EQUATIONS OF LINES & PLANES

I. Main Topics
   A. Direction cosines
   B. Lines
   C. Planes

II. Direction cosines
   A. The cosines of the angles between a line and the coordinate axes
   B. The coordinates of the endpoint of a vector of unit length
   C. The ordered projection lengths of a line of unit length onto the x, y, and z axes

III. Lines
   A. Defined by 2 points
      Two-point form: \[ \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \]
      where \((x_1, y_1)\) and \((x_2, y_2)\) are two known points on the line
   B. Defined by 1 point (e.g., \((x_0, y_0, z_0)\)) and a direction
      Slope-intercept form (2-D): \(y = mx + b\)
      General form (2-D): \(Ax + By + C = 0\)
      \[ y = mx + b \]
      \[ mx - y + b = 0 \]
      \[ (1/n)(mx - y + b) = 0 \]
      \[ Ax + By + C = 0 \]
      **Parametric form**: \(x = x_0 + t\alpha, y = y_0 + t\beta, z = z_0 + t\gamma,\)
      where \(\alpha = \cos \omega_x, \beta = \cos \omega_y,\) and (for 3-D) \(\gamma = \cos \omega_z;\)
      \(\alpha, \beta, \) and \(\gamma\) are direction cosines. Note that in 2-D, \(\cos \omega_x = \sin \omega_y\)
   C. Defined by the intersection of two planes
IV Planes

1 Defined by three points

2 Defined by two intersecting lines

3 Defined by two parallel lines

4 Defined by a line and a point not on the line

5 Defined by a distance and direction (or pole) from a point

A General form: \( Ax + By + Cz + D = 0 \)

B Normal form: \( \alpha x + \beta y + \gamma z = d \), where
\[
\alpha = \frac{A}{\pm \sqrt{A^2 + B^2 + C^2}}, \quad \beta = \frac{B}{\pm \sqrt{A^2 + B^2 + C^2}}, \quad \gamma = \frac{C}{\pm \sqrt{A^2 + B^2 + C^2}},
\]
\[d = \frac{-D}{\pm \sqrt{A^2 + B^2 + C^2}}\]

The sign of the denominator is opposite to the sign of \( D \), so \( d > 0 \).
\[\alpha = \cos \omega_x, \ \beta = \cos \omega_y, \ \text{and} \ \gamma = \cos \omega_z.\]

C Vector expression of normal form: \( n \cdot V = d \), where

\( V \) is a vector from a given point \( O \) to the plane,

\( n \) (bold) is the unit normal to the plane given by direction cosines \( \alpha, \beta, \) and \( \gamma; n \) also goes through point \( O \).

d (unbolded) is the distance from the point to the plane along the normal vector \( n \), and

\( \cdot \) refers to the dot product:
\[<x_1, y_1, z_1> \cdot <x_2, y_2, z_2> = x_1 x_2 + x_2 y_2 + z_1 z_2\]

The equation of "C" can be understood as follows: "The distance from the reference point to a plane (as measured along a direction perpendicular to the plane) is \( d \)." If the normal points from the reference point to the plane, then \( d > 0 \). Otherwise, \( d < 0 \).
Direction Cosines from Geologic Angle Measurements
(Spherical coordinates)

Positive z-axis up
y = north; x = east
xy plane is horizontal plane

RIGHT-HANDED
COORDINATES

Remember: Trends are azimuths and are measured in a horizontal plane. Plunges are inclinations and are measured in a vertical plane.

The direction cosines $\alpha$ and $\beta$ are determined from OA', the length of OA' being $|OA'| = |OA| \cos \phi = \cos \phi$.

$$\alpha = \cos \omega_x = (\cos \phi)(\sin \theta)$$
$$\beta = \cos \omega_y = (\cos \phi)(\cos \theta)$$
$$\gamma = \cos \omega_z = -(\sin \phi)$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

Plunge $= \phi$
Trend $= \theta$

Positive z-axis down
x = north; y = east
xy plane is horizontal plane

RIGHT-HANDED
COORDINATES

Remember: Trends are azimuths and are measured in a horizontal plane. Plunges are inclinations and are measured in a vertical plane.

The direction cosines $\alpha$ and $\beta$ are determined from OA', the length of OA' being $|OA'| = |OA| \cos \phi = \cos \phi$.

$$\alpha = \cos \omega_x = (\cos \phi)(\cos \theta)$$
$$\beta = \cos \omega_y = (\cos \phi)(\sin \theta)$$
$$\gamma = \cos \omega_z = +(\sin \phi)$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1$$
The angle between the x-axis and OP is $\omega_x$. The angle between the y-axis and OP is $\omega_y$. The angle between the z-axis and OP is $\omega_z$.

$\alpha = \cos \omega_x = \frac{x}{d}$

$\beta = \cos \omega_y = \frac{y}{d}$

$\gamma = \cos \omega_z = \frac{z}{d}$

$\alpha$, $\beta$, and $\gamma$ are the direction cosines of the angles between the normal to the plane and the x-, y-, and z- axes, respectively.

If $\mathbf{n}$ is a unit vector ($|\mathbf{n}| = 1$) normal to the plane through point P, then $\mathbf{n} \cdot \mathbf{V} = d$