FINITE STRAIN AND INFINITESIMAL STRAIN

I Main Topics (on infinitesimal strain)
A The finite strain tensor \([E]\)
B Deformation paths for finite strain
C Infinitesimal strain and the infinitesimal strain tensor \(\varepsilon\)

II The finite strain tensor \([E]\)
A Used to find the changes in the squares of lengths of line segments in a deformed body.
B Definition of \([E]\) in terms of the deformation gradient tensor \([F]\)
Recall the coordinate transformation equations:
1 \[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} \text{ or } [X'] = [F][X]
\]
2 \[
\begin{bmatrix}
dx' \\
dy'
\end{bmatrix} = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
dx \\
dy
\end{bmatrix} \text{ or } [dX'] = [F][dX]
\]
If \[
\begin{bmatrix}
dx \\
dy
\end{bmatrix} = [dX], \text{ then } [dx \ dy] = [dX]^T; \text{ transposing a matrix is switching its rows and columns}
\]
3 \[(ds)^2 = (dx)^2 + (dy)^2 = [dx \ dy][dx] = [dX]^T [dX] = [dX]^T [I][dX],
\]
where \(I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) is the identity matrix.
4 \[(ds')^2 = (dx')^2 + (dy')^2 = [dx' \ dy'][dx'] = [dX']^T [dX']
\]
Now \(dX'\) can be expressed as \([F][dX]\) (see eq. II.B.2). Making this substitution into eq. (4) and proceeding with the algebra
5 \[(ds)^2 = [[F][dX]]^T [F][dX] = [dX]^T [F]^T [F][dX]
\]
\]
7 \[(ds)^2 - (ds')^2 = [dX]^T [F]^T [F] - I][dX]
\]
8 \[
\frac{1}{2}
\left[(ds)^2 - (ds')^2\right] = \left(\frac{1}{2}\right)[dX]^T \left[F]^T [F] - I\right][dX] = [dX]^T [E][dX]
\]
9 \[E = \left(\frac{1}{2}\right)[F]^T [F] - I = \text{ finite strain tensor}\]
III Deformation paths

Consider two different finite strains described by the following two coordinate transformation equations:

\[
\begin{bmatrix}
x_1' \\
y_1'
\end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 x + b_1 y \\ c_1 x + d_1 y \end{bmatrix} = [F_1][X]
\]

\text{Deformation 1}

\[
\begin{bmatrix}
x_2' \\
y_2'
\end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_2 x + b_2 y \\ c_2 x + d_2 y \end{bmatrix} = [F_2][X]
\]

\text{Deformation 2}

Now consider deformation 3, where deformation 1 is acted upon (followed) by deformation 2 (i.e., deformation gradient matrix F1 first acts on [X], and then F2 acts on [F1][X]):

\[
\begin{bmatrix}
x'' \\
y''
\end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

\text{Deformation 3}

Next consider deformation 4, where deformation 2 is acted upon (followed) by deformation 1 (i.e., deformation gradient matrix F2 first acts on [X], and then F1 acts on [F2][X]):

\[
\begin{bmatrix}
x''' \\
y'''
\end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

\text{Deformation 4}

A comparison of the net deformation gradient matrices in C and D shows they generally are different. Hence, the net deformation in a sequence of finite strains depends on the order of the deformations. (If the b and c terms [the off-diagonal terms] are small, then the deformations are similar)
Coordinate Transformation 1

\[
\begin{pmatrix}
    x' \\
    y'
\end{pmatrix} = \begin{pmatrix}
    2 & 1 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
\]

\[
\begin{pmatrix}
    2 & 1 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    1 \\
    1
\end{pmatrix} = \begin{pmatrix}
    2 \\
    0
\end{pmatrix}
\]

Coordinate Transformation 2

\[
\begin{pmatrix}
    x' \\
    y'
\end{pmatrix} = \begin{pmatrix}
    1 & 0 \\
    0 & 2
\end{pmatrix}
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
\]

\[
\begin{pmatrix}
    1 & 0 \\
    0 & 2
\end{pmatrix}
\begin{pmatrix}
    0 \\
    2
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0
\end{pmatrix}
\]

Coordinate transformation 3 (transformation 1 followed by transformation 2)

\[
\begin{pmatrix}
    x'' \\
    y''
\end{pmatrix} = \begin{pmatrix}
    2 & 1 \\
    0 & 2
\end{pmatrix}
\begin{pmatrix}
    1 & 0 \\
    2 & 1
\end{pmatrix}
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
\]

\[
\begin{pmatrix}
    2 & 1 \\
    0 & 2
\end{pmatrix}
\begin{pmatrix}
    0 \\
    2
\end{pmatrix} = \begin{pmatrix}
    2 \\
    0
\end{pmatrix}
\]

Coordinate transformation 4 (transformation 2 followed by transformation 1)

\[
\begin{pmatrix}
    x'' \\
    y''
\end{pmatrix} = \begin{pmatrix}
    2 & 1 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    1 & 0 \\
    0 & 2
\end{pmatrix}
\begin{pmatrix}
    x \\
    y
\end{pmatrix}
\]

\[
\begin{pmatrix}
    2 & 1 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    2 \\
    2
\end{pmatrix} = \begin{pmatrix}
    2 \\
    0
\end{pmatrix}
\]
IV Infinitesimal strain and the infinitesimal strain tensor $[\varepsilon]$

A What is infinitesimal strain?

Deformation where the displacement derivatives are small relative to one (i.e., the terms in the corresponding displacement gradient matrix $[J_u]$ are very small), so that the products of the derivatives are very small and can be ignored.

B Why consider infinitesimal strain if it is an approximation?

1 Many important geologic deformations are small (and largely elastic) over short time frames (e.g., fracture earthquake deformation, volcano deformation).

2 The terms of the infinitesimal strain tensor $[\varepsilon]$ have clear geometric meaning (clearer than those for finite strain)

3 Infinitesimal strain is much more amenable to sophisticated mathematical treatment than finite strain (e.g., elasticity theory).

4 The net deformation for infinitesimal strain is independent of the deformation sequence.

5 Example

\[
F5 = \begin{bmatrix} 1.02 & 0.01 \\ 0 & 1.01 \end{bmatrix} \quad F6 = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.02 \end{bmatrix} \quad J_u5 = \begin{bmatrix} 0.02 & 0.01 \\ 0 & 0.01 \end{bmatrix} \quad J_u6 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}
\]

Deformation 5 followed by deformation 6 gives deformation 7:

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1.01 & 0.00 \\ 0.00 & 1.02 \end{bmatrix} \begin{bmatrix} 0.00 \\ 1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.0302 & 0.0100 \\ 0.0000 & 1.0302 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Deformation 6 followed by deformation 5 gives deformation “7a”:

\[
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1.02 & 0.01 \\ 0.00 & 1.01 \end{bmatrix} \begin{bmatrix} 0.00 \\ 1.02 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.0302 & 0.0101 \\ 0.0000 & 1.0302 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

The net deformation is essentially the same in the two cases.
C The infinitesimal strain tensor (Taylor series derivation)

Consider the displacement of two neighboring points, where the distance from point 0 to point 1 initially is given by \(dx\) and \(dy\). Point 0 is displaced by an amount \(u_0\), and we wish to find \(u_1\). We use a truncated Taylor series; it is linear in \(dx\) and \(dy\) (\(dx\) and \(dy\) are only raised to the first power).

1. \(u_1^x = u_0^x + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \ldots\)
2. \(u_1^y = u_0^y + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \ldots\)

These can be rearranged into a matrix format:

3. \[
\begin{bmatrix}
  u_1^x \\
  u_1^y \\
\end{bmatrix}
= \begin{bmatrix}
  u_0^x \\
  u_0^y \\
\end{bmatrix}
+ \begin{bmatrix}
  \frac{\partial u}{\partial x} \\
  \frac{\partial u}{\partial y} \\
\end{bmatrix}
\begin{bmatrix}
  dx \\
  dy \\
\end{bmatrix}
= \begin{bmatrix}
  U^0 \\
  J_u \\
\end{bmatrix}[dX]
\]

Now split \(J_u\) into two matrices: the symmetric infinitesimal strain matrix \([\varepsilon]\), and the anti-symmetric rotation matrix \([\omega]\) by using \(J_u^T\),

\[
\begin{bmatrix}
  e & f \\
  g & h \\
\end{bmatrix}
= \begin{bmatrix}
  e + e & f + g \\
  g + f & h + h \\
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
  J_u + J_u^T \\
  J_u - J_u^T \\
\end{bmatrix}
= \begin{bmatrix}
  1 & 1/2 \begin{bmatrix}
  J_u^T & -J_u \\
  J_u & J_u^T \\
\end{bmatrix} + 1/2 \begin{bmatrix}
  J_u - J_u^T \\
  J_u^T - J_u \\
\end{bmatrix}
\]

Now the displacement expression can be expanded using \([\varepsilon]\) and \([\omega]\)

5. \[
\begin{bmatrix}
  \varepsilon \\
  \omega \\
\end{bmatrix}
= \begin{bmatrix}
  \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \\
  \frac{\partial u}{\partial y} + \frac{\partial u}{\partial y} \\
\end{bmatrix}
\begin{bmatrix}
  \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \\
  \frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \\
\end{bmatrix}
\]

Equations (3) and (5) show that the deformation can be decomposed into a translation, a strain, and a rotation.

D Geometric interpretation of the infinitesimal strain components
**Infinitesimal Deformation of Line Elements AB and AD**
(modified from Chou and Pagano, 1967)

**Line length changes**

\[
\varepsilon_{xx} = \frac{A'D' - AD}{AD} = \frac{dx + \frac{\partial u_x}{\partial x} \, dx}{dx} - \frac{dx}{dx} = \frac{\partial u_x}{\partial x}
\]

\[
\varepsilon_{yy} = \frac{A'B' - AB}{AB} = \frac{dy + \frac{\partial u_y}{\partial y} \, dy}{dy} - \frac{dy}{dy} = \frac{\partial u_y}{\partial y}
\]

**Angle changes**

\[
\tan \Psi_1 = \frac{\frac{\partial u_x}{\partial y} \, dy}{dy} = \frac{\partial u_x}{\partial y} - \Psi_1 \text{ if } \frac{\partial u_x}{\partial y} \text{ is small}
\]

\[
\tan \Psi_2 = \frac{\frac{\partial u_y}{\partial x} \, dx}{dx} = \frac{\partial u_y}{\partial x} - \Psi_2 \text{ if } \frac{\partial u_y}{\partial x} \text{ is small}
\]

\[
\Psi = \Psi_1 - \Psi_2 = \text{change in right angle} \quad \text{(the minus sign accounts for } \Psi_2 \text{ being negative)}
\]

\[
\gamma = \tan \Psi \quad \varepsilon_{xy} = \varepsilon_{yx} = (1/2) \tan \Psi
\]

\[
\gamma = \text{engineering shear strain} \quad \varepsilon = \text{tensor shear strain}
\]
Infinitesimal Strains

**a. Normal strain** $e_{xx}$

- Deformed
- $\frac{du_x}{dx}$

**b. Normal strain** $e_{yy}$

- $\frac{du_y}{dy}$

**c. Shear strains** $e_{xy}, e_{yx}$

- $e_{xy} = e_{yx} = \frac{1}{2}(\Psi_1 - \Psi_2)$
- But for small angles, $\Psi = \tan \Psi$

**d. Rotations** $\omega_{yx}, \omega_{xy}$

- $\omega_{xy} = \omega_{yx} = \frac{1}{2}(\Psi_1 + \Psi_2)$
- But for small angles, $\Psi = \tan \Psi$

---

Note that "simple shear strain" involves a shear and *rotation*. Here $\Psi_1$ is zero and $\Psi_2$ is negative.

† The shear strain $e_{xy} = e_{yx}$ is half the shear strain $\gamma$.

* Positive angles are measured about the z-axis using a right hand rule. In (b) the angle $\Psi_2$ is clockwise (negative), but $du_x$ is positive. In (d) $\Psi_2$ is counter-clockwise, and $du_x < 0$. 
E Relationship between \([\varepsilon]\) and \([E]\)

From eq. II.B.9, \([E]\) is defined in terms of deformation gradients:

1 \[ [E] = \frac{1}{2} \left[ [F]^T [F] - I \right] \] finite strain tensor

The tensor \([E]\) also can be solved for in terms of displacement gradients because \(F = J_u + I\).

2 \[ [E] = \frac{1}{2} \left[ [J_u + I]^T [J_u + I] - I \right] \]

3 \[ [E] = \frac{1}{2} \left[ \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]^T \left[ \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

4 \[ [E] = \frac{1}{2} \left[ \begin{bmatrix} \frac{\partial u_x}{\partial x} + 1 & \frac{\partial u_x}{\partial y} + 1 \\ \frac{\partial u_y}{\partial x} + 1 & \frac{\partial u_y}{\partial y} + 1 \end{bmatrix} \right] \]

5 \[ [E] = \frac{1}{2} \left[ \begin{bmatrix} \frac{\partial u_x}{\partial x} + 1 & \frac{\partial u_x}{\partial y} + 1 \\ \frac{\partial u_y}{\partial x} + 1 & \frac{\partial u_y}{\partial y} + 1 \end{bmatrix} \right] \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 1 \]

If the displacement gradients are small relative to 1, then the products of the displacements are very small relative to 1, and in infinitesimal strain theory they can be dropped, yielding \([\varepsilon]\):

6 \[ \varepsilon = \frac{1}{2} \left[ \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} \right] = \frac{1}{2} \left[ [J_u]^T + [J_u] \right] \]

This suggests that for multiple deformations, infinitesimal strains might be obtained by matrix addition (i.e., linear superposition) rather than by matrix multiplication; the former is simpler. Also see equation IV.C.5.
7 Example of IV.B.5: $[\varepsilon]$ from superposed vs. sequenced deformations

\[
F_5 = \begin{bmatrix} 1.02 & 0.01 \\ 0 & 1.01 \end{bmatrix} \quad J_u^5 = \begin{bmatrix} 0.02 & 0.01 \\ 0 & 0.01 \end{bmatrix} \quad F_6 = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.02 \end{bmatrix} \quad J_u^6 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}
\]

a Linear superposition, assuming infinitesimal strain (approx.)

\[
\begin{align*}
\text{»} F_5 &= \begin{bmatrix} 1.02 & 0.01; 0.00 & 1.01 \end{bmatrix} \\
F_5 &= \begin{bmatrix} 1.0200 & 0.0100 \\ 0 & 1.0100 \end{bmatrix} \\
E_5 &= \frac{1}{2} [J_u^5]'[F_5] - I \\
\Rightarrow E_5 &= 0.5*(F_5*'F_5\text{-eye}(2))
\end{align*}
\]

\[
\begin{align*}
\text{»} F_6 &= \begin{bmatrix} 1.01 & 0.00; 0.00 & 1.02 \end{bmatrix} \\
F_6 &= \begin{bmatrix} 1.0100 & 0 \\ 0 & 1.0200 \end{bmatrix} \\
E_6 &= \frac{1}{2} [J_u^6]'[F_6] - I \\
\Rightarrow E_6 &= 0.5*(F_6*'F_6\text{-eye}(2))
\end{align*}
\]

\[
\begin{align*}
\left(\frac{1}{2}[J_u^5] + [J_u^5]'\right) \\
\Rightarrow E_7 &= E_5 + E_6 \\
E_7 &= \text{Linear superposition of strains (Infinitesimal approximation)} \\
&= \begin{bmatrix} 0.0302 & 0.0051 \\ 0.0051 & 0.0303 \end{bmatrix}
\end{align*}
\]

b Sequenced deformation (exact)

\[
\begin{align*}
\text{»} F_7 &= F_6*F_5 \\
F_7 &= \begin{bmatrix} 1.0302 & 0.0101 \\ 0 & 1.0302 \end{bmatrix} \\
E_7 &= 0.5*(F_7*'F_7\text{-eye}(2)) \\
E_7 &= \text{Convert def. gradients to strain} \\
&= \begin{bmatrix} 0.0307 & 0.0052 \\ 0.0052 & 0.0307 \end{bmatrix}
\end{align*}
\]

See eq. IV.B.5

Good match with approximation
8 Recap

The infinitesimal strain tensor can be used to find the change in the square of the length of a deformed line segment connecting two nearby points separated by distances $dx$ and $dy$,

$$\frac{1}{2}\left\{(ds')^2 - (ds)^2\right\} = [dX]^T [\varepsilon][dX]$$

and, with the rotation tensor, to find the change in displacement of two points in a deformed medium that are initially separated by distances $dx$ and $dy$:

$$[\Delta U] = \frac{1}{2}[\varepsilon][dX] + \frac{1}{2}[\omega][dX]$$

9 For infinitesimal strains the displacements are essentially the same no matter whether the pre- or post-deformation positions are used.