n_j = \cos \theta_{nj} = a_{nj}$
19. Principal Stresses

III Principal stresses (eigenvectors and eigenvalues)

A

\[
\begin{bmatrix}
\tau_x \\
\tau_y
\end{bmatrix}
= \begin{bmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{bmatrix}
\begin{bmatrix}
\nu_x \\
\nu_y
\end{bmatrix}
\]

Cauchy's Formula

B

Vector components

Let \( \lambda = \left| \begin{array}{cc} \sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}\end{array} \right| \)

C

The form of (C) is \([A][X=\lambda[X]\), and \([\sigma]\) is symmetric

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

From previous notes

III Eigenvalue problems, eigenvectors and eigenvalues (cont.)

\( \text{J Characteristic equation: } |A-\lambda I|=0 \)

\( \text{3 Eigenvalues of a symmetric 2x2 matrix } \quad A = \begin{bmatrix} a & b \\
b & d \end{bmatrix} \)

a \( \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - b^2)}}{2} \)

b \( \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a + 2ad + d)^2 - 4ad + 4b^2}}{2} \)

c \( \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a - 2ad + d)^2 + 4b^2}}{2} \)

d \( \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a - d)^2 + 4b^2}}{2} \)

Radical term cannot be negative. Eigenvalues are real.
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

From previous notes

L Distinct eigenvectors $(X_1, X_2)$ of a symmetric 2x2 matrix are perpendicular

Since the left sides of (2a) and (2b) are equal, the right sides must be equal too. Hence,

4 $\lambda_1 (X_2 \cdot X_1) = \lambda_2 (X_1 \cdot X_2)$

Now subtract the right side of (4) from the left

5 $(\lambda_1 - \lambda_2) (X_2 \cdot X_1) = 0$

• The eigenvalues generally are different, so $\lambda_1 - \lambda_2 \neq 0$.
• This means for (5) to hold that $X_2 \cdot X_1 = 0$.

Therefore, the eigenvectors $(X_1, X_2)$ of a symmetric 2x2 matrix are perpendicular

19. Principal Stresses

III Principal stresses (eigenvectors and eigenvalues)

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \lambda \begin{bmatrix} n_x \\ n_y \end{bmatrix}$$

D Meaning

1 Since the stress tensor is symmetric, a reference frame with perpendicular axes defined by $n_x$ and $n_y$ pairs can be found such that the shear stresses are zero

2 This is the only way to satisfy the equation above; otherwise $\sigma_{xy} n_y \neq 0$, and $\sigma_{xx} n_x \neq 0$

3 For different (principal) values of $\lambda$, the orientation of the corresponding principal axis is expected to differ
19. Principal Stresses

V Example

Find the principal stresses

given \( \sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4 \text{ MPa} & \sigma_{xy} = -4 \text{ MPa} \\ \sigma_{yx} = -4 \text{ MPa} & \sigma_{yy} = -4 \text{ MPa} \end{bmatrix} \)

First find eigenvalues (in MPa)

\[
\lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a - d)^2 + 4b^2}}{2} \\
\lambda_1, \lambda_2 = -4 \pm \sqrt{64} = -4 \pm 4 = 0, -8
\]
19. Principal Stresses

IV Example

\[ \sigma_{ij} = \begin{bmatrix} \sigma_{xx} = -4 \text{ MPa} & \sigma_{xy} = -4 \text{ MPa} \\ \sigma_{yx} = -4 \text{ MPa} & \sigma_{yy} = -4 \text{ MPa} \end{bmatrix} \]

\[ \lambda_1, \lambda_2 = -4 \pm \frac{\sqrt{64}}{2} = -4 \pm 4 = 0, -8 \] Eigenvalues (MPa)

Then solve for eigenvectors (X) using \([A-\lambda I]X = 0\)

For \(\lambda_1 = 0\):

\[ \begin{bmatrix} -4 - 0 & -4 \\ -4 & -4 - 0 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -4n_x - 4n_y = 0 \Rightarrow n_x = -n_y \]

For \(\lambda_2 = -8\):

\[ \begin{bmatrix} -4 - (-8) & -4 \\ -4 & \sigma_{yy} - (-8) \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4n_x - 4n_y = 0 \Rightarrow n_x = n_y \]

Note that \(X_1 \cdot X_2 = 0\)

Principal directions are perpendicular
19. Principal Stresses

V Example
Matrix form/Matlab

>> sij = [-4 -4; -4 -4]
sij =
   -4   -4
   -4   -4
>> [v,d]=eig(sij)
v =
 0.7071  -0.7071
 0.7071   0.7071
d =
  8.0000   0.0000
   0.0000   0.0000

Eigenvectors
(in columns)
Corresponding
eigenvalues
(in columns)

Summary of Strain and Stress

• Different quantities with different dimensions
  (dimensionless vs. force/unit area)
• Both can be represented by the orientation and
  magnitude of their principal values
• Strain describes changes in distance between points
  and changes in right angles
• Matrices of co-axial strain and stress are symmetric:
  eigenvalues are orthogonal and do not rotate
• Asymmetric strain matrices involve rotation
• Infinitesimal strains can be superposed linearly
• Finite strains involve matrix multiplication