GEOMETRIC THERMODYNAMICS AS A TOOL FOR
ANALYSIS AND PREDICTION IN OCEANOGRAPHY

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ABSTRACT

The physical parameters that are important to oceanographers often have a stochastic nature
and can be represented as the sum of a deterministic average and a random component of zero
mean. Coastline shapes, water depth and fluid density are examples of such quantities. When
the random components are small, perturbation methods can be used to calculate their effects
on the mean flow. However, in certain cases it is the derivative of the random component which
is of importance and that can have a very large magnitude. Consequently, the ostensibly small
stochastic part may well be more influential than the smooth average component. In this paper
we present a technique for quantifying roughness that can be easily implemented for experimental
data sets and apply the method to some bathymetric examples. Moreover, to examine how
such randomness will influence ocean flows we consider the problem of predicting the dispersion
relations for topographic Rossby waves propagating in the presence of a rough ocean floor. The
random depth and its derivative act as coefficients in the equations governing topographic Rossby
waves. In this paper we analytically and numerically examine the solutions to those equations
and consider how they change as the roughness of the bottom increases.

1. INTRODUCTION

Many physical characteristics of importance to the oceanographer have a stochastic
nature and can be represented as the sum of a deterministic average and a random
component of zero mean. Quantities that come to mind include coastline shapes, water
depth and fluid density.

When the random components are small, perturbation methods can be used to
calculate their effects on the mean flow (see for example Mysak (1978)). However, in
certain cases it is the derivative of the random component which is of importance and
that can have a very large magnitude. Consequently, the influence of the ostensibly small
stochastic part may well be of the same, or even larger order, than that of the
smooth average component.

For example, it has long been recognized that variations in the sea floor topography
allows for the propagation of a class of disturbances known as topographic Rossby
waves (see for example Pedlosky (1989)). These flows are spatially extensive and have
large temporal periods. The critical coefficient in the governing equations for such waves depends on the derivative of the undisturbed water depth. This depth might well be considered random and rather small ($O(1)$ km.) when compared to the spatial extent of the waves in question ($O(100)$ km.). However, the derivative of the depth which appears in the equation can not be treated as a small term.

In past studies, the ocean floor was often treated as a plane with a slight slope and indeed mathematical analysis then predicts the existence of topographic waves. While it true that many regions of the ocean can be characterized by having a small mean slope, it is not apparent that one can ignore other variations in topography in favor of this slope.

A natural question arises then as to whether one can quantitatively characterize the roughness inherent in bathymetric data. Can one give some reasonable estimate of the “average slope” of such a data set? In recent years, fractal techniques have proved popular in similar quests by researchers in many fields. However, dimension estimates depend on infinite scaling properties that are often not physically justifiable as real surfaces are self similar only over a limited range of scales. Moreover, as a practical matter, fractal analyses are simply not formulated with discrete data sets in mind.

In this paper we take an alternative approach based on the geometric thermodynamic theory for curves and surfaces. The essential ideas are developed in the next section and are illustrated there by a number of simple examples. We merely note now that the theory allows one to compute a temperature for a curve or surface. The temperature of a straight line is zero and the value increases as the curve roughness increases. The quantity can be successfully measured even for data sets of finite resolution and efficient algorithms to accomplish this are presented in section 3.

Tools to analyze rough data sets are very useful but it is even more intriguing to apply those tools to predict how roughness influences oceanographic flows. To that end we take up the problem of computing the dispersion relations that govern linear topographic Rossby wave disturbances for an ocean of random depth. The governing equations, some properties of their solutions, and the numerical methods used to solve them are described in section 4.

To render the problem computationally tractable, we restrict attention to topographies that vary in one direction only, i.e. we consider oceans with corrugated floors. Past investigations by Thomson (1975), Odulo and Pelinovsky (1978) and others have shown that if the waves are constrained to propagate in the same direction as the bottom relief then even for simple floors with periodic ripples wave dissipation and reflection are observed. In particular, the second study showed that the characteristic damping time for Rossby waves is $T_d \sim (\Delta h/h_0)^{-2}$. Typical values for the ocean are fairly large—in the 4 months to 3 years range. In this paper we will consider waves propagating in a direction that is not parallel to the bottom topography and in this case the waves do not dissipate.

It should be pointed out from the start that while the ocean floor can be modelled stochastically, it is far removed from a white noise state. In this paper most of the
simulations were done for synthetic bottom profiles though some preliminary analysis has been carried for data sets collected in the North Atlantic and Pacific. Some of the methods we used to synthesize rough bottom profile are presented in section 5 and their geometric thermodynamic characteristics are computed.

In section 6 we present results for bathymetries of various temperatures.

2. GEOMETRIC THERMODYNAMICS

In this section we explain how one can formulate a thermodynamic theory for geometric objects and how one can use that theory to construct quantitative estimates of the "roughness" of curves. We illustrate the concepts with a number of simple examples.

\[
\begin{align*}
\omega & \quad \delta K \\
\theta & \quad x \\
\text{Figure 1} & \text{ Random straight line } \omega \text{ intersecting curve } \Gamma \text{ at three points. The convex hull of } \Gamma \text{ is the set of points that lie inside the dashed boundary line.}
\end{align*}
\]

The fundamental quantity we measure for a curve is the average number of intersections it has with randomly chosen straight line segments. In general, the rougher the curve, the larger this number will be. To formalize the idea let \( \Gamma \) be a rectifiable curve in the plane and let \( \Omega(\Gamma) \) be the set of all straight lines intersecting \( \Gamma \). Directly measuring the number of intersections between a random element \( \omega \in \Omega(\Gamma) \) and \( \Gamma \) is a computationally intensive task but Blaschke (1936) has shown that if one picks \( \omega \) at random, with the natural (and as it turns out unique) distribution \( m \) that is invariant under rigid motions of the plane, then the average number of intersections between the line \( \omega \) and the curve \( \Gamma \) is given by

\[
\frac{2|\Gamma|}{|\partial K|}, \tag{1}
\]

where the convex hull of \( \Gamma \), \( K \) has boundary \( \partial K \) and we use \(|\cdot|\) to denote the length of a curve. A detailed definition of the term convex hull is given in the next section but its intuitive meaning should be clear from figure (1).
An easy derivation of Blaschke’s formula for smooth (piecewise differentiable) curves can be found in Santalo (1976) and depends on the formula

\[
\int_{\Omega(\Gamma)} n_\Gamma(w) \, d\omega = \int_0^{\pi} ds \int_0^{\Gamma} |\sin \theta| \, d\theta = 2|\Gamma|.
\]

(2)

Here the line \( \omega \) is parameterized by its perpendicular length, \( s \), from the origin and by the angle \( \theta \) subtended by the normal to \( \omega \) with the \( x \) axis. \( n_\Gamma(\omega) \) is defined as the number of points at which the line \( \omega \) intersects \( \Gamma \) (\( n_\Gamma(\omega) = 3 \) in figure (1)). Some manipulation of this formula quickly gives

\[
\frac{1}{m(\Omega(\Gamma))} \int_{\Omega(\Gamma)} n_\Gamma(\omega) \, d\omega = \frac{2|\Gamma|}{|\partial K|},
\]

(3)

as claimed at the beginning of the section.

Steinhaus (1954) observed that while the quantity on the right hand side of (3) only makes sense for rectifiable curves, the left hand side, representing the average number of intersections with lines, makes sense for any planar set, whatever its complexity. The set need not be a curve representing a single valued function or even a curve at all. He then suggested that the average be considered as a measure for the “length” of such a set. This is the starting point of our paper and suggests a way of measuring the roughness of interfaces that can be much more general than those described by functions of one or two variables.

DuPain, Kamae and Mendes-France (1986) extended Steinhaus’s approach by applying ideas from the field of statistical mechanics. They considered the family \( M^*(\Gamma) \) of all probability measures on \( \Omega(\Gamma) \) which gave the same average number of intersections of lines \( \omega \) with \( \Gamma \) as is given by the isotropic homogeneous measure \( m \). For a curve \( \Gamma \) which has the property that for any positive integer \( k \) there exists a line \( \omega \) which intersects \( \Gamma \) exactly \( k \) times, one can associate a geometric entropy function

\[
\sigma : M^*(\Gamma) \to \mathbb{R}
\]

by defining

\[
\sigma(m) = - \sum_{k=1}^{\infty} m(\omega : |\omega \cap \Gamma| = k) \log m(\omega : |\omega \cap \Gamma| = k)
\]

(4)

where \( |\omega \cap \Gamma| \) stands for the number of intersections between \( \omega \) and \( \Gamma \).

By a straightforward application of the method of Lagrange multipliers one can then find a “Gibbs” measure \( g \in M^*(\Gamma) \) which maximizes the geometric entropy over \( M^*(\Gamma) \). It turns out that

\[
g(\omega : |\omega \cap \Gamma| = k) = Ce^{-\beta k},
\]

(5)

where

\[
e^\beta = \frac{2|\Gamma|}{2|\Gamma| - |\partial K|}.
\]

(6)
The maximum geometric entropy is then
\[
\sigma(g) = \log \left( \frac{2|\Gamma|}{|\partial K|} \right) + \frac{\beta}{e^\beta - 1},
\] (7)
and \( C^{-1} \) is the partition function with \( C = e^\beta - 1 \).

Other geometric "thermodynamic" quantities can easily be defined including the geometric temperature \( \tau = \beta^{-1} \), the geometric pressure \( \Pi = |\partial K|^{-1} \), the geometric volume \( V = |\Gamma| \), the geometric heat \( Q = (e^\beta - 1)^{-1} \) and the geometric free energy \( F = \beta^{-1} \log (e^\beta - 1) \). A particular quantity that we shall make further use of is the geometric internal energy
\[
U = \frac{2|\Gamma|}{|\partial K|} = \frac{e^\beta}{e^\beta - 1}.
\] (8)

Although the construction used above is only valid for a limited class of curves, the quantities \( \beta \) and \( \sigma \) can easily be extended to all rectifiable curves. Mann, Rains and Wojtczynski (1991) contains further details of the application of these ideas to the roughness of surfaces but in this paper we will only consider one dimensional objects.

Note that if \( \Gamma \) is itself a straight line segment then \( U = 1 \) and \( \tau = 0 \). Thus the least interesting curves, straight lines, all have zero temperatures!

To gain some familiarity with the concepts outlined above let us consider some other simple examples where \( \Gamma \) is a portion of an infinite curve with small scale roughness. It is then reasonable to replace \( |\partial K| \) by \( 2L \) where \( L \) is the distance between the end points. This is because for a periodic extension of \( \Gamma \) the convex hull is an infinite strip and the section of \( \partial K \) corresponding to \( \Gamma \) has perimeter approximately equal to \( 2L \). In a later section we will explain algorithms that can be used to precisely measure the convex hull for more complicated situations. With that approximation \( U = |\Gamma|/L \).

Example 1: Sawtooth Curves

Let \( \Gamma_N \) be the periodic sawtooth curve with period \( a \) and amplitude \( \epsilon h_0 \) shown on the left of figure (2).

Figure 2 Symmetric and asymmetric sawtooth curves
By inspection

$$|\Gamma_N| = 2N \sqrt{\left(\frac{a}{2}\right)^2 + (\epsilon h_0)^2},$$

(9)

while $|\partial K| \approx 2Na$. Hence

$$U \approx \sqrt{1 + (2\epsilon h_0 a^{-1})^2}.$$  

(10)

Each line making up the sawtooth has the same slope in absolute magnitude so the average value for the absolute slope is

$$\delta = 2\epsilon h_0 a^{-1}.$$  

(11)

Note that

$$\delta = \sqrt{U^2 - 1}.$$  

(12)

Another observation will be of some consequence later is that the profile on the right of figure (2), which is not invariant with respect to a change of direction $x \rightarrow -x$, still gives rise to the same values for $U$ and $\delta$.

**Example 2: Sinusoidal Curves**

For our next example we consider the sinusoidal profile

$$\Gamma_N(x) = h_0 \left(1 + \epsilon \sin \frac{2\pi}{a} x\right), \quad 0 \leq x \leq Na.$$  

(13)

For this curve the average value of the absolute value of the slope is easily calculated as

$$\delta = \frac{1}{Na} \int_0^{Na} \frac{2\pi \epsilon h_0}{a} \cos \frac{2\pi}{a} x \left| \frac{2\pi \epsilon h_0}{a} x \right| dx = \frac{4\epsilon h_0}{a}.$$  

(14)
The length of the curve is
\[ |\Gamma_N| = \int_0^{Na} \sqrt{1 + \left( \frac{\Gamma'_N(x)}{\sqrt{a}} \cos y \right)^2} \, dx = \frac{2Na}{\pi} \int_0^{\pi/2} \sqrt{1 + \left( \frac{2\pi \chi_0}{a} \cos y \right)^2} \, dy, \]  
(15)

while \( |\partial K| \approx 2Na \). Hence
\[ U \approx 2 \int_0^{\pi/2} \sqrt{1 + \left( \frac{2\pi \chi_0}{a} \cos y \right)^2} \, dy \approx 2 \int_0^{\pi/2} \sqrt{1 + \left( \frac{\pi \delta}{2} \cos y \right)^2} \, dy. \]  
(16)

Setting \( p = \pi \delta / 2 \) this integral can be expressed in terms of the elliptic function of second kind \( E \) in the form
\[ U = \sqrt{1 + p^2} \, E \left( \frac{\pi}{2}, \frac{p}{\sqrt{1 - p^2}} \right), \]
\[ = \sqrt{1 + p^2} \left[ 1 - \frac{1}{2} \left( \frac{p^2}{1 - p^2} \right) - \cdots - \frac{(2n - 1)!!}{2^n n!} \frac{1}{2n - 1} \left( \frac{p^2}{1 - p^2} \right)^n \right], \]
(17)

so that in the first approximation
\[ U \approx \sqrt{1 + \left( \frac{\pi \delta}{2} \right)^2}, \]  
(18)

and once again the average slope is proportional to \( \sqrt{U^2 - 1} \)
\[ \delta \approx \frac{2}{\sqrt{U^2 - 1}}. \]  
(19)

3. MEASURING THE CONVEX HULL

A domain \( D \subset \mathbb{R}^n \) is said to be convex if for every pair of points \( p_1, p_2 \in D \), the line segment \( \overline{p_1 p_2} \) is entirely contained in \( D \). Given an arbitrary set of points \( S \subset \mathbb{R}^n \), the convex hull \( \text{conv}(S) \) of \( S \) is defined to be the smallest convex domain containing \( S \). The hull of a bounded set will always be a convex polytope.

For any finite set of points on the plane it is easy to visualize the convex hull by imagining that the points are marked on a board with protruding nails. To find the hull, stretch a rubber band so that it encloses all of the points and release it. The band will be caught on the nails located at the extreme points of the set and form the polygonal boundary of the convex hull.

In order to apply Blaschke's formula (1) to general sets of points we must implement an algorithm for computing the convex hull. Several algorithms for this purpose exist for
planar sets of points and we will merely mention a couple of techniques here. The reader is referred to the text by Preparata and Shamos (1985) for further details of the theory.

The *package wrapping* technique is the simplest algorithm for extracting the subset of the points that form the convex hull. While it is not the fastest method for sets of points on the plane, it deserves attention because it is one of the few that can be generalized to deal with higher dimensional data. This is an important consideration because we will eventually want to handle large three dimensional sets of topographic data.

The method parallels how a human might draw the boundary of the convex hull. Start with some point that is guaranteed to be on that boundary, say the one with smallest $y$ coordinate. Fix one end of a horizontal line to this point and rotate it upwards until it encounters another point in the set. That point must also belong to the convex hull. Use it as a new anchor for the horizontal line and repeat the procedure. Continue in this fashion until you form a package that completely wraps around the original set of points. The package is precisely the boundary of the convex hull.

Of course, instead of sweeping horizontal lines around to see which point in the set they hit first, one actually looks at all the segments between the current anchor and the other points not yet accounted for by the convex hull boundary. The end point of the segment that subtends the smallest angle with the $x$ axis will be the next point in the hull and it will also be the new anchor. The major computational costs associated with the algorithm are the calculation of lots of angles followed by some form of sorting procedure on those angles. It can be shown that the technique takes $O(N^2)$ operations for sets with $N$ points. The constant in front of the $N^2$ is large however.

Several improvements on the basic algorithm can be made. For one thing, it is possible to cheaply eliminate many of the points before we call the convex hull routine. One way to do this is to construct an extreme quadrilateral by searching for those points in the set that have the largest and smallest $x$ and $y$ coordinates. This search can be done in $3N$ operations and will typically yield four different vertices. Points that lie inside the region defined by those vertices cannot be on the convex hull boundary. By eliminating them (another linear time process) one effectively reduces $N$, the number of points submitted to the more expensive package wrapping technique. If one happens to know something about the distribution of the set of points even better quadrilaterals can be chosen to maximize this effect.

Additional savings come from the realization that a lot of time is spent computing angles. A naive implementation might calculate $\theta = \arctan(\Delta y/\Delta x)$ but evaluating the arctangent is relatively expensive, particularly on RISC machines that do not perform the computation in hardware. In any case, the precise value of the angle is of little interest here as we are only using it as a key in the sorting process. What is required is a cheap alternative to the arctangent that preserves the ordering properties of that function. A good candidate for this purpose is $\Delta y/(|\Delta y| + |\Delta x|)$ with appropriate modifications for positive and negative values of $\Delta x$ and $\Delta y$. 
The two techniques just mentioned, eliminating "obvious" points and replacing an expensive calculation with a cheaper one, can both be used to good effect for data in any dimension. Further savings are possible for planar points. Graham (1972) suggested that one first form any simple closed polygon that contains all the points. Having found this polygon one then proceeds to eliminate from it those points that do not belong to the convex hull. The major cost of this effort is the initial construction of the closed polygon and this is done by a sorting procedure based on angles from say the lowest point in the set. The number of operations for the Graham scan is thus dominated by the sorting process which can be done in $N \log N$ operations for $N$ input points. Even more sophisticated divide and conquer algorithms are available in the literature but we have found that even for large sets of data the Graham scan technique coupled with interior elimination provides adequate efficiency.

Figure (3) shows the calculation of the convex hull boundary for 100 points which were chosen to be uniformly distributed in a square.

![Figure 3](image.png)

Figure 3. These four plots show: (a) the original set of points, (b) the set with the "interior" points removed, (c) the package wrapping algorithm in progress, (d) the completed convex hull boundary.

**Roughness Calculations for Bathymetric Data**

The topographic wave dispersion relation computations carried out in the later sections of this paper are for synthesized bottom profiles only. Indeed, at this early stage of our investigations we are primarily interested in profiles having a controllable degree
of roughness. However, it is naturally interesting to examine the degree of roughness that is present in real bathymetric data. Therefore we have analyzed some of the high quality tracklines that are present in the large database assembled by the National, Geophysical and Solar-Terrestrial Data Center/NOAA (NGSDC/NOAA 1977). The reader is referred to Dworski and Holloway (1983) for a statistical study of this data.

We present a sample calculation here. The data came from a cruise by the R/V Melville II from Adak to Tokyo in October 1973. Bathymetry data at the start and the end of cruise were ignored until a reading of 5000 meters was encountered. Some 2408 depths were recorded corresponding to about one reading every 1.75 kilometers along the track. On the Mercator map the trackline is approximately a straight line starting at (178°W, 52°N) near Adak in the North Pacific and proceeding south west to (143°E, 35°N) west of Tokyo. Figure (4) shows the total depth profile.

![Bathymetric data from the cruise of the R/V Melville II from Adak to Tokyo in October 1973.](image)

Looking at figure (4) it is clear that the data are rougher in some sections than in others. In the following table we present some thermodynamic characteristics for the curve as a whole and then separately for four 1000 kilometer sections along its length. We note that the thermodynamic statistics are all perfectly well defined for sections of the curve—in the future we expect to make use of this trait to focus our computational energies on those parts of the boundaries that are likely to provide the greatest challenge for flow simulations. The fact that the theory is well posed for even the crudest of data sets makes it a useful diagnostic for adaptive computations.
| Section            | Number of data points | Number eliminated by interior check | $U = 2|\Gamma|/|K|$ | Temperature $\tau$ |
|-------------------|-----------------------|-------------------------------------|----------------|------------------|
| Full track        | 2408                  | 1726                                | 1.0007         | 0.1378           |
| 1000-2000 km.     | 520                   | 457                                 | 1.0017         | 0.1566           |
| 2000-3000 km.     | 544                   | 495                                 | 1.0009         | 0.1416           |
| 3000-4000 km.     | 575                   | 295                                 | 1.0001         | 0.1033           |
| 4000-5000 km.     | 564                   | 260                                 | 1.0004         | 0.1271           |

In the table we report on the number of points that were present in the experimental data and also the number of those that were eliminated by the interior check procedure before the convex hull routine was called. On the average, some 70% of the data points were eliminated by this check and in fact the calculations could easily be carried out in near real time on a moderate workstation or personal computer. We note that the temperature of the first 1000 kilometer stretch is the largest which corresponds well with our intuitive sense that data sections that are visually "roughest" should give rise to larger temperatures.

4. LINEAR TOPOGRAPHIC ROSSBY WAVES

In this section we consider perturbations of the rest state for a rotating inviscid ocean of variable depth. The motions to be considered will be characterized as having a large horizontal extent when compared to the maximum water depth and therefore use is made of the hydrostatic approximation. The curvature of the earth is ignored and we will denote by $f$ the local vertical component of the earth's rotation vector—the Coriolis parameter which we take to be a constant.
The equations linearized about the rest state are (cf. LeBlond and Mysak(1978))

\[
\begin{align*}
    u_t - f v &= -g \eta_x, \\
    v_t + f u &= -g \eta_y, \\
    \eta_t + (hu)_x + (hv)_y &= 0,
\end{align*}
\]  

(20)

where \( u, v \) are the perturbation velocity components in the \( x, y \) directions, \( h(x, y) \) measures the undisturbed water depth and \( \eta(x, y, t) \) measures the departure of the free surface from the rest state.

These are easily reduced to the following set

\[
\begin{align*}
    \partial_t \left[ (\partial_{tt} + f^2) \eta - g \nabla \cdot (h \nabla \eta) \right] - gfJ(h, \eta) &= 0, \\
    (\partial_{tt} + f^2) u &= -g(\partial_{xt} + f \partial_y) \eta, \\
    (\partial_{tt} + f^2) v &= -g(\partial_{yt} - f \partial_x) \eta,
\end{align*}
\]  

(21)

where \( J(h, \eta) = h_x \eta_y - h_y \eta_x \).

Even in the case of a flat ocean floor when \( h(x, y) = \text{constant} \), the equations admit wave solutions. These gravity waves have relatively short periods which are \( \leq 1/f \) and are not of interest in the current study. It is convenient to eliminate them from the start and to concentrate on the longer period waves that are only seen in the presence of a non-trivial topography. A scaling analysis shows that for motions with long temporal periods (typically 50 or more days) the \( \partial_{tt} \) terms in (21) are negligible. Moreover, for the periodic boundary data under consideration, the equations for \( u \) and \( v \) decouple entirely from the \( \eta \) equation allowing us to concentrate on

\[
\partial_t \left[ f^2 \eta - g \nabla \cdot (h \nabla \eta) \right] - gfJ(h, \eta) = 0.
\]  

(22)

If \( L \) is a characteristic horizontal length and \( D \) is say the maximum undisturbed ocean depth we can introduce non-dimensional (starred) variables as follows

\[
\begin{align*}
    x &= \frac{L}{2\pi} x^*, \quad y = \frac{L}{2\pi} y^*, \quad h = Dh^*, \quad \eta = D\eta^*, \\
    t &= f^{-1} t^*, \quad u = \frac{L_f}{2\pi} u^*, \quad v = \frac{L_f}{2\pi} v^*,
\end{align*}
\]  

(23)

and arrive at the following equation for \( \eta^* \)

\[
\partial_{t^*} \left[ \rho_T^2 \eta^* - \rho_L \nabla^* \cdot (h^* \nabla^* \eta^*) \right] - J^*(h^*, \eta^*) = 0.
\]  

(24)

All derivatives are now taken with respect to the non-dimensional variables while

\[
\rho_T = \frac{\sqrt{L/2\pi} g}{f^{-1}}, \quad \rho_L = \frac{D}{L/2\pi}
\]  

(25)
are small non-dimensional time and length ratio parameters respectively. From now on we shall drop the stars and all references will be to the non-dimensional equations and variables.

It is our intent to solve (24) for random, periodic \( h(x, y) \). Note that it is the derivatives of this random function that are important in the current context. To render the problem computationally tractable we will only consider the case where \( h = h(y) \). A normal mode decomposition of the following form is then employed

\[
\eta(x, y, t) = \hat{\eta}(y) e^{i\alpha x} e^{i\sigma t}
\]

yielding the equation

\[
\sigma \left( \left( \rho_T^2 + \alpha^2 \rho_L h \right) \hat{\eta} - \rho_L (h \hat{\eta})' \right) + \alpha h' \hat{\eta} = 0.
\]

where the prime denotes the derivative with respect to \( y \).

**Properties of the Governing Equation**

Introducing new parameters

\[
\lambda = -\alpha / \rho_L \sigma, \quad \rho = \rho_T^2 / \rho_L.
\]

equation (27) becomes

\[
\frac{d}{dy} \left[ h(y) \frac{d\hat{\eta}}{dy} \right] + \left[ \lambda h'(y) - (\alpha^2 h(y) + \rho) \right] \hat{\eta} = 0.
\]

which is to be solved with periodic boundary data. In this format the equation is similar to a periodic Sturm Liouville system (see for example Birkhoff and Rota (1978)) except for the important fact that \( h'(y) \), the coefficient multiplying the eigenvalue, is not necessarily positive. Nevertheless, many of the results for Sturm-Liouville systems still apply. For example we can easily prove the following orthogonality theorem.

**Theorem:**

Eigenfunctions corresponding to different eigenvalues are orthogonal with respect to \( dh \) i.e. if \( \hat{\eta}^{(1)} \) and \( \hat{\eta}^{(2)} \) are eigenfunctions belonging to distinct eigenvalues \( \lambda^{(1)} \) and \( \lambda^{(2)} \) then

\[
\int_0^{2\pi} \hat{\eta}^{(1)}(y) \hat{\eta}^{(2)}(y) h'(y) dy = 0
\]

(30)
Proof: Define the operator $L$ by

$$L[\hat{\eta}] = \frac{d}{dy} \left[ h(y) \frac{d\hat{\eta}}{dy} \right] - (\alpha^2 h(y) + \rho)\hat{\eta}. \tag{31}$$

Then

$$L \left[ \hat{\eta}^{(i)} \right] = -\lambda^{(i)} h'(y) \hat{\eta}^{(i)} \text{ for } i = 1, 2. \tag{32}$$

It is easily verified directly that

$$\hat{\eta}^{(1)} L \left[ \hat{\eta}^{(2)} \right] - \hat{\eta}^{(2)} L \left[ \hat{\eta}^{(1)} \right] = \frac{d}{dy} \left\{ h(y) \left[ \hat{\eta}^{(1)}(y) \frac{d\hat{\eta}^{(2)}}{dy} - \hat{\eta}^{(2)}(y) \frac{d\hat{\eta}^{(1)}}{dy} \right] \right\}. \tag{33}$$

Integrating from 0 to $2\pi$ substituting for the operators on the left hand side yields

$$\left( \lambda^{(1)} - \lambda^{(2)} \right) \int_{a}^{b} h'(y) \hat{\eta}^{(1)}(y) \hat{\eta}^{(2)}(y) \, dy = h(y) \left[ \hat{\eta}^{(1)}(y) \frac{d\hat{\eta}^{(2)}}{dy} - \hat{\eta}^{(2)}(y) \frac{d\hat{\eta}^{(1)}}{dy} \right]_{y=0}^{2\pi} \tag{34}$$

which is zero due to the periodic nature of the coefficients and the eigenfunctions. Hence if the eigenfunctions are distinct we have the orthogonality result.

Other properties of interest include

- Eigenpairs come in conjugates i.e. if $(\lambda, \hat{\eta}(y))$ is an eigenpair so also is $(\lambda^*, \hat{\eta}^*(y))$.
- The eigenvalues $\lambda$ and thus the wave speeds $\sigma$ are purely imaginary. Thus the waves are not dissipative.
- The number of zeros in the eigenfunctions increases as $\sigma$ decreases.

The latter result has both physical and computational significance. As was mentioned earlier, the topographic waves of greatest importance are those with the largest wavelengths. In the $x$-direction this concern with wavelength causes us to pay particular attention to small values of the wavenumber parameter $\alpha$. By the same token we wish to characterize the eigenfunctions $\hat{\eta}(y)$ according to the number of oscillations they make in the non-dimensional interval $y \in [0, 2\pi]$ and concentrate on those that have the fewest oscillations, and thus the fewest zeros in that domain. This idea is depicted in figure (5).
Figure 5 “Long” and “short” wavelength solutions.

This ability to label the eigenfunctions is crucial in the computational setting where the bottom profile is modeled by a random process. Each different realization of the bottom topography yields a different spectrum of $\sigma$’s. It is only by labelling the $\sigma$’s by the number of zeros in matching $\tilde{\eta}$’s that we can do any sort of reasonable statistical analysis on the dispersion relations. Essentially it allows us to compare like with like from run to run.

The details of the proofs of these and other properties of a mathematical nature will be published later. We note that in particular we can deduce some asymptotic results for the Lyapunov exponent and rotation number associated with this equation when the bottom profile $h(y)$ is a piecewise linear curve such that the slope $h'(y)$ is a “telegraphic” random process—formally this is a stationary ergodic Markov process where the slope switches between two states $\{+H, -H\}$ at nodes that are exponentially distributed along the $y$ axis.

**Prior Results**

Most of the previous work on this problem was done for deterministic bottom profiles. In particular the two profiles shown in the figure below were investigated by a number of researchers and we mention a couple of relevant results from those studies now.

In the case of a small constant slope profile

$$h(y) = [1 - \epsilon y]$$

(35)
the following quantized set of dispersion relations are easily found (see for example Pedlosky)

\[
\sigma_n(\alpha) = i \frac{\epsilon}{\rho_L} \left[ \frac{\alpha}{n^2 + \alpha^2 + \rho_T^2} \right]
\]  

(36)

The corresponding topographic waves propagate along the positive \( x \)-direction

For the second case in the figure, that of a small purely sinusoidal bottom profile with period \( 2\pi/\mu \)

\[
h(y) = 1 - \epsilon \sin \mu y
\]

(37)

the governing equation is of the Hills type. For small \( \alpha \) we get periodic solutions (periods \( 2\pi/n \)). Rhines and Bretherton (1973) found that asymptotically \( (\rho_T = 0) \)

\[
\sigma_n(\alpha) = \pm i \frac{\epsilon}{\rho_L} \frac{1}{\sqrt{(n/\mu)^2 + \alpha^2}}
\]

(38)

Topographic waves propagating in both directions along the corrugations of the bottom profile. This is not surprising as the bottom slope varies periodically from positive to negative. This result can serve as a useful test of the numerics.

**Numerical Simulations**

Next we turn to numerical simulations carried out for other bottom profiles. Having expressed everything is in terms of nondimensional coordinates, we assume that the bottom topography is a periodic extension of the fundamental interval \( y \in [0, 2\pi) \) and use the Fourier series expansions

\[
h(y) = \sum_{-\infty}^{\infty} \hat{h}_k e^{iky}, \quad \eta = \sum_{-\infty}^{\infty} \hat{\eta}_k e^{iky}
\]

(39)

to reduce the differential eigen-problem (27) to the generalized algebraic eigen-problem

\[
A \hat{\eta} = i \lambda B \hat{\eta}
\]

(40)

where

\[
A_{jk} = (\alpha^2 + jk) \hat{h}_{j-k} + \rho \delta_{jk}
\]

\[
B_{jk} = (j-k) \hat{h}_{j-k}
\]

(41)

It is convenient to introduce the bottom profile by the relation

\[
h(y) = 1 - \epsilon b(y).
\]

(42)

In terms of the Fourier coefficients of this profile we have

\[
A_{jk} = (\alpha^2 + jk) \hat{b}_{j-k} - \epsilon^{-1} (\alpha^2 + jk + \rho) \delta_{jk}
\]

\[
B_{jk} = (j-k) \hat{b}_{j-k}
\]

(43)

The eigenvalues produced from equation (40) depend on all the parameters in the problem

\[
\lambda = \lambda(\alpha, \epsilon, \rho, \{ \hat{b}_j \})
\]

(44)

and also on the resolution chosen for the eigenfunction and the bottom profile. The actual values were produced using the standard \( QZ \) algorithm on the generated \( A, B \) matrices.
5. SIMULATING RANDOM BOUNDARIES

Most of the results presented in this paper are for simulated models of a rough ocean floors. There is of course an element choice in the way one simulates rough surfaces. Various methods are discussed by Ogilvy (1991). Our own choice was motivated by both practical and theoretical considerations:

- It is natural to refer vertical distances to the maximum depth of the undisturbed ocean, as was done above in the non-dimensionalization process. Consequently we want \( b_k \geq 0 \) for all \( k \). This condition is ensured by using an exponential distribution.
- There is strong evidence from experimental bathymetry data that the floor of the ocean is non-Gaussian (see for example, Dworski and Holloway (1983)). Indeed our simulated profiles are somewhat reminiscent of experimental measurements.
- From a theoretical point of view, it is desirable to work with a process for which all of the moments are finite as is the case for the exponential distribution. Although this does not play a major role here, several theoretical statistical results for moving average processes of the type described below only hold under the assumption that the moments of higher order exist (see for example, Grenauher and Rosenblatt (1956)).

The principal tool we have used to produce synthetic bottom profiles are wide-sense, discrete-"time", stationary stochastic processes where at any point \( y_k = k \Delta y \) in physical space the bottom boundary is represented by the moving average

\[
b(y_k) \equiv b_k = \sum_{j=-\infty}^{\infty} a_j V_{k-j}, \quad k = 0, 1, 2, \ldots
\]

(45)

The \( V_j \) were chosen to be independent random variables having a common exponential distribution function so that the probability that \( V_j \) is less than \( v \) is given by

\[
P(V_j \leq v) = 1 - e^{-v}.
\]

(46)

The infinite sums must be truncated for computations and in this paper we take as weights

\[
a_j = \begin{cases} 
1/W & \text{for } j = 0, 1, \ldots, W - 1, \\
0 & \text{otherwise}
\end{cases}
\]

(47)

where \( W \) is the averaging width. Thus a set of \( N_b \) points were generated according to the prescription

\[
b_k = \sum_{j=k}^{k+W-1} \frac{1}{W} V_j \quad \text{for } k = 0, 1, \ldots, N_b - 1.
\]

(48)

The Fourier transform of these values then gives the \( \hat{b}_k \) coefficients that are used to produce the \( A \) and \( B \) matrices above.
Note that in practice the quantities of interest are $\epsilon V_j$ which are also exponentially distributed, but with parameter $1/\epsilon$ so that

$$P(\epsilon V_j \leq v) = P\left(V_j \leq \frac{v}{\epsilon}\right) = 1 - e^{-\frac{v}{\epsilon}}.$$  

(49)

Then the mean and variance for the corresponding $\epsilon b_k$'s are

$$E\{\epsilon b_k\} = \epsilon, \quad \text{Var}\{\epsilon b_k\} = \epsilon^2$$  

(50)

while the correlations are given by

$$\text{Cor}(b_j, b_k) = \begin{cases} 1 - (j - k)/W & \text{if } |j - k| \leq W, \\ 0 & \text{otherwise.} \end{cases}$$  

(51)

Larger values of the parameter $\epsilon$ increase the mean value of the bottom profile and also the deviations from that mean while increasing $W$ makes points on the boundary more correlated and tends to smooth it out. This is observed in the figure (6) which shows profiles $\epsilon b(y)$ for some different values of $\epsilon$ and $W$. In each case $N_b = 256$. The number $\tau$ reported on each graph is the geometric temperature which was discussed earlier. We point out that larger values of $\tau$ are clearly associated with "wilder" boundaries.

![Figure 6](image-url)  

Some realizations of bottom profiles for different values of $\epsilon$ and $W$ with $N_b = 256$. The scale on each is identical to that shown on the lowest left graph.
6. DISPERSION RELATION RESULTS

In this section we take up the problem of numerically solving the algebraic eigenvalue problem (40). Note that the input function \( b(y) \) is real valued and thus \( \hat{b}_j = \hat{b}^*_{-j} \), where the superscript star denotes the complex conjugate. Using this, it is easy to show that the matrix \( A \) is Hermitian while the matrix \( B \) is skew Hermitian. Actually, for the results presented in this section, we also assumed that the bottom profile is symmetric, \( b(y) = b(-y) \). The matrices are then real which simplifies the numerical calculation of the eigenvalues somewhat.

The eigenvalues can be considered as functions of all the parameters in the problem, \( \lambda = \lambda(\alpha, \epsilon, \rho, \{ \hat{b}_j \}) \). The \( \hat{b}_j \) depends on the floor data \( b_k \equiv b(y_k) \) which in turn are determined by the averaging width \( W \) described in the previous section. In real long wave flows the parameter \( \rho \) is tiny and we have taken it to be zero in all our simulations. Therefore \( \lambda = \lambda(\alpha, \epsilon, W) \).

There are numerical resolution parameters to be considered also—how many Fourier modes, or equivalently how many points in physical space, are used to resolve the bottom boundary and the disturbance \( \hat{\eta}(y) \)? The point of view we have taken is that if \( N_b \) modes are used for \( b(y) \) then one should increase the number of modes used for \( \hat{\eta}(y) \) until convergence is seen. In our study several such resolution studies were performed. To capture the "longest" mode (the \( \hat{\eta}(y) \) with the fewest zeros) it was found that using an expansion with for \( \hat{\eta}(y) \) with \( N_b \) modes was always more than adequate. Typical values of \( N_b \) were 128, 256, and 512. Note that extracting the eigenvalues is \( O\left(N^3\right) \) process so going to higher resolutions is prohibitively expensive.

For each choice of the parameters \( \epsilon \) and \( W \) one can generate many realizations of a bottom topography, each of which has approximately the same thermodynamic properties. Figure (7) shows how the temperature of the bottom profile changes for twenty different realizations each for \( \epsilon = 0.025, 0.050, 0.075 \) and \( W = 4 \). In practice each trial corresponds to choosing a fresh seed for a random number generator. It is clear from the plot that increasing \( \epsilon \) guarantees an increase in \( \tau \) although there is also some variability from realization to realization.
Although for fixed values of $\epsilon$ and $W$ the curve temperature remains fairly close to some constant value there can be quite a range for the curve ordinates. This is depicted in (8) which shows the mean, and the upper and lower bounds found for $\epsilon b(y_k)$ over 20 realizations, each with $\epsilon = 0.05$, $W = 4$. Also clearly visible in this plot is the symmetry assumption mentioned earlier. That assumption will be removed in a later paper. Also note the mean profile has $\epsilon b(y) \approx \epsilon$ as we would expect.

For each individual bottom profile we look at a range of wavenumbers $\alpha$, fill the matrices $A, B$ and solve the eigenvalue problem. The eigenvalues are sorted according to their size and the largest ones are output—we already know that the corresponding eigenfunction will have the fewest zeros and thus correspond to the largest wave. We then can make a plot of the dispersion relation which shows the wave speed $\sigma$ as a function of the $x$-wavenumber $\alpha$.

Large numbers of eigenvalue problems are tackled in this process. The total number can be expressed as $N_W N_r N_{\tau} N_{\alpha}$ where the four $N$'s respectively represent the number
of averaging widths tried, the number of $\epsilon$'s used, the number of realizations generated, and the number of axial wavenumbers investigated per realization. Many of these runs are independent so if a distributed network of workstations is available they can be used with good effect to reduce the computational burden.

In figure (9) we show the mean value of the dispersion curve (for the longest wave) found for three different values of $\epsilon$. In each case $W = 4$ and runs were done for 20 realizations of the bottom topography. Also reported on the graph is the mean value of the temperature of the bottom in each of the cases. Clearly as the temperature rises so also does the mean value of the wave speed $\sigma$.

A more detailed statistical study of the variation of $\sigma$ with $\tau$ will the subject of another paper. However, it is clear that there is a correlation between the roughness parameter and the predicted wave speed of long waves.

Of course there is also some variability in the computed dispersion relations for different realizations at a fixed value of $\epsilon$. In figure (10) plots are shown of the minimum and maximum eigenvalues found across all the bottom profiles run. This is done for $\epsilon = 0.025$ and 0.050 corresponding to profiles with temperatures close to $\tau = 0.24$ and $\tau = 0.34$ respectively. The region enclosed by minimum and maximum plots on the left is clearly different from that enclosed on the right.

Figure 9 The dispersion relation for the longest wave averaged over 20 different realizations with the same $\epsilon$. $W=4$ in each case.
Figure 10 The range of values found for the eigenvalues over all the runs for two different values of $\epsilon$.

7. SUMMARY

We have presented a method for quantifying roughness of curves and surfaces that is easily implemented for experimental data sets. In contrast to fractal analyses, the technique is based on probabilistic concepts that validly apply to data sets of finite resolution. Intuition as to the meaning of the temperature of a curve was developed by means of simple examples, and efficient algorithms and implementations were discussed for more realistic data.

While there is no doubt that geometric thermodynamics is a useful identification tool it is an open question as to whether it can be used in a predictive fashion in oceanography. To that end we are currently studying the testbed problem of topographic waves in an ocean of random depth and have presented some early results in this paper for synthesized bottom profiles. It will be interesting to make use of real bathymetric data in these simulations also.

The coefficients in the governing equations are the derivatives of random functions and are therefore not necessarily small. One question that we are now investigating is whether it possible to replace a complex boundary with a much simpler one having the same “average” slope where that quantity is proportional to $U = 2|\Gamma|/|\partial K|$. Unfortunately, as was mentioned at the end of section 2, $U$ by itself is insensitive to some features of a boundary that we would not expect the flow to be insensitive to. Work on this and other points is ongoing and we will present a more detailed analysis in a future paper.

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