Locally Isotropic Turbulence as a Gauge Field Theory

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Abstract. A new model for turbulence is proposed. The model is expressed in general Lagrangian coordinates, which need not be the initial positions of fluid particles. The dynamics respect local \( SO(3) \) symmetry, as a consequence of gauging the global \( SO(3) \) symmetry of gas dynamics. That is, the partial derivatives in the Euler equations of gas dynamics are replaced with covariant partial derivatives involving \( so(3) \)-valued connections. Thus, an affine differential structure is imposed on the Lagrangian manifold, as an alternative to the Riemannian structure associated with induced metrics. It is assumed that the connections are random. Hence the stationary and isotropic turbulence in this model is sustained by interactions with suitably-structured connections, rather than by the suitably-structured forcing that must be present in conventional models of such turbulence. The random connections or gauge fields are assigned a Gibbs-type probability distribution for which the energy is the Euclidean Yang-Mills form. The instantons or local extrema of the form can twist large-scale eddies into the flow. Simulations of these eddies are being attempted, by combining computational fluid dynamics with lattice gauge theory. Artificial, first-order phase transitions may be avoided by suitable choice of lattice action. Second-order phase transitions imply a rigorous continuum limit, but the existence of such a transition for the chosen lattice action remains controversial. Numerical experiments are in progress.

1. Introduction

Flow at high Reynolds’ number is extremely complex, both in space and time. We regard this complexity as a glimpse of an abstract ensemble of flows. Such randomness can be introduced into fluid dynamics through an ensemble of forcing fields \( F \), or an ensemble of initial conditions \( I \). Thus the momentum equation becomes

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}
\]

subject to

\[
\mathbf{u}(\mathbf{x}, 0) = \mathbf{I}(\mathbf{x}).
\]

The notation is standard. There is no need to consider the conservation laws for mass and energy just yet, nor the equations of state. We assume that all moments exist for all fields. Denote the ensemble average by \( \langle \cdot \rangle \). Then (1) and (2) become in the simple case of constant density \( \rho \),

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} \right) = -\nabla p - \rho \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \mathbf{F},
\]

subject to

\[
\langle \mathbf{u}(\mathbf{x}, 0) \rangle = \langle \mathbf{I}(\mathbf{x}) \rangle.
\]

Note that when \( \rho \) is constant, the flow is solenoidal:

\[
\nabla \cdot \mathbf{u} = 0.
\]

The primes denote departures from the mean, and the average is taken with respect to the joint distribution of \( \mathbf{F} \) and \( \mathbf{I} \). The ‘turbulence problem’ is the expression of the Reynolds’ stress tensor \( \langle \mathbf{u}' \mathbf{u}' \rangle \) in terms of the first order moments \( \langle \mathbf{u}_k \rangle \). Nearly all ‘closure theories’ reduce to ‘mixing length theory’, which is a parody of weakly nonuniform gas kinetics. That is, a down gradient eddy flux of momentum is assumed; for example:

\[
\langle \mathbf{u}' \mathbf{u}' \rangle = -K \left( \frac{\partial \langle \mathbf{u}_1 \rangle}{\partial x_j} + \frac{\partial \langle \mathbf{u}_j \rangle}{\partial x_i} \right)
\]

for some constant ‘eddy viscosity’ \( K \). It then follows that the divergence of the Reynolds stress becomes

\[
-\nabla \cdot (\langle \mathbf{u}' \mathbf{u}' \rangle) = K \nabla^2 \langle \mathbf{u} \rangle,
\]

and so the averaged equation (3) has the same form as (1), which is the equation for each ensemble member or realization.

The simple closure (6) has not met with universal success. Preserving the form of (1) is a very strong assumption. The correct closure, should one exist, may not even yield a partial differential equation, much less a parabolic equation. The mean flow \( \langle \mathbf{u} \rangle \) may actually satisfy, say, an integro-differential equation that in no way resembles (1). How might we guess its form? Some new starting point or guiding principle is surely needed. There is the Hopf equation for the exact evolu-
2. Local isotropy and stationarity

These tendencies for turbulence are usually expressed in terms of a uniform, constant (and therefore ignorable) mean velocity, and a two-point velocity covariance tensor

\[ C_{ij}(x, y, t, s) \equiv \langle u_i(x, t)u_j(y, s) \rangle \]

that is invariant with respect to spatial translations, rotations, and reflections, and temporal translations and reflections:

\[ C_{ij} = C_{ij}([x - y], [t - s]). \]

This symmetry was assumed by Kraichnan in his derivation of the inertial-range and dissipation-range spectra. Any model for turbulence such as (1), (2) would require that the random forcing \( F \) be isotropic and stationary, in order for the turbulent velocity field \( u \) to be isotropic and stationary, asymptotically in the limit as \( t \to \infty \). In particular, the turbulence will decay in the absence of \( F \). *Isotropic and stationary turbulence can only be sustained by ensemble of suitably-structured forcing fields.* A strikingly different concept of isotropy was demonstrated in a movie on turbulence by R. W. Stewart. The camera zooms into a real, turbulent smoke plume, which then fills the screen. A small circular region is cut, and rotated through an obtuse angle. The subsequent appearance of the plume is practically unchanged. The symmetry is most easily expressed in the Lagrangian coordinates of the smoke particles. Let \( a \) be a 3-vector that labels the particles. The initial position of the particles is a convenient choice, but not the only one. The Lagrangian coordinates are then \( (a, t) \), where \( t \) is the time elapsed since a particle is first labeled. The subsequent particle position is

\[ x = x(a, t); \]

the fluid pressure field

\[ p = p(a, t). \]

Let \( R \) be a rotation matrix, that is, \( R \) is a 3 \times 3 orthonormal matrix:

\[ RR^T = R^T R = 1. \]

If we consider just rotations without reflections, then

\[ \det(R) = 1. \]

A rotation of the entire plume is a transformation

\[ x \to x' = Rx. \]

Note that the prime here does not denote a turbulent fluctuation. A rotation of the plume within a small circle centered on the particle having Lagrangian coordinates \( (a, t) \) is in effect a local rotation of axes:

\[ x(a, t) \to x'(a, t) = R(a, t)x(a, t) \]

where \( R \) is, as indicated, a function of the Lagrangian coordinates. The question arises: are the global rotation (13) and local rotation (14) both symmetries of the dynamics?

3. Lagrangian dynamics

The momentum equations for an inviscid fluid are

\[ \frac{\partial^2 x_i}{\partial t^2} = -(\rho J)^{-1} \frac{\partial \rho}{\partial a_k} \frac{\partial J}{\partial a_k} \left( \frac{\partial x_j}{\partial a_i} \right) \]

where \( x_i \) (1 = 1, 2, 3) is a Cartesian coordinate of the particle position \( x \), \( \rho = \rho(a, t) \) is the fluid density, while \( p = p(a, t) \) is the pressure and \( J \) the Jacobian of the coordinate transformation:

\[ J = \frac{\partial x_1}{\partial a_1} \frac{\partial x_2}{\partial a_2} \frac{\partial x_3}{\partial a_3} \]

\[ J \left( \frac{\partial x_j}{\partial a_i} \right) \]

Note that \( J \) is the scalar triple product of the Lagrangian strains, and only depends upon partial derivatives of \( x \). The second partial derivative with respect to time (the acceleration) in (15) is taken with \( a \) fixed. Conservation of mass is a scalar law:

\[ \frac{\partial}{\partial t} (\rho J) = 0. \]

Conservation of entropy is also a scalar law:

\[ \frac{\partial \eta}{\partial t} = 0. \]
The combined first and second laws of thermodynamics are
\[ T d\eta = de - \frac{P}{\rho^2} d\rho \]  
where \( T \) is the temperature and \( e \) the internal energy. For an ideal, calorifically perfect gas,
\[ p = R\rho T, \quad e = c_v T \]  
where \( R \) is the gas constant and \( c_v \) the specific heat at constant volume. The following standard results are easily derived from (18), (19), (20) and (21):
\[ p\rho^{-\gamma} = p_0\rho_0^{-\gamma}, \quad pJ^x = p_0J_0^x, \quad eJ^{-1} = e_0J_0^{-1} \]  
where \( \gamma = c_p/c_v, \rho_p = c_v + R, \) and \( p_0 = p_0(a) \equiv p(a, 0) \), etc. It is possible to chose \( p_0 \) etc. such that \( p_0\rho_0^{-\gamma} \) etc. are all independent of \( a \), and denoted \( (p\rho^{-\gamma})_0 \). Notice that \( e \) depends only upon the flow through \( J \), that is, through \( \left( \frac{\partial x_i}{\partial a_j} \right) \):
\[ e = e(J) = e \left( \frac{\partial x_i}{\partial a_j} \right) \]  
The momentum equation (15) is awkward. It is the Euler-Lagrange equation for the much more elegant, fluid-dynamical action
\[ L_f = pJ \left\{ \frac{1}{2} \frac{\partial x}{\partial t} \frac{\partial x}{\partial a_j} - e(J) \right\} \]
\[ = L_f \left( \frac{\partial x}{\partial t}, \frac{\partial x}{\partial a_j} \right) \]  
by virtue of (23) (Seliger and Whitham, 1968). The kinetic energy and the Jacobian in (24) are scalars, so \( L_f \) is invariant under the global rotation (13):
\[ L'_f = L_f. \]  
However \( L_f \) is changed by the local rotation (14), owing to contributions from partial derivatives of \( R(a, t) \). It would seem, therefore, that local isotropy is not consistent with the dynamics of flow realizations.

4. Gauging the symmetry

There is a standard method in mathematical physics for rendering an action invariant with respect to a local symmetry. It involves the introduction of gauge fields (see, e.g., Weinberg 1996). First, recall that pure rotations without reflection comprise the three parameter Lie group \( SO(3) \). Again, \( R \) is in \( SO(3) \) if
\[ RR^T = R^T R = I, \quad \det(R) = +1. \]
This is a nonAbelian group: in general \( RS \neq SR \) for \( R, S \) in \( SO(3) \). Examples of the three parameters are the Euler angles \( \Theta = (\theta_1, \theta_2, \theta_3) \). Any element of \( SO(3) \) has an exponential expression:
\[ R = \exp(\theta_i \tau_i) \]  
where \( \tau_i \) \((i = 1, 2, 3)\) are the generators of \( SO(3) \):
\[ \tau_i = \frac{\partial R}{\partial \theta_i} \bigg|_{\theta=0}. \]  
The \( \tau_i \) are skew \( 3 \times 3 \) matrices, satisfying the commutator relations
\[ [\tau_i, \tau_j] = -\epsilon_{ijk} \tau_k. \]  
In fact, \( (\tau_i)_{jk} = \epsilon_{ijk} \). The generators comprise the Lie algebra \( so(3) \), and are independent of \( (a, t) \). If the rotation (27) is local, it is the parameters which vary:
\[ \theta = \theta(a, t), \quad R = R(\theta) = R(\theta(a, t)) = R(a, t). \]  
Covariant partial derivatives are defined by
\[ D_j x \equiv \frac{\partial x}{\partial t} + g_0 A_j x, \]
\[ D_j x \equiv \frac{\partial x}{\partial a_j} + g_0 A_j x, \]  
(\( j = 1, 2, 3 \)), where \( g_0 \) is a nonnegative "coupling constant". The matrices \( A_t = A_t(a, t), A_j = A_j(a, t) \) are any elements of the Lie algebra \( so(3) \), that transform as follows under the action of an element \( S = S(\theta) \) of the Lie group \( SO(3) \):
\[ A'_t = SA_t S^T - g_0^{-1} \frac{\partial S}{\partial \theta} S^T, \]
\[ A'_j = SA_j S^T - g_0^{-1} \frac{\partial S}{\partial a_j} S^T. \]  
Thus \( A_t, A_j \) are 'affine connections'. A covariant or 'gauged' fluid-dynamical action may be defined as
\[ L_g = \rho J \left\{ \frac{1}{2} D_j x D_j x - e(J) \right\} \]  
where
\[ J \equiv D_1 x D_2 x \times D_3 x \]
and
\[ eJ^{-1} = e_0J_0^{-1}. \]  
Note that \( L_g \) depends not only upon the simple partial derivatives of \( x \), but also \( x \) itself and the connections or 'gauge fields' \( A_t, A_j \). It follows from (33) and (34) that \( L_g \) is locally gauge-invariant:
\[ L'_g = L_g. \]
Setting to zero the first variation of \( L_g \) with respect to \( x \):
\[ \frac{\delta L_g}{\delta x^i} = 0, \]
yields the gauged momentum equations:

\[-D^l_i D_l x = -\left(\rho J\right)^{-1} \left( \frac{\partial}{\partial \alpha_k} \left( p \frac{\partial J}{\partial \frac{\partial x}{\partial \alpha_k}} \right) - p \frac{\partial J}{\partial x_l} \right),\]

where the adjoint covariant derivative is

\[D^l_i = -\frac{\partial}{\partial t} + g_0 A^T_i.\]

Since \( A_t \) is skew, it has only three independent nonvanishing components, and it defines a vector product:

\[g_0 A_t x = \Omega_t \times x.\]

Sattinger and Weaver (1986) noticed that the left hand side of (40) is

\[-D^l_i D_l x = \frac{\partial^2 x}{\partial t^2} + 2\Omega_t \times \frac{\partial x}{\partial t} + \frac{\partial \Omega_t}{\partial t} \times x + \Omega_t \times \left( \Omega_t \times x \right),\]

which is the acceleration of a particle at \( x(a, t) \), in a frame rotating at angular velocity \( \Omega_t(a, t) \) with respect to some inertial frame.

Suppose that there is a group element \( R(a, t) \) such that

\[A_t = g_0^{-1} R^T \frac{\partial R}{\partial t}, \quad A_j = g_0^{-1} R^T \frac{\partial R}{\partial t}.\]

Then, as \( x \rightarrow x' = Rx \), (33) and (34) imply

\[A_t \rightarrow A'_t = 0, \quad A_j \rightarrow A'_j = 0,\]

and so

\[L_g \rightarrow L'_g = L'_f.\]

That is, the ‘pure-gauge’ connections (44) amount to no more than the effects of a reference frame that is rotating in time and ‘twisting’ about the Lagrangian labeling axes. In general there is no group element \( R \) such that all of (44) is true, but any one of \( A_t, A_j \) can always be eliminated. Given \( A_t \), any solution of

\[\frac{\partial R}{\partial t} = g_0 RA_t \]

will eliminate \( A_t \); then one is working in the ‘temporal gauge’. ‘Axial’ gauges may be obtained by eliminating one of the \( A_j \). Gauge-invariant quantities such as the covariant kinetic energy \( k = \frac{1}{2} \rho J D_l x D_l x \) are unaffected by the choice of gauge. They may be more conveniently calculated in a particular gauge, such as the temporal gauge.

5. Affine geometry

If we use the notation \( A_{ij}^k \) for \( (A_j)_{ik} \), then (32) becomes

\[(D_j x)_i = \frac{\partial x_i}{\partial a_j} + g_0 A_{ij}^k x_k.\]

This resembles the parallel-transport operator for a Riemannian manifold, with some metric \( g^{ij} \). A vector is parallel-transported along the curve \( a_j = a_j(s) \) if

\[\frac{d a_i}{d s} D_j x = 0.\]

If the parallel transport of two vectors \( u_j \) and \( v_k \) leaves unchanged their inner product

\[g(u_j, v_k) \equiv u_j u^j \equiv u_j g^{jk} v_k,\]

then the connection in (48) is ‘metric-compatible’. Given the metric \( g^{ij} \), only the well-known Levi-Civita connection \( \{ e_i \} \) is metric-compatible and symmetric in the lower indices (e.g., Nakahara, 1990). The \( so(3) \)-valued connections \( A_j \) are skew matrices, that is

\[A_{ij}^k = -A_{kj}^i\]

and in general are not symmetric in the lower indices. Any connection may be split into a symmetric part and an antisymmetric part:

\[A_{ij}^k = P_{ij}^k + Q_{ij}^k\]

where

\[P_{ij}^k = \frac{1}{2}(A_{ij}^k + A_{ki}^j), \quad Q_{ij}^k = \frac{1}{2}(A_{ij}^k - A_{ki}^j).\]

The antisymmetric part \( Q_{ij}^k \) is also known as the torsion tensor. If \( A_{ij}^k \) is metric compatible, then

\[A_{ij}^k = \{ e_i \}^k + K_{ij}^k\]

where the contorsion tensor \( K_{ij}^k \) is determined by the metric and the torsion tensor (Nakahara, 1990). In particular the contorsion has no symmetry in its lower indices, and vanishes if the torsion vanishes. The decomposition (54) would seem upon detailed inspection to be invalid if \( A_{ij}^k = -A_{ki}^j \), and so the \( so(3) \)-valued connections \( A_j \) would seem to be incompatible with any metric. In particular the structure would seem incompatible with the metric induced at any instant on the \( a_j \) by the particle paths \( x_j = x_j(a_k, t) \):

\[ds^2 = dx_j dx_j = da_i \frac{\partial x_j}{\partial a_i} \frac{\partial x_j}{\partial a_k} da_k.\]

That is, the induced metric is

\[g^{ik} = \frac{\partial x_j}{\partial a_i} \frac{\partial x_j}{\partial a_k}.\]
metric-compatible Levi-Civita connections for the induced metric (56), and are almost certainly not even compatible with it. In more familiar terms, the $a_j$ are not merely ‘curvilinear coordinates’. Nevertheless (31), (32) and the parallel-transport condition (49) do define a differential structure on the four-dimensional manifold of Lagrangian coordinates $(a, t)$. Together with the field of rotations $R = R(a, t)$, the entire structure is a principal fiber bundle with connections $A_i, A_j$ (Nakahara, 1990). It need not have a metric. In the absence of an induced differential structure, the fluid-mechanical conservation laws could only be expressed in terms of the covariant derivatives (31) and (32), such as in (35).

Even if the induced differential structure were adopted, there is no a priori necessity to define parallel transport using the symmetric metric-compatible connection. There are asymmetric connections that preserve inner products during parallel transport of the pair of vectors. Furthermore, the geodesics are determined solely by the metric and the symmetric torsion-free part of the connection. Recall that geodesics are curves along which the tangent vector is parallel-transported, and which are local extrema of the distance between points. It may be noted that the standard development of general relativity is torsion-free (Wald, 1984), but torsioned developments do exist and they may explain certain cosmological phenomena.

Choosing $\frak{so}(3)$-valued connections leads to terms in the gauged momentum equations (40) that are not present in the standard, Lagrangian fluid dynamical equation (15). Transforming (40) back to the Eulerian coordinates $(x, t)$ would yield forces not present in standard Eulerian fluid dynamics. These forces would have complicated Eulerian form, so the transformation is not given here. The Lagrangian form of these additional forces is complicated enough. All involve $p, \mathbf{x},$ and $A_j$; the term of highest degree is $O(p^4 \alpha^2)$. We require a quantitative concept for the size of a connection. Pure-gauge connections can be gauged to nothing; see (44), (45), so matrix norms for the $A_j$ do not suffice. The field strength $F_{\mu\nu}$ of the connections $A_\mu$ is defined by

$$F_{\mu\nu} = \frac{\partial}{\partial a_\mu} A_\nu - \frac{\partial}{\partial a_\nu} A_\mu + g_0 [A_\mu, A_\nu].$$

It transforms as

$$F'_{\mu\nu} = SF_{\mu\nu}S^T,$$

as a consequence of (33) and (34), and so cannot be ‘gauged away’.

It may be shown that the field strength $F_{\mu\nu} = 0$ if, and only if, the $A_\mu$ are ‘flat’ or pure-gauge connections as in (44).

A scalar for the size of the connections is the Euclid-

ian Yang-Mills action

$$\mathcal{L}_Y = \mathcal{L}_Y(A_\mu) \equiv -\frac{1}{2} \text{tr} (F_{\mu\nu}F^{\mu\nu}).$$

(59)

It is gauge-invariant:

$$\mathcal{L}'_Y = \mathcal{L}_Y.$$  

(60)

This is an easy consequence of (58), and the fact that the trace of a product of matrices does not depend upon the ordering of the matrices in the product.

6. Random geometry

It is proposed that turbulence be modeled with the gauged momentum equation (40). The gauge fields are assumed to be random, with distribution

$$\mathcal{P}(A_\mu) \propto \exp(-\int_M \mathcal{L}_Y)$$

(61)

where $M$ is the Lagrangian 4-manifold, and the integral involves some volume form on $M$, such as that associated with an induced metric. The dynamics of each realization respect the local $SO(3)$ symmetry (14), since the connections are $\frak{so}(3)$-valued and transform as (33), (34). The integrated action in (61) is quartic in the connections, and has local extrema. These “instantons” are integrable, that is, they are localized over $M$ and so can ‘twist’ large eddies into the fluid flow. The instanton solutions can have any length scale (Ryder, 1985).

7. Numerical simulation

The gauged momentum equations are even more complicated than the standard momentum equations, owing to the presence of gauge fields coupled to the fluid by the constant $g_0$. In addition, the gauge fields are random. Their distribution $\mathcal{P}$ depends upon $g_0$, which multiplies the commutator in the field strength (57). There is a vast literature on perturbative analysis of $\mathcal{L}_Y$ for small $g_0$, but that limit merely returns us to the intractable standard equations. It is more interesting to explore the range of finite $g_0$, in which nonperturbative phenomena such as instantons are to be expected. The exploration may be made numerically, by combining computational fluid dynamics with ‘lattice gauge theory’.

A computational grid over $(a, t)$ is a 4-D ‘lattice’. The lattice spacings could be measured with the induced metric. The fluid dynamical fields $x(a, t), p(a, t)$ etc. are defined at the lattice vertices; the gauge fields are defined on the links, midway between vertices. One rod through the lattice is shown below:
The rod is in the $a_\mu$ or just the “a” direction; the subscripts $k, k \pm \frac{1}{2}, k \pm 1$ are counters along this rod. The covariant derivative may be approximated by

$$D_x \equiv \frac{\partial x}{\partial a} + A x \simeq \frac{R_{k+\frac{1}{2}} x_{k+1} - R_{k-\frac{1}{2}} x_{k-1}}{2\Delta a}$$

where

$$R_{k+\frac{1}{2}} = \exp(-g_0\Delta a A_{k+\frac{1}{2}})$$

is a group element, on the link. The group elements define parallel transports on the lattice. Thus $R_{k-\frac{1}{2}}$ transports $x_{k-1}$ from $a_{k-1}$ to $a_k$, while $R_{k+\frac{1}{2}}$ transports $x_{k+1}$ from $a_{k+1}$ to $a_k$. A subtraction may then be performed at $a_k$ as in (62). Only the $SO(3)$ group elements appear on the lattice ($\Delta a > 0$), but the $so(3)$ Lie algebra elements are recovered in the naive continuum limit as $\Delta a \to 0$. The limit is naive until it is established that $x_{k+\frac{1}{2}} \to x_k$, $A_{k+\frac{1}{2}} \to A_{k+\frac{1}{2}}$, etc.

It would be straightforward to discretize the gauged momentum equation (40), using (62). It is better to discretize the action $L_g$, and then vary with respect to the values of $x$ at the vertices. The numerical integration procedure is the generation of a 4-D lattice of random group elements $R$, and then the stepping of the discrete, gauged momentum equations in the time direction. It is convenient for a finite lattice to be periodic in each direction with respect to the group elements. There is no inconsistency in simultaneously imposing, say, initial conditions and rigid boundary conditions on the fluid particle paths.

Only the first step will be reported here, namely, the generation of a lattice of random group elements. The fluid has no influence on the geometry, unlike the stress-energy momentum tensor in general relativity, or the fermions in quantum field theory.

Only the group elements $R$ are prescribed on the lattice, so the Yang-Mills action for the connections $A_\mu$ must be replaced with an action for the $R$. The Wilson action is defined in terms of plaquettes, or rectangles of links: see Fig. 1. The group element on each link yields the parallel transport in the direction of coordinate increase, so the ‘plaquette operator’ $R_\square$ for transport around the plaquette is

$$R_\square \equiv R_4^T R_3 R_2 R_1.$$  
(64)

Then the plaquette action is

$$L_A^\square \equiv 1 - \frac{1}{3} tr(R_\square),$$  
(65)

Figure 1. A ‘plaquette’, defined by four coplanar vertices in a four-dimensional lattice. The group elements $R_1, \ldots, R_4$ define parallel transport along the links, in the indicated directions. Thus, parallel transport clockwise around the plaquette, starting at the lower left vertex, is given by $R_\square = R_4^T R_3 R_2 R_1$.

and the Wilson action for the lattice is

$$L_W^\square \equiv \sum_{R_\square} L_A^\square$$  
(66)

where the summation extends over all plaquettes. The superscript $A$ will be explained subsequently. In the naive continuum limit as $\Delta a \to 0$,

$$L_A^\square = \frac{g_0^2}{6} \frac{1}{2} tr(F_{\mu \nu} F_{\mu \nu}) \Delta a^4 + O(\Delta a^6).$$  
(67)

A partition function for the lattice of group elements is

$$Z(\beta) = \int_{SO(3)} \exp(-\beta L_W^\square)$$  
(68)

where the integration extends over the group space of $SO(3)$ (the space of parameters such as $\theta$), and over the lattice. Thus, as $\Delta a \to 0$, the partition function becomes the distribution (61) if $\beta = \frac{g_0^2}{\theta}$. It is convenient to define a lattice ‘temperature’ by

$$T \equiv \beta^{-1} = \frac{g_0^2}{\theta}.$$  
(69)

Hence strong coupling corresponds to high temperature.

The lattice ‘energy’ $E$ and ‘specific heat’ $C$ may be defined using (68) in the usual way:

$$E \equiv -\frac{\partial}{\partial \beta} \log_e Z = T^2 \frac{\partial}{\partial T} \log_e Z,$$

and

$$C \equiv \frac{\partial E}{\partial T}.$$  
(71)

There is a problem with the partition function $Z(\beta)$ defined in (68):
\[ \Pi_1(SO(3)) = \mathbb{Z}_2, \quad (72) \]

that is, the first homotopy group of \( SO(3) \) is the binary group. In other words, the group space of \( SO(3) \) is doubly connected; there are two equivalence classes of closed paths in the group space. The members of the first class can all be deformed continuously to one point; those of the second class can all be deformed to one open arc. Two copies of the arc can be deformed continuously to the point, hence discrete ‘charges’ can be assigned to the two classes, such that the charges combine according to the multiplication table for \( \mathbb{Z}_2 \):

\[
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0
\end{array}
\quad (73)
\]

A continuous distribution of group elements around a plaquette defines a closed path in the group space, so a charge value may be assigned to the plaquette. A stack of plaquettes of like charge is in effect a vortex or monopole field. The possibility arises that there may be phase transitions associated with the partition function \((68)\), owing to the two monopole states. Numerical experiments (Halliday and Schwimmer, 1981; Greensite and Lautrop, 1981) show that there is in fact a first-order phase transition, or simple discontinuity in \( E \) as a function of \( T \) at \( T = 0.44 \) (\( \beta = 2.3 \)). The change in \( E \) at that temperature is the ‘latent heat’ for the phase transition. Such non-analyticity would make difficult the continuation of lattice properties to the continuum limit. The \( SO(3) \) monopoles do not exist in the continuum limit, as only the \( \mathfrak{so}(3) \) Lie algebra elements \( \mathbf{A}_\mu \) remain.

The problem of monopoles and first-order phase transitions may be avoided by changing the lattice action. The procedure actually involves generating a lattice of elements in the Lie group \( SU(2) \), of unitary matrices with positive determinant:

\[
U = \begin{pmatrix} a & b \\ -b^* & a \end{pmatrix}, \quad (74)
\]

\[
UU^H = U^H U = 1, \quad \det(U) = |a|^2 + |b|^2 = 1, \quad (75)
\]

where the superscript \( H \) denotes the conjugate transpose. The plaquette action is

\[
\mathcal{L}_P^F = 1 - \frac{1}{2} \text{tr}_F(U); \quad (76)
\]

the Wilson action for the lattice is

\[
\mathcal{L}_W = \sum_{\square} \mathcal{L}_P^F. \quad (77)
\]

The superscript \( F \) indicates the use of \((74)\), the fundamental representation of the Lie group \( SU(2) \). The partition function for the lattice is

\[
Z(\beta) = \int_{SU(2)} \exp(-\beta \mathcal{L}_W^F). \quad (78)
\]

There is an explicit mapping of elements \( U \) in \( SU(2) \) onto elements \( R \) in \( SO(3) \) (Ryder, 1985) so the \( SU(2) \) lattice is also an \( SO(3) \) lattice, except with partition function \((78)\) instead of \((68)\). The mapping is two-to-one:

\[
U \rightarrow R, \quad -U \rightarrow R; \quad SO(3) = SU(2)/\mathbb{Z}_2. \quad (79)
\]

Also, \( SU(2) \) and \( SO(3) \) have the same Lie algebra:

\[
\mathfrak{su}(2) = \mathfrak{so}(3). \quad (80)
\]

Indeed, \( SO(3) \) is the adjoint representation of \( SU(2) \). Thus the adjoint Wilson action \((65)\) and fundamental Wilson action \((76)\) have the same naïve continuum limit, which is the desired Yang-Mills action \((59)\):

\[
\mathcal{L}_Y^F(U), \mathcal{L}_Y^R(R) \rightarrow \mathcal{L}_Y^A(A). \quad (81)
\]

The merit of \( SU(2) \) is that its group space is simply-connected:

\[
\Pi_1(SU(2)) = 0; \quad (82)
\]

there can be no monopoles, and so first-order phase transitions for \((78)\) are unexpected. None have been detected. In summary, the strategy here is to create an \( SO(3) \) lattice, for which the partition function is \((78)\). This is achieved by creating an \( SU(2) \) lattice distributed according to \((78)\) and then transforming the \( SU(2) \) elements into \( SO(3) \) elements.

8. Continuum limits

Before proceeding to describe calculations with the \( SU(2) \) lattice, it is necessary to discuss higher-order phase transitions, and genuine continuum limits. A second-order phase transition involves a lattice energy \( E \) which is a continuous but nonanalytic function of the lattice temperature \( T \). The specific heat \( C \equiv \frac{dE}{dT} \) might be of the form

\[
C(T) \propto |T - T_0|^{-\alpha} \quad (83)
\]

where \( 0 < \alpha < 1 \), in which case \( C \) has an unbounded cusp at \( T = T_0 \), or

\[
C(T) \cong C_0 - C_1 |T - T_0|^\alpha \quad (84)
\]

where \( 0 < \alpha < 1 \), in which case \( C \) has a bounded cusp at \( T_0 \). The partition function \( Z \) for a finite lattice is
an analytic function of $T$, so phase transitions can only arise as the lattice becomes infinite. In the limit, the lattice spacing should be assumed to vanish, at such a rate that continuum scales can be recovered. It may be argued that such nonanalyticity implies the divergence of correlation lengths $\nu$, measured in lattice spacings, as $T$ approaches the critical point $T_0$ and as $\Delta a \to 0$. If $\nu$ diverges so that $\nu \Delta a$ has a finite limit $l$, the $l$ must be the scale of some continuum field. It makes sense, therefore, to seek a genuine continuum limit at a critical temperature.

One continuum limit for $SU(2)$ has been determined analytically. The coupling constant $g_0$ is assumed to depend upon $\Delta a$, and it is assumed that $g_0(\Delta a) \to 0$ as $\Delta a \to 0$. A weak-coupling expansion ($g_0 \ll 1$) does not lead to difficulties, so there appears to be a continuum limit as $T = g_0^2/6 \to 0$. This is interpreted by physicists as a model for asymptotic freedom, or the observed decoupling of particles at close range inside nuclei.

9. Pseudo-random lattices

Monte Carlo simulations of the 4-D $SU(2)$ lattice were reported by Creutz (1980, 1983). His very efficient code was based on a very efficient heat-bath algorithm. Only small lattices ($10^4$) could be managed at that time. Nevertheless, the simulations of heat capacity agreed with analytical calculations for low temperature and high temperature:

$$C(T) \to 3/4 \quad \text{as} \quad T \to 0 \quad (85)$$

$$C(T) \sim (4T^2)^{-1} \quad \text{as} \quad T \to \infty . \quad (86)$$

Creutz’ code may now be run quickly on a workstation, for lattices as large as $10^4$ and $32^4$. The energy $E$ and specific heat $C$ are plotted in Fig. 2 as functions of temperature $T$. There is a peak in $C$ near $T = 0.44$, as detected by Creutz (1980), but no evidence for an unbounded cusp. The possibility of a bounded cusp remains, and a more refined search is needed (Swendsen et al., 1992).

10. Interim conclusions

It is unlikely that the existence or otherwise of a second-order critical point can be established rigorously for the four-dimensional $SU(2)$ lattice. For that matter, the existence or otherwise of classical solutions of the equations of gasdynamics has yet to be established, and almost certainly cannot owing to shock formation. Nevertheless, computational fluid dynamics is very useful. Equally, numerical evidence of large-scale eddies, coupled to instantons in an affine differential structure on the Lagrangian manifold, may assist in the comprehension of locally isotropic turbulence.

Figure 2. The energy and specific heat of four-dimensional $SU(2)$ lattices of dimension $16^4, 32^4$. The existence of a bounded critical point, near the lattice temperature $T = 0.44$, is controversial.

Acknowledgments. I am very grateful to Dr. Michael Creutz of the Brookhaven National Laboratory, both for the provision of his heat-bath $SU(2)$ lattice code and for many clarifying discussions. This work is supported by the NASA Office of Mission to Planet Earth, Grant NAGW-4680. The calculations were performed on an SP-2 computer, generously provided to the College of Oceanic and Atmospheric Sciences by the IBM Corporation. It was my pleasure to participate in the 1997 'Aha Huliko'a workshop.

References


