FREQUENCY SHIFTS OF ROSSBY WAVES
IN $\beta$-PLANE TURBULENCE

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ABSTRACT

Frequency shifts in Rossby wave propagation due to nonlinear interactions in geostrophic (beta-plane) turbulence are studied by direct numerical simulations and a statistical closure theory. The shifts are of systematic sign and can be quite large as compared with the linear Rossby frequency. Under certain conditions, the closure equations yield a simple approximation for the shifts. An explanation of the shifts is given by a model that includes oscillating random sweeping and strain of large eddies.

INTRODUCTION

The beta-plane model is one of the simplest turbulence models that takes into account planetary gradient of Coriolis effect, obeying

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (\zeta, \psi)}{\partial (x, y)} - \beta \frac{\partial \psi}{\partial x} = \nu \nabla^2 \zeta, \quad (1)$$

where $\psi$ is the stream function related to the fluid velocity as $u = (\partial \psi / \partial y, -\partial \psi / \partial x)$, $\zeta = -\nabla^2 \psi$ is the vorticity, and $\nu$ the viscosity. The $\beta$ term represents Coriolis effect. In the absence of the nonlinear Jacobian term, Eq.(1) exhibits just the Rossby wave propagation, while in the absence of the $\beta$ term, Eq.(1) is the two-dimensional Navier-Stokes equation. Thus the model provides a simple prototype of wave/turbulence system.

We consider in this paper the frequency shifts of the the Eulerian two-time correlation function in homogeneous and quasi-stationary beta-plane turbulence. Under periodic boundary conditions, in the Fourier space given by

$$u(x, t) = \sum_k u(k, t) \exp(ik \cdot x),$$
the Eulerian correlation spectrum $U$ defined by

$$U(k, \tau, t) = \langle u(k, \tau) \cdot u(-k, t) \rangle,$$  

(2)

obeys

$$\left[ \frac{\partial}{\partial \tau} + \nu k^2 + i \omega_0(k) \right] U(k, \tau, t) = T(k, \tau, t) \equiv -\langle J(k, \tau) \psi(-k, t) \rangle,$$  

(3)

where $\omega_0(k) = -\beta k_x/k^2$ is the linear frequency, and $J$ is the Fourier transform of the Jacobian term in Eq.(1).

The real part of $T$ represents the energy transfer due to nonlinear interactions;

$$\left[ \frac{\partial}{\partial t} + 2\nu k^2 \right] U(k, t) = 2 \text{Re} T(k, t),$$

where $U(k, t) \equiv U(k, t, t)$ and $T(k, t) \equiv T(k, t, t)$, while the imaginary part normalized by the energy spectrum $U$ gives

$$\text{Im} \frac{T(k, t)}{U(k, t)} = \Delta \omega(k) \equiv \text{Re}[\bar{\omega}(k)] + \omega_0(k),$$  

(4a)

where

$$\bar{\omega}(k) = \int \omega U(k, \omega) d\omega / \int U(k, \omega) d\omega,$$  

(4b)

$$U(k, t + \tau, t) = \int U(k, \omega) \exp(i\omega \tau) d\omega.$$

Here and hereafter, we assume quasi-stationarity of turbulence such that the $t$-dependence of $U(k, t + \tau, t)$ is negligible compared with its $\tau$-dependence, and omit the argument $t$ at will. In the absence of the nonlinear interactions, $\Delta \omega$ is zero, and $\Delta \omega$ is therefore a measure of the frequency shifts due to nonlinear interactions.

Direct numerical simulations (DNS) of beta-plane turbulence in planar and spherical geometries so far have suggested that the shifts are westward and nearly proportional to $k_x$ (see the review by Holloway, 1986). There have been theoretical studies on renormalized frequencies that take into account the nonlinear interactions (Legras, 1980; Carnevale and Martin, 1982). However, the reason for the observed shifts remained unknown.

The primary purpose of this paper is to study the frequency shifts by DNS and a two-point closure theory, and to understand the reason. As for the closure theory, we use the Lagrangian renormalized approximation (LRA; Kaneda, 1981), which is derived by systematic Lagrangian renormalized expansions.
STATISTICAL APPROXIMATION (LRA)

In a wide class of two-point closure theories of turbulence, the transfer function $T$ for homogeneous turbulence in a quasi-stationary state is given by an equation of the form

$$T(k) = \frac{1}{2} \sum_{p,q}^{\Delta} \frac{|p \times q|^2}{k^2 p^2 q^2} \theta(-k, p, q)$$

$$\times [(p^2 - q^2)^2 U(p) U(q) - 2(p^2 - q^2)(k^2 - q^2)U(k)U(q)]$$

(5)

where $\sum_{p,q}^{\Delta}$ denotes the sum over $p, q$ satisfying $p + q = k$. The principal difference between various closure theories comes from the difference of the triple relaxation factors $\theta$.

The application of the LRA to the $\beta$-plane model equation (1) yields Eq.(5) with

$$\theta(-k, p, q) \equiv \int_{0}^{\infty} G(-k, \tau) G(p, \tau) G(q, \tau) d\tau$$

(6)

where $G$ is an appropriately defined Lagrangian response function obeying

$$\left[ \frac{\partial}{\partial \tau} + \nu k^2 + i\omega_0(k) \right] G(k, \tau) = -2 \sum_{p,q}^{\Delta} \frac{|p \times q|^4}{k^2 p^2 q^2} \int_{0}^{\tau} G(-q, s) ds U(-q) G(k, \tau)$$

(7)

$$G(k, 0) = 1,$$

(Kaneda, 1981; Kaneda and Gotoh, 1988).

In terms of $\phi$ defined by

$$G(k, \tau) = \exp[-\phi(k, \tau)],$$

Eq.(7) may be written as

$$\frac{\partial^2}{\partial \tau^2} \phi(k, \tau) = 2 \sum_{p,q}^{\Delta} \frac{|p \times q|^4}{k^2 p^2 q^2} \exp[-\phi(-q, \tau)] U(q),$$

(8)

where

$$\phi(k, 0) = 0, \quad \phi_r(k, 0) = \nu k^2 + i\omega_0(k),$$

and we have used $U(q) = U(-q)$. For small $\tau$, $\phi$ may therefore be expanded as

$$\phi(k, \tau) = [\nu k^2 \tau + \frac{A(k)}{2} \tau^2 + ...] + i[\omega_0(k) \tau + \frac{B(k)}{6} \tau^3 + ...],$$

(9a)
where
\begin{equation}
A(k) = 2 \sum_{p,q} \Delta |\hat{k} \times \hat{q}|^4 \frac{k^2 q^2}{p^2} U(q), \tag{9b}
\end{equation}
\begin{equation}
B(k) = -2\beta \sum_{p,q} \Delta |\hat{k} \times \hat{q}|^4 \frac{k^2 q^2}{p^2} U(q), \tag{9c}
\end{equation}
\(\hat{k} = k/k\), and we have used \(p \times q = k \times q\) for \(k = p + q\).

In the following section, we need an estimation for the imaginary part of the triple relaxation factor \(\theta(-k, p, q)\) with \(k \gg q\). If \(\phi(p) \sim \phi(k)\) for \(p = k - q\) and \(k \sim p \gg q\), then Eq.(6) gives
\begin{equation}
\theta(-k, p, q) \sim \theta(-k, k, q) = \int_0^\infty \exp[-2\phi_R(k, \tau) - \phi_R(q, \tau) - i\phi_I(q, \tau)]d\tau, \tag{10}
\end{equation}
for \(k \gg q\), where \(\phi_R\) and \(\phi_I\) are the real and imaginary parts of \(\phi\), respectively, and we have used \(\phi_R(k) = \phi_R(-k)\), and the term \(\phi_I(k)\) has disappeared because \(\phi_I(k) + \phi_I(-k) = 0\).

In order to get a rough estimate of the imaginary part, we assume that the terms of higher order in \(\tau\) may be neglected when Eq.(9a) is substituted into Eq.(10), i.e., we may substitute
\begin{equation}
\phi_R(k) \sim \nu k^2 \tau + A(k) \tau^2, \quad \phi_I(k) \sim \omega_0(k) \tau + B(k) \tau^3, \tag{11}
\end{equation}
into Eq.(10). This substitution does not imply that the value of \(\phi\) itself is assumed to be well approximated by Eq.(11) in the entire range of \(\tau\). It is clear that Eq.(11) may be a poor approximation for large \(\tau\), although it may be good for small \(\tau\). The substitution implies that we assume the magnitude of the integrand in Eq.(10) to be sufficiently small for large \(\tau\) (where Eq.(11) may be wrong), and the error caused by the substitution to be not serious.

For the sake of simplicity, we further assume that the anisotropy of the energy spectrum may be negligible, or we may discard the anisotropic part of \(U\), for example by the smallness of \(\beta\). Then \(A(k)\) is a function of only the magnitude \(k\) and, Eq.(9c) may be reduced to
\begin{equation}
B(k) = \omega_0(k) C(k),
\end{equation}
where \(C\) is a function of only \(k\).

The substitution of Eq.(11) into Eq.(10) then gives
\begin{equation}
\theta(-k, p, q) \sim \int_0^\infty \exp\{-\nu[2k^2 + q^2] \tau - [2A(k) + A(q)] \tau^2 - i\omega_0(q) \tau[1 + C(q) \tau^2]\}d\tau,
\end{equation}
where we have put \( A(k) = A(k) \). If the viscous term is negligible, and
\[
2A(k) \gg A(q), \quad 2A(k) \gg |C(q)|, \quad [2A(k)]^{1/2} \gg |\omega_0(q)|, \quad (12a, b, c)
\]
this gives
\[
\text{Im} \theta(-k, p, q) \sim -\gamma(k) \omega_0(q), \quad (13a)
\]
where
\[
\gamma(k) \equiv \int_0^\infty \tau \exp[-2A(k)\tau^2]d\tau. \quad (13b)
\]

Although it is not easy to estimate \( A \) and \( C \) under general conditions, simple estimations are possible under certain idealized conditions and assumptions as follows. Let \( E \) and \( Z \) be the total energy and enstrophy defined by
\[
E \equiv \sum_q U(q), \quad Z \equiv \sum_q q^2 U(q),
\]
respectively, and let us assume that most energy and enstrophy are from wavenumbers near \( k_E \) and \( k_Z \), respectively, so that \( E \) and \( Z \) may be approximated as
\[
E \sim \sum_{q < K_E} U(q), \quad Z \sim \sum_{q < K_Z} q^2 U(q), \quad (14a, b)
\]
where \( K_E \) and \( K_Z \) are of the same order with \( k_E \) and \( k_Z \), respectively. If \( k \gg k_Z \), and the dominant contributions to \( A(k) \) and \( B(k) \) in Eqs.(9b) and (9c) come from the domain \( q < K_Z \), then it is shown after some algebra that Eqs.(9b,c) and (14b) give
\[
A(k) \sim \frac{3}{4} Z, \quad C(k) \sim \frac{1}{4} Z, \quad (15a, b)
\]
while the dimensional consideration based on Eqs.(9b,c) and (14a) gives
\[
A(q) = O(k_{kE}^2 E), \quad C(q) = O(k_{kE}^2 E),
\]
for \( q \sim k_E \ll k \), provided that the dominant contributions to \( A(q) \) and \( C(q) \) are from the energy containing range.

The conditions (12a) and (12b) are then well satisfied if
\[
Z \gg k_{kE}^2 E. \quad (16)
\]
The numerical factor \( 2 \) in front of \( A(k) \) in Eq.(12) is insignificant in the estimation of the order of magnitude of the terms, but may be significant numerically in real DNS of limited resolution, where the strong inequalities in Eqs.(12) and (16) may hold only in a weak sense, i.e., the strong inequality "\( \gg \)" is to be changed to the weaker "\( > \)".
SIMPLIFIED APPROXIMATION

Let \( T^<(k|K) \) be the contribution from the interactions among the modes \((k, p, q)\) in Eq.(5) with \( p \) or \( q < K \). The contribution can be estimated in the same way as Kraichnan (1966). Since

\[
(p^2 - q^2)[(p^2 - q^2)U(p) - (k^2 - q^2)U(k)] \sim -k^2(q \cdot \nabla_k)[k^2U(k)],
\]

for \( k = p + q \) and \( k \sim p \gg q \), we have for \( k \gg K \),

\[
T^<(k|K) \sim - \sum_{q<K}^\Delta \theta(-k, p, q)|\hat{k} \times \hat{q}|^2U(q)(q \cdot \nabla_k)[k^2U(k)],
\]  \hspace{1cm} (17)

where \( \sum_{q<K}^\Delta \) denotes the sum over \( q \) satisfying \( k = p + q \) and \( q < K \).

In order to get a simplified approximation for the frequency shift \( \Delta \omega(k) \) for large wavenumber \( k \), we introduce here the following three assumptions.

(I): The dominant contributions to \( \text{Im} \ T(k) \) for \( k \gg K_Z \) come from nonlocal interactions with low wavenumbers, so that

\[
\text{Im} \ T(k) \sim \text{Im} \ T^<(k|K_Z).
\]

(II): \( \beta \) is not very large so that the anisotropy of the energy spectrum \( U \) is weak and we may therefore neglect its anisotropic part, i.e., we may put

\[
U(q) \sim U(q).
\]

(III): The imaginary part of the triple relaxation for \( k \gg K_Z \) may be approximated by Eq.(13) in the estimation \( \text{Im} \ T^<(k|K_Z) \).

Under the assumption (III), Eq.(17) yields

\[
\text{Im} \ T^<(k|K_Z) \sim \gamma(k) \sum_{q<K_Z} \omega_0(q)|\hat{k} \times \hat{q}|^2U(q)(q \cdot \nabla_k)[k^2U(k)],
\]  \hspace{1cm} (18)

and under the isotropic assumption (II) this may be further reduced to

\[
\text{Im} \ T(k|K_Z) \sim -\frac{\beta \hat{k}_x \gamma(k)}{8} \sum_{q<K_Z} U(q) \frac{\partial [k^2U(k)]}{\partial k}.
\]  \hspace{1cm} (19)

A rough estimate of \( \gamma(k) \) may be obtained by substituting Eq.(15a) into Eq.(13b), which results in

\[
\gamma(k) = \frac{2}{3Z}.
\]  \hspace{1cm} (20)
If we further assume Eq.(14a), then the assumption (I) and Eq.(19) give

$$\text{Im} \, T(k) \sim -\frac{\beta \dot{k}_x E}{12Z} \frac{\partial [k^2 U(k)]}{\partial k},$$

and therefore

$$\Delta \omega(k) = \frac{\text{Im} \, T(k)}{U(k)} = -\frac{\beta \dot{k}_x E}{12Z U(k)} \frac{\partial [k^2 U(k)]}{\partial k}. \tag{21}$$

If \( U(k) \sim k^{-m} \), then Eq.(21) yields for \( k \gg k_Z \),

$$\Delta \omega(k) = \frac{(m-2)\beta E}{12Z} k_x. \tag{22}$$

The comparison of Eq.(22) with the linear frequency yields

$$\frac{\Delta \omega(k)}{\omega_0(k)} = -\frac{(m-2)}{12} \frac{k^2}{k_Z^2},$$

while the comparison with the random sweeping frequency, yields

$$\frac{\Delta \omega(k)}{\omega^*(k)} = \frac{(m-2)}{6} \frac{k_Z^2}{k} \frac{k_x^2}{k^2},$$

where \( k_Z = \zeta'/u' \) is a representative wavelength for a flow with an rms velocity \( u' = E^{1/2} \) and an rms vorticity \( \zeta' = Z^{1/2} \) and \( k_R = (\beta/2u')^{1/2} \) is a representative wavelength obtained by comparing representative wave speed with \( u' \) (Rhines, 1975).

Equations (21) and (22) suggest that the frequency shifts are independent of the amplitude of the turbulent flow. However, it is to be remembered that Eq.(21) is based on the assumption (III) or Eq.(13), and in the derivation of Eq.(13) we have assumed Eq.(12) and that the viscosity is negligible. In the limit of weak nonlinearity, Eq.(13) does not hold, but

$$\text{Im} \theta(-k, p, q) \sim -\frac{\omega_0(q)}{(2\nu k^2 + \nu q^2)^2 + \omega_0^2(q)}.$$

This can be justified by noting that neglecting the right-hand side of Eq.(8) yields

$$\phi(k, \tau) = \nu k^2 \tau + i\omega_0(k) \tau.$$

Retracing the derivation of Eq.(21) then gives for \( k \gg k_Z \),

$$\Delta \omega(k) = -\frac{\beta \dot{k}_x E}{32(\nu k^2)^2 U(k)} \frac{\partial [k^2 U(k)]}{\partial k},$$
instead of Eq.(21), provided that the assumptions (I) and (II) are still valid in this limit, and \( \nu k^2 \gg |\omega_\omega(q)| \) for \( k \gg K_Z > q \).

It is also to be noted here that if \( m < 2 \) then the integration of Eq.(18) or Eq.(19) over \( q \) does converge at low \( q \). This implies that the dominant contributions come from local or high wavenumbers, and this is incompatible with the assumption (I). Hence \( m \) must satisfy \( m > 2 \) unless the form \( U(k) \sim k^{-m} \) is assumed to be valid only in a local sense.

The approximation (22) has the advantage of simplicity, as compared with Eq.(5), but is based on several assumptions. It is therefore interesting to compare the simplified approximation (22) with the estimate obtained from Eq.(5) without using the assumptions. In the next section, we try such a comparison as well as the comparison of the theory with DNS.

DNS AND NUMERICAL SOLUTION OF THE LRA

Fields satisfying Eq.(1) under periodic boundary conditions were generated by alias-free spectral method with wavenumber increment \( \Delta k = 1 \) in each of \( k_x \) and \( k_y \) directions, and retained wavevector domain \( k < K_{max} \), where \( K_{max} \) is about 85. The initial values of the Fourier components \( u(k) \) were chosen to be normally distributed with given initial isotropic spectrum \( U(k, t = 0) \). In the runs reported here, we used \( \nu = 0.004 \) and \( E(k) \equiv \pi kU(k, t = 0) = Ck \exp(-2k/k_0) \), where \( k_0 \) is a constant, and the constant \( C \) is so normalized that \( E = 1 \) in each realization. In a series of run (Series B), \( k_0 \) was fixed at \( k_0 = 5.0 \) and \( \beta \) was changed as \( \beta = 2.5, 5.0 \) and 10.0. These runs are called here as B25, B5 and B10, respectively. In another series (Series K), \( \beta \) was fixed at \( \beta = 5.0 \) and \( k_0 \) was changed as \( k_0 = 2.5, 5.0 \) and 10.0. These runs are called as K25, K5 and K10, respectively.

In order to avoid the initial rapidly changing phase, we started to take time averages after \( t = 0.8 \). The averages here are time averages from \( t = 0.8 \) to \( t = 1.0 \). In all the runs, the time averaged spectrum \( k^4 U(k) \) was observed to be nearly isotropic, and the slope of \( U \) was steeper than \( k^{-4} \) at high \( k \). The representative wavenumbers \( k_Z = \zeta'/u' = \sqrt{Z/E} \) and the total enstrophy in the runs were as follows:

\[
\begin{array}{cccccc}
\text{B5/K5} & \text{B25} & \text{B10} & \text{K25} & \text{K10} \\
 k_Z & 4.64 & 4.64 & 4.64 & 2.95 & 6.55 \\
 Z & 17.5 & 17.5 & 17.6 & 8.13 & 25.4
\end{array}
\]

Thus \( k_Z \) and \( Z \) are larger for larger \( k_0 \) in Series K, as would be expected.

If we take the characteristic eddy-damping time scale as \( \tau_D \sim \sqrt{4/3Z} \), which
is suggested from Eqs.(11) and (15a), and the eddy turn over time as $\tau_T \sim 2\pi/\zeta'$,
then, for example, for K5/B25 they are given by $\tau_D \sim 0.28$, and $\tau_T \sim 1.5$. Thus
the time interval of the averages is comparable to or shorter than the damping and
eddy turn over times. (The time interval is limited in our DNS, because in order
to avoid extra complexity caused by the introduction of external driving force, we
are considering here only freely decaying turbulence in which the statistics cannot
be stationary in a strict sense. Better statistics could be obtained by increasing
the number of realizations. However, a preliminary test of taking averages over 6
realizations suggested that the results are qualitatively not significantly different
from those by one realization.)

The LRA approximation Eq.(5) for $T(k)$ with Eq.(6) was also estimated by
numerical computation. The sums over $p, q$ in Eqs.(5) and (8) were computed by
an alias free spectral method based on the use of Fast Fourier Transform (FFT) as in
a previous study, (Gotoh and Kaneda, 1991). In order to avoid large fluctuations in
the simulated energy spectrum, we substituted to $U(k)$ the isotropic band-averaged
as well as time- averaged spectrum. The wavenumber increment in the computation
is $\Delta k = 1$ as in DNS, and the retained wavevector domain was $k < K_{max} \sim 85$.
As a preliminary check, we computed $\Delta \omega$ by two ways; one is by using the single-
precision FFT and the other by double-precision FFT. Although the value of $\Delta \omega(k)$
at high wavenumbers was found to be very sensitive to the precision, no significant
difference was observed at $k$ less than about 40. We therefore present results only
for $k_x < 40$, in the followings. The sensitivity at high $k$ is presumably because $U(k)$
is there very small and $\Delta \omega$ has the denominator $U(k)$ as in Eq.(4a).

Figures 1,2 and 3 show the frequency shifts by DNS and the LRA in Series B,
while Figs. 1,4 and 5 show the shifts in Series K. In the figures, the values by the
simplified approximation (22) are also plotted, where the value $m = 7$, which was
guessed from the energy spectrum at $k \sim 20$ or so, is used. The energy spectrum
is not rigorously of power low form in the DNS, and this exponent should not be
taken too seriously.

Although it is difficult to make detailed quantitative comparisons due to rela-
tively large fluctuations in the simulated values of $\Delta \omega$ taken from short time interval
as noted above, the figures show that the slope $\Delta \omega/k_x$ increases with $\beta$ in Series
B, and decreases with $k_0$, i.e., with the total enstrophy in Series K. The DNS re-
results suggest that the shifts are nearly proportional to $k_x$, the slopes in the figures
are positive (i.e.,$\Delta \omega/k_x > 0$) and $\Delta \omega$ exhibits only weak dependence on $k_y$. The
positivity of the slopes means that the shifts are in the direction of westward phase
propagation. These results are in agreement with previous studies, (cf. Holloway,
1986). The results of the LRA as well as the simplified approximation (22) are seen
to agree qualitatively with DNS.
Figure 1. Frequency shift $\Delta \omega$ by LRA, DNS and simplified approximation (22) with $m=7$ for Case B5/K5 ($k_0=5$, $\beta=5$) at $k_y=0, 10, 20,$ and 30.

Figure 2. Same as Figure 1, but for Case B25 ($k_0=5$, $\beta=2.5$) at $k_y=0$ and 20.
Figure 3. Same as Figure 2, but for Case B10 ($k_0=5$, $\beta=10$).

Figure 4. Same as Figure 2, but for Case K25 ($k_0=2.5$, $\beta=5$).

Figure 5. Same as Figure 2, but for Case K10 ($k_0=10$, $\beta=5$).
OSCILLATING RANDOM VELOCITY GRADIENT MODEL

In order to get some idea on the physics underlying the frequency shifts discussed in the previous sections, let us consider the following model equation for the vorticity $\zeta(k)$ of small eddies;

$$
\left[ \frac{\partial}{\partial t} + \lambda V \right] \zeta(k, t) = -\mu(k) \zeta(k, t) + f(k, t),
$$

where the parameter $\lambda$ is introduced for the later convenience, $\mu$ a real time-independent deterministic damping factor satisfying $\mu(k) = \mu(-k)$, $f$ a statistically homogeneous and stationary white-noise random process with zero mean and $f(-k) = f^*(k)$, and $V$ an operator defined by

$$
V = V(k, t) = ik_a U_a(t) + ik_a S_{ab}(t) \frac{\partial}{\partial k_b},
$$
in which $U_a$ and $S_{ab}$ are wavevector-independent random variables with zero mean.

The $U_a$- and $S_{ab}$-terms are supposed to be models for the effects of random sweeping and random straining of the vorticity field by large eddies, respectively. Such a representation of the effects of the large eddies has been used in studies of the role of large eddies on small eddies, (see, for example, Townsend, 1976). The $\mu$-term is supposed to represent the viscous damping as well as the eddy-damping due to the nonlinear interactions that are not taken into account by the $V$-term.

Under the existence of the uniform strain term ($S_{ab}$ term), the two-time Eulerian correlation function, unlike single-time correlation, of $\zeta$ obeying Eq.(23) is not homogeneous. This implies that $<\zeta(x, t)\zeta(x', t')>$ depends on the space variables $x$ and $x'$ not only through $x - x'$ unless $t = t'$, (cf. Gotoh and Kaneda, 1991). We consider here the Fourier transform of $<\zeta(x, t)\zeta(x', s)>$ with respect to $x$ for $x' = 0$, and define the spectrum $U(k, \tau, t)$ as

$$
U(k, \tau, t) = \int <\zeta(k, \tau)\zeta(p, t)> \frac{d^2 p}{k^2}.
$$

This definition of $U$ is equivalent to Eq.(2) for homogeneous turbulence.

Multiplying Eq.(23) with $\zeta(p, s)$ and taking the average yield

$$
\frac{\partial}{\partial t} <\zeta(k, t)\zeta(p, s)> |_{t=s} = T_\zeta - \mu(k) <\zeta(k, t)\zeta(p, t)> + <f(k, t)\zeta(p, t)>,
$$

where

$$
T_\zeta \equiv T_\zeta(k, p, t) = -\lambda <V(k, t)\zeta(k, t)\zeta(p, t)>. $$
Since the imaginary parts of the second and third terms on the right-hand-side of \( \text{Eq.}(24) \) are zero, \( \text{Eq.}(24) \) gives

\[
\text{Re}[\tilde{\omega}(k)] = \frac{\text{Im} T_\zeta(k)}{k^2 U(k)},
\]

where

\[
T_\zeta(k) \equiv \int d\mathbf{p} < T_\zeta(k, \mathbf{p}, t) >,
\]

and \( \tilde{\omega}(k) \) is defined in the same way as \( \text{Eq.}(4b) \) through the frequency spectrum \( U(k, \omega) \). Because the linear frequency \( \omega_0(k) \), i.e. the frequency in the absence of \( V \)-term, is zero in \( \text{Eq.}(24) \), \( \text{Eq.}(25) \) also represents the frequency shift \( \Delta \omega(k) \) due to nonlinear interactions, i.e.,

\[
\Delta \omega(k) = \frac{\text{Im} T_\zeta(k)}{k^2 U(k)},
\]

in the present model (cf. \( \text{Eq.}(4a) \)).

Since \( \text{Eq.}(23) \) is linear in \( \zeta \), it is possible to solve \( \zeta \) analytically, but the expression for \( \Delta \omega \) would be then quite complicated. Hence we try here a perturbative expansion of \( \Delta \omega \) in powers of \( \lambda \). When \( \lambda = 0 \), \( \text{Eq.}(23) \) is just the wellknown Langevin equation. Let \( \zeta_0 \) be the zeroth order solution of \( \text{Eq.}(23) \) with \( \lambda = 0 \), and

\[
< \zeta_0(k, t) \zeta_0(q, t) > = \delta(k + q) k^2 U_0(k).
\]

By discarding terms of \( 0(\lambda^3) \) and putting \( \lambda = 1 \), we obtain after some straightforward algebra,

\[
\frac{T_\zeta(k)}{k^2} = - \int_0^\infty d\tau < U_\alpha(0) S_{ab}(-\tau) > \exp[-2\mu(k)\tau] \hat{k}_a \hat{k}_a \frac{\partial}{\partial k_b} [k^2 U_0(k)],
\]

provided that the term second order in \( S_{ab} k_a (\partial/\partial k_b) \) is negligible.

A specific model of the correlation between \( U \) and \( S \) in \( \text{Eq.}(26) \) may be obtained by assuming

\[
U_a(t) \sim \sum_{q<K} u_a(q, t), \quad S_{ab}(t) \sim - \sum_{q<K} q_b u_a(q, t),
\]

with

\[
< U_\alpha(t) S_{ab}(s) > = \sum_{q<K} < u_\alpha(q, t) q_b u_a(-q, s) >,
\]

\( \text{Eq.}(27) \)
where $K \ll k$. Without loss of generality, we may put

$$< u_0(q, 0)u_0(q, -\tau) >= D_{aa}(q)U(q)\exp[-\phi(q, \tau)],$$

(28)
in which $\phi = \phi_R + i\phi_I$ is a function of $q$ and $\tau$, and the factor $D_{aa}(q) = \delta_{aa} - \hat{q}_a\hat{q}_a$ ensures the incompressibility condition of the velocity field.

Substituting Eq.(27) with Eq.(28) into Eq.(26) gives

$$\Delta \omega(k) = \frac{\text{Im} T_z(k)}{k^2 U(k)} = -\sum_{q<K} \text{Im} \theta(k, q)|\hat{k} \times \hat{q}|^2 U(q)(q \cdot \nabla_k)[k^2 U(k)]/U(k).$$

(29)
to the lowest order in $\lambda$, where

$$\theta(k, q) = \int_0^{\infty} \exp[-2\mu(k)\tau - \phi(q, \tau)]d\tau,$$

(30)
and we have used $D_{aa}(q)\hat{k}_a\hat{k}_a = |\hat{k} \times \hat{q}|^2$. The right-hand side of Eq.(29) multiplied by $U(k)$ is of the same form as the imaginary part of Eq.(17) except that $\text{Im} \theta(k, p, q)$ is replaced by $\text{Im} \theta(k, q)$ in Eq.(29).

If we choose $\phi(q, \tau) = [\mu(q) + i\omega(q)]\tau$, then Eq.(30) may be written as

$$\theta(k, q) = \frac{1}{2\mu(k) + \mu(q) + i\omega(q)}.$$  

(31)
If $\omega(q) \sim \omega_0(q)$ and $\mu(k) \gg \mu(q), |\omega_0(q)|$ for $k \gg K > q$, then

$$\text{Im} \theta(k, q) \sim -\gamma(k)\omega_0(q), \quad \gamma(k) = \frac{1}{4\mu^2(k)}.$$

By choosing $\mu$ as

$$\frac{1}{\mu^2(k)} \sim \frac{8}{3Z},$$

(32)
(i.e., $\gamma(k) = 2/(3Z)$ as in (20)), and retraing the derivation of (21) from (18), we can recover Eq.(21) from Eq.(29) under the isotropic assumption (11). When $U(k) \sim k^{-m}$, Eq.(29) becomes identical to Eq.(21).

The above model suggests that the correlation $< U_a(0)S_{ab}(\tau) >$ between the random sweeping velocity and strain of large eddies may yield the systematic westward frequency shifts of small eddies. Equation (26) shows that the frequency shifts are smaller for larger damping factor $\mu(k)$ of small eddies. The small eddies have a characteristic life time of order $1/\mu(k)$ associated with the damping factor in Eq.(23). Equation (32) or (11) with (15a) suggests that the life time is shorter for larger total vorticity $Z$ under certain conditions. This results in smaller frequency shifts for larger $Z$. The result Eq.(29) with Eq.(31) shows that the increase of frequency $\omega(q)$ of large eddies yields larger frequency shifts of small eddies when $|\omega(q)| \ll 2\mu(k)$, but the frequency shifts decrease with the increase of $\omega(q)$ in the opposite limit $|\omega(q)| \gg 2\mu(k)$.
OTHER QUANTITIES

A) Frequency Shifts of Eulerian Response Function

In this paper we have considered the frequency shift $\Delta \omega$ of the Eulerian two-time correlation function $U$. It might be tempting to relate the shift with that of the Eulerian response function $G$ (or the so-called Eulerian renormalized propagator), which may be defined, corresponding to our use of Eq.(1), as

$$G(k,t,s)\delta(k+q) \equiv \langle \hat{G}(k,q,t,s) \rangle,$$

where $\hat{G}$ is defined as

$$\delta \zeta(k,t) = \int d^2q \int_{-\infty}^{t} ds \hat{G}(k,q,t,s) \delta f(q,s),$$

in which $\delta f$ is an infinitesimal disturbance added to the right-hand-side of Eq.(1) and $\delta \zeta$ is the response to the disturbance, and $\hat{G}$ obeys

$$\left[ \frac{\partial}{\partial t} + \nu k^2 + i\omega_0(k) \right] \hat{G}(k,k',t,s) = \sum_{p,q}^{\Delta} (q_x p_y - q_y p_x) \left( \frac{1}{p^2} - \frac{1}{q^2} \right) \zeta(p,t) \hat{G}(q,q',t,s),$$

$$\hat{G}(k,k',t,t) = 1. \tag{33a}$$

$$\tag{33b}$$

Because $\hat{G}$ is deterministic at $t = s$ and satisfies Eq.(33b) and $\langle \zeta \rangle = 0$, Eq.(33a) gives

$$\left[ \frac{\partial}{\partial t} + \nu k^2 + i\omega_0(k) \right] G(k,k',t,s) = 0, \text{ at } t = s.$$

Unlike to the frequency shift $\Delta \omega$, there is therefore no contribution from the nonlinear interactions to $\Delta \omega_G$, where $\Delta \omega_G$ is defined similarly to Eq.(4a) with $T$ replaced by the average of the right-hand side of Eq.(33a). Thus it is wrong to assume the so-called fluctuation-dissipation approximation

$$U(k,t,s) = U(k)G(k,t,s),$$

as far as the shift $\Delta \omega_G$ is concerned, and the shift $\Delta \omega$ of Eulerian two-time correlation should not be confused with the shift $\Delta \omega_G$ of the response function.
B) Frequency Shifts of Lagrangian Correlation Function

Another quantity which might be related to the shift $\Delta \omega$ is the shift of Lagrangian correlation. Let $U_L(k, \tau, t)$ be the Fourier transform with respect $r$ of the Lagrangian two-time velocity correlation $\langle \mathbf{v}(x + r, t; \tau) \cdot \mathbf{v}(x, t; \tau) \rangle$, where $\mathbf{v}(x, t; \tau)$ is the velocity at time $\tau$ of the fluid particle that was at $x$ at time $t$.

Because
\[
\frac{\partial}{\partial \tau} \langle \mathbf{v}(x + r, t; \tau) \cdot \mathbf{v}(x, t; \tau) \rangle = \langle \frac{\partial}{\partial \tau} \mathbf{v}(x + r, t; \tau) \rangle \cdot \mathbf{u}(x, t),
\]
and
\[
\frac{\partial}{\partial \tau} \mathbf{v}(x, t; \tau)|_{\tau=t} = \left[ \frac{\partial}{\partial t} + (\mathbf{u}(x, t) \cdot \nabla) \right] \mathbf{u}(x, t) = -\nabla p - (\text{terms linear in } \mathbf{u}),
\]
it is shown that
\[
\left[ \frac{\partial}{\partial \tau} + \nu k^2 + i \omega_0(k) \right] U_L(k, \tau, t) = 0, \quad \text{at } \tau = t, \quad (34)
\]
where we have used $\langle \nabla p \cdot \mathbf{u} \rangle = 0$ in homogeneous turbulence, in which $p$ is the pressure. Unlike the Eulerian spectrum $U$ in Eq.(3), there is therefore no contribution from the nonlinear interactions to the $\tau-$ derivative at $\tau = t$ of the Lagrangian spectrum $U_L$. Thus the frequency shift $\Delta \omega$ should not be confused with that of Lagrangian correlation $U_L$.

In the LRA, $U_L$ is given by $U_L(k, \tau, t) = G(k, \tau, t)U(k, t)$ and the LRA with Eq.(7) is consistent with Eq.(34). Because
\[
(\partial/\partial t)U(k, t) = (\partial/\partial \tau)U_L(k, \tau, t) + (\partial/\partial \tau)U_A(-k, \tau, t), \quad \text{at } \tau = t,
\]
and $U(k, t)$ is real, Eq.(34) also implies that there is neither contribution from the nonlinear interactions to $\text{Im}(\partial/\partial \tau)U_A(k, \tau, t)$ at $\tau = t$, where $U_A(k, \tau, t)$ is the Fourier transform of $\langle \mathbf{v}(x + r, \tau; \tau) \cdot \mathbf{v}(x, \tau; t) \rangle$ and $(\partial/\partial \tau)U_A(-k, \tau, t)$ is the key quantity in the Abridged Lagrangian History Direct Interaction Approximation by Kraichnan (1965).

C) Frequency Shifts in Inviscid Truncated System

The inviscid truncated model of Eq.(1) with a retained wavevector domain $D$ has an equilibrium state characterized by the equilibrium energy spectrum
\[
U(k) = \frac{1}{a + bk^2} \quad \text{in } D,
\]
where $a$ and $b$ are constants (Salmon et al., 1976). Since Eq.(5) gives

$$
\frac{T(k)}{U^2(k)} = \frac{1}{2} \sum_{p,q \in D} \theta(-k,p,q) \frac{|p \times q|^2}{k^2 p^2 q^2} U(p) U(q) \left\{ \frac{p^2 - q^2}{U(k)} - \frac{k^2 - q^2}{U(p)} - \frac{k^2 - q^2}{U(q)} \right\},
$$

(a similar expression has been derived by Carnevale et al., 1981), it is shown that if $U$ is given by the equilibrium spectrum and if the triple relaxation factor satisfies the symmetry $\theta(-k,p,q) = \theta(-k,q,p)$ between $p$ and $q$, then $T(k)$ is identically zero, i.e., not only the real but also the imaginary part of $T(k)$ are zero. The triple relaxation factor of the LRA given by Eq.(6) in fact satisfies the symmetry, and the LRA therefore yields $\Delta \omega(k) = 0$ at the inviscid equilibrium state.

D) Complex Eddy Viscosity

There are various ways to define eddy viscosity. Following Kraichnan (1976), we consider here the following definition of the eddy viscosity $\nu_T$;

$$
\nu_T(k|K) \equiv -T^>(k|K)/[k^2 U(k)],
$$

where $T^>(k|K)$ is the contribution to $T(k)$ from the interactions among the modes $(k,p,q)$ with $p$ or $q > K$. By assuming $U(q) \ll U(k)$ for $q \gg k$, and noting that Eq.(5) gives

$$
\frac{T^>(k|K)}{U(k)} \sim \frac{1}{2} \sum_{q > K} \theta(-k,p,q) \frac{|\hat{p} \times \hat{q}|^2}{k^2} (q^2 - p^2) \{[p^2 U(p) - q^2 U(q)] + k^2 [U(q) - U(p)]\},
$$

for $k \ll K$, we obtain

$$
\nu_T(k|K) = \sum_{q > K} \theta(-k,p,q) (\hat{k} \times \hat{q})^2 \frac{\hat{k} \cdot q}{q^2} (\hat{k} \cdot \nabla_q)[q^2 U(q)],
$$

for $k \ll K$, where the triple relaxation factor may be approximated as

$$
\theta(-k,p,q) \sim \theta(-k,q,q) = \int_0^\infty \exp\{-2\phi_R(q,\tau) - \phi_R(-k,\tau) - i \phi_I(-k,\tau)\} d\tau,
$$

provided that $\phi(p) \sim \phi(-q)$ for $p = k - q$ and $k \ll q \sim p$.

If we suppose $U(q) \sim U(q)$ and

$$
\phi_I(k,\tau) \sim a(k) \omega_0(k) \tau, \quad |\phi_I(k,\tau)| \ll |\phi_R(q,\tau)|,
$$

then
for \( \tau = O(\tau_R(k)) \) in Eq.(35), then

\[
\text{Im} \nu(k|k_L) = \frac{\alpha(k) \omega(k)}{8} \sum_{q > k_L} \frac{\gamma(q)}{q} \frac{\partial[q^2 U(q)]}{\partial q},
\]

where \( \tau_R(k) \) is the characteristic time scale of \( \phi_R(k, \tau) \), and

\[
\gamma(q) = \int_0^\infty \tau \exp[-\phi(q, \tau)] d\tau.
\]

Thus the imaginary part of the viscosity \( \nu_T \) may be nonzero.

CONCLUSION

The results obtained in the present paper may be summarized as follows.

I]. The DNS and the LRA agree in the following points:

1. the shifts are westward, i.e., \( \Delta \omega > 0 \) for \( k_x > 0 \),
2. the shifts are nearly proportional to \( k_x \),
3. the shifts increase with \( \beta \),
4. the shifts increase with \( E/Z \), but are independent of either amplitude under certain conditions.

II]. The above properties may be explained by a model that includes

1. oscillating random sweeping and strain of large eddies,
2. eddy-damping of small eddies.

These are represented by the \( V- \) and \( \mu- \) terms in the model (23). The LRA as well as the model suggests that the shifts may occur even if the energy spectrum is nearly isotropic.

III]. The time dependence of Eulerian correlation should not be confused with those of Eulerian response function and/or Lagrangian correlation. It is wrong to assume the fluctuation-dissipation relation for Eulerian correlation. An analysis of the nonlocal interactions suggests that eddy viscosity may be complex.

The present paper treats only cases of small \( \beta \), and the effects of high \( \beta \) and strong anisotropy are remained to be studied. The role of coherent structure, which was not taken into account in the theory, remains an open question.
REFERENCES