BASIC CONCEPTS OF KINEMATICS AND DEFORMATION

I Main Topics (see chapters 14 and 18 of Means, 1976)
   A Fundamental principles of continuum mechanics
   B Position vectors and coordinate transformation equations
   C Displacement vectors and displacement equations
   D Deformation

II Fundamental principles of continuum mechanics
   A Relates natural world to the realm of mathematics
   B Densities of mass, momentum, and energy exist (no “holes”)
   C Number of particles is sufficiently large that the notion of an average bulk material behavior is meaningful
   D Examples of continuous properties
      1 Density \( \rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \)
      2 Hydraulic conductivity ("permeability")
   E Scale matters (see B, C, and D)

   E Variability
      1 Heterogeneity: material property depends on position
      2 Anisotropy: material property depends on orientation

II Position vectors and coordinate transformation equations
   A \( \mathbf{X} \) = initial (undeformed) position
   B \( \mathbf{X}' \) = final (current, or deformed) position (at time \( \Delta t \))
   C Coordinate transformation equations
      1 \( \mathbf{X}' = f(\mathbf{X}) \) Lagrangian: final position set in terms of initial
      2 \( \mathbf{X} = g(\mathbf{X}') \) Eulerian: initial position set in terms of final
III Displacement vector ($U$)

A $U = X' - X$

1. x-component: $u_x, u_1$, or just $u$
2. y-component: $u_y, u_2$, or just $v$
3. z-component: $u_z, u_3$, or just $w$

B $U = U(X)$ Lagrangian: displacement in terms of initial position

C $U = U(X')$ Eulerian: displacement in terms of final position

IV Deformation: rigid body motion + change in size and/or shape

A Rigid body translation

1. No change in the length of line connecting any points
2. All points displaced by an equal vector (equal amount and direction); no displacement of points relative to one another
3. $[X'] = [U] + [X]$ matrix addition ($U$ is a constant)

B Rigid body rotation

1. No change in the length of line connecting any points
2. All points rotated by an equal amount about a common axis; no angular displacement of points relative to one another
3. $[X'] = [a][X]$ matrix multiplication; rows in $[a]$ are dir. cosines!

C Change in size and/or shape (distortional strain)

1. The lengths of at least some line segments connecting points in a body change (i.e., the relative positions of points changes)
2. $U$ is not a constant throughout the body (i.e., $U$ varies)
3. Change in linear dimension
   a. Extension (or elongation) \[ \varepsilon = \frac{\Delta L}{L_o} = \frac{L_1 - L_o}{L_o} \text{ dimensionless!} \]
   b. Stretch \[ s = \frac{L_1}{L_o} = \frac{L_1 - L_o}{L_o} + 1 = \varepsilon + 1 \text{ dimensionless!} \]
4. Change in right angles: $\gamma = \tan \psi$
D Change in volume (dilational strain)

1. Dilation \[ \Delta = \frac{\Delta V}{V_o} = \frac{V_1 - V_o}{V_o} \]

Example 1: Consider a rectangular box with sides of length \( a_o, b_o, c_o \), the sides lying along the 1,2,3 axes. Its volume is \( a_o b_o c_o \), or

\[
V_0 = \begin{bmatrix}
a_o & 0 & 0 \\
0 & b_o & 0 \\
0 & 0 & c_o \\
\end{bmatrix} = a_o b_o c_o
\]

Suppose the box is stretched along the 1,2,3 axes such that its new dimensions are \( a_1, b_1, c_1 \). Its new volume \( V_1 \) is

\[
V_1 = \begin{bmatrix}
a_1 & 0 & 0 \\
0 & b_1 & 0 \\
0 & 0 & c_1 \\
\end{bmatrix} = a_1 b_1 c_1
\]

\[
\Delta = \frac{V_1 - V_0}{V_0} = \frac{a_1 b_1 c_1 - a_o b_o c_o}{a_o b_o c_o} = S_1 S_2 S_3 - 1 \approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3
\]

(The expression at the right side applies for small strains [\(< \sim 1\%\)]

Example 2: Volumetric strain with no shape change

\[
V_1 = (\Delta + 1)V_0
\]

All sides of the box are strained equally \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon \).

\[
[X'] = (\varepsilon + 1)[X] = S[X]
\]
1-D Inhomogeneous Deformation (From Means, 1976)

\[ x' = 2x \quad \text{Lagrangian coordinate transformation equation} \]
\[ x = x'/2 \quad \text{Eulerian coordinate transformation equation} \]
\[ u = x \quad \text{Lagrangian displacement equation} \]
\[ u = u(x) = x' - x = 2x - x = x \]
\[ u = x'/2 \quad \text{Eulerian displacement equation} \]
\[ u = u(x') = x' - x = x - x/2 = x/2 \]
Rigid Body Motion

Rigid Body Translation

Rigid Body Rotation

Basic Measures of Strain

Elongation

\[ \varepsilon = \frac{(L_1 - L_0)}{L_0} \]

\[ S = \frac{L_1}{L_0} \]

Shear Strain

\[ \gamma = \tan \Psi \]

Dilation

\[ \Delta = \frac{(V_1 - V_0)}{V_0} \]

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