MEASUREMENTS OF CHAOS IN THE OCEAN

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Abstract

The theory of dissipative chaos appears to promise great insights into the behavior of natural systems like the ocean. Results based upon model simulations show the possibility that phenomena such as El Niño are chaotic. Chaotic phenomena also demonstrate that certain traditional methods are not appropriate for chaotic systems. For example a perturbation from the linear solution provides no insight into the behavior of the nonlinear system if that system is chaotic, even if the nonlinear terms are small. The existence of chaos implies an inherent limit to the predictability of a system, this is one reason why it is important to determine if a system is chaotic.

However, when one attempts to make estimates of measures of chaos (dimensions, Lyapunov exponents, etc.) from oceanographic data one is faced with the fact that the methods that quantify chaotic properties of systems from data require an enormous number of degrees of freedom for any reasonable degree of confidence. Again traditional analysis techniques can make matters worse and not better. An example of this is the use of a smoothing filter: the filter can increase the dimension of the resulting data set by as much as 1.

1 What chaos might contribute

There are several ways that ideas from chaotic dynamics may contribute to an understanding of the ocean. The primary question is whether or not any oceanic phenomena are chaotic.

If an oceanic phenomenon is chaotic, that will automatically impose inherent limits to the predictability of the system. If this is so, it is important to be able to quantify what the predictability limit is.

1.1 Are phenomena such as El Niño chaotic?

A first question to ask is whether any oceanic phenomena are actually driven by chaotic dynamics. The identification of chaos in the ocean would mean that the relatively complicated behavior that is observed could be described in terms of a system with a small number of degrees of freedom. This possibility that El Niño is chaotic has been investigated by looking at the available data (Fraedrich, 1988), and by model studies (Vallis, 1986).
Figure 1. The Southern Oscillation Index, a monthly time series of sea level pressure differences between Tahiti and Darwin, Australia (These data are scaled to standardized dimensionless units so that the series has a zero mean and a unit standard deviation.)

Figure 1 shows the Southern Oscillation Index; its irregularity is visually reminiscent of chaotic time series. This time series has fewer than 500 data points in it, which is unfortunately too few to make reliable calculations of the dimension of the underlying system. Model studies of El Niño indicate that it is possible to mimic time series such as the Southern Oscillation Index with models that are chaotic. Figure 2 shows the Vallis (1986) model. This very simple model produces an El Niño event with about the right periodicity. The system is chaotic and has a Lyapunov dimension of 2.088 (see Fig. 3).

1.2 Chaotic Lagrangian trajectories?

The irregular nature of drifter trajectories is suggestive of either turbulence or chaos, (see Fig. 4). The possibility that these trajectories are fractal has been investigated by several people (Osborne, Brown and others). The major problem with these analyses is that the data records are short (typically about 1000 points), while the methods used in chaotic analysis require one or two orders of magnitude more data for confident estimates.
Figure 2. The Vallis (1986) ENSO model. Top: west-east section of the equatorial Pacific Ocean, defining symbols used in the model. Center: model equations. Bottom: the chaotic attractor resulting from the model equations with parameters $A = 1 \text{ year}^{-1}$ and $B = 2 \text{ m}^2 \text{ s}^{-2} \text{ °C}^{-1}$.

\[
\frac{du}{dt} = B(T_E - T_W)/2\Delta x - C(u - u^*)
\]

\[
\frac{dT_W}{dt} = \frac{u}{2\Delta x}(T - T_E) - A(T_W - T^*)
\]

\[
\frac{dT_E}{dt} = \frac{u}{2\Delta x}(T_W - T) - A(T_E - T^*)
\]
Figure 3. The Lyapunov spectrum of the Vallis attractor. The panels show the convergence of a numerical estimate of the respective Lyapunov exponents as a function of time. The noted asymptotic value is the final estimate of the exponent. The time units are nondimensional and correspond to one unit being equivalent to one week. The Lyapunov dimension (calculated using the Kaplan-Yorke equation (28)) of this system is $D_A = 2.087$. 
Figure 4. Complete trajectories of RAFOs floats in the Gulf Stream. The tick marks are at daily intervals; the typical float track is 45 days long. Floats in the upper panel were deployed on the 15°C surface, the middle panel at 12°C, and the lower at 9°C.
Early calculations by Osborne et al. (1986) for a year of measurements of three surface drifters indicated a correlation dimension of about 1.4. More recent calculations on SOFAR float trajectories (Brown and Smith, 1990) are more ambiguous. Based on available observations, the current conclusion is that float trajectories are probably not chaotic. They are more likely to be controlled by turbulent processes.

1.3 Limits to predictability

If a system is chaotic, then trajectories that are nearby in phase space will diverge exponentially. Increasing the accuracy of the observations does not help, since predictability only increases linearly with the number of digits.

Another possible situation that can impose limits on predictability is the possibility that the boundary between the states of the ocean/atmosphere is fractal. As an illustration of this possibility, consider the determination of the basins of attraction (i.e., the root that is reached for a given starting point) for the problem of finding the roots of

\[ z^3 - 1 = 0 \]

for complex \( z \), by using Newton's method. Here Newton's method for this complex polynomial is the "physics" for a system which ultimately reaches one of three states. It turns out that the boundaries of the regions that reach a given root are fractal and have the remarkable property that any boundary point is a boundary between all three domains, these boundary points define a set known as a Julia set (see Fig. 5). The implication for predictability is that for measurements with a given finite error, there are some regions that are perfectly predictable and other regions where there is no predictability at all.

1.4 Perturbation expansion of chaotic models

One common technique in solving nonlinear systems is to do a perturbation expansion about some small parameter. We demonstrate here that a conventional perturbation expansion may not be helpful when the system is chaotic because the perturbation solution has no chaotic behavior.

Look at the Lorenz system of equations,

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \quad (1) \\
\dot{y} &= -y - xz + rz \quad (2) \\
\dot{z} &= xy - bz. \quad (3)
\end{align*}
\]
Figure 5. The basins of attraction for the roots of $z = (-1 - i\sqrt{3}) / 2$, for complex $z$, using Newton's method. The starting points that converge to the root $z = 1$ are colored grey, points that converge to the root $z = (-1 + i\sqrt{3}) / 2$, and points that converge to the root $z = (-1 - i\sqrt{3}) / 2$ are black. (The center of the figure is at the origin.)

The parameter $r$ is the ratio of the Rayleigh number divided by the critical Rayleigh number. The parameter $\sigma$ is the Prandtl number. The third parameter $b$ is related to the horizontal wave number of the system. Typical values, $r = 28$, $\sigma = 10$, $b = 8/3$, dimension = 2.05. A common second set of values, $r = 45.92$, $\sigma = 16$, $b = 4$, dimension = 2.067.

The interesting cases are where the Rayleigh number ratio $r$ is large, which suggests that we could expand the system of equations around a parameter proportional to the reciprocal of $r$ (which would be small).
If we define

$$\varepsilon = r^{-\frac{1}{2}}$$  \hspace{1cm} (4)

and let

$$x' = \varepsilon x$$
$$y' = \varepsilon^2 \sigma y$$
$$z' = \sigma (\varepsilon^2 z - 1)$$
$$t' = t / \varepsilon,$$  \hspace{1cm} (5)

Then equations (1) - (3) become (after dropping the primes)

$$\dot{x} = y - \varepsilon \sigma x$$  \hspace{1cm} (6)
$$\dot{y} = -xz - \varepsilon y$$  \hspace{1cm} (7)
$$\dot{z} = xy - \varepsilon b(z + \sigma).$$  \hspace{1cm} (8)

Now consider the expansion of \(x, y, \) and \(z\) in terms of the parameter \(\varepsilon\)

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$
$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$
$$z = z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \cdots.$$  \hspace{1cm} (9)

Introducing (9) in (6) - (8), gives the order 0 equations,

$$\dot{x}_0 = y_0$$  \hspace{1cm} (10)
$$\dot{y}_0 = -x_0 z_0$$  \hspace{1cm} (11)
$$\dot{z}_0 = x_0 y_0$$  \hspace{1cm} (12)

and at order \(\varepsilon\), the system,

$$\dot{x}_1 = y_1 - \sigma x_0$$  \hspace{1cm} (13)
$$\dot{y}_1 = -x_0 z_1 - x_1 z_0 - y_0$$  \hspace{1cm} (14)
$$\dot{z}_1 = x_1 y_1 + x_0 y_0 - b(z_0 + \sigma).$$  \hspace{1cm} (15)

The interdependence of the order 0 equations can removed with some algebraic manipulation. Use (11) and (12) to eliminate \(x_0\)

$$y_0 \dot{y}_0 + z_0 \dot{z}_0 = 0$$
$$\frac{d}{dt} \left[ y_0^2 + z_0^2 \right] = 0.$$  \hspace{1cm} (16)

Integrating this,
\[ y_0^2 = -z_0^2 + [y_0^2(0) + z_0^2(0)] \]  
(17)

where the terms in the brackets of equation (17) are the initial values of \( y_0 \) and \( z_0 \). We will define this (constant) term as

\[ C_{23} = y_0^2(0) + z_0^2(0) \]  
(18)

If we go back to (10) and (12) to eliminate \( y_0 \),

\[ \dot{z}_0 = x_0 \dot{x}_0 = \frac{1}{2} x_0^2 \]
\[ \frac{d}{dt} \left( z_0 - \frac{1}{2} x_0^2 \right) = 0. \]  
(19)

Integrating this gives

\[ z_0 = \frac{1}{2} x_0^2 - \left[ \frac{1}{2} x_0(0) - z_0 \right]; \]  
(20)

here the terms in the brackets of equation (20) are the initial values of \( x_0 \) and \( z_0 \). We will define this term as

\[ C_{13} = \frac{1}{2} x_0(0) - z_0(0). \]  
(21)

Using (17) in (10) gives

\[ (\dot{x}_0)^2 = -z_0^2 + C_{23}. \]  
(22)

Now using (19)

\[ (\dot{x}_0)^2 = -\frac{1}{4} x_0^4 + C_{13} x_0^2 + [C_{23} - C_{13}^2]. \]  
(23)

Given the solution to this equation, \( y_0 \) can be solved for by using (10). Then given \( x_0 \) and \( y_0 \), \( z_0 \) can be solved for by using (12). An equation for \( z_0 \) can also be derived by using manipulations similar to that used in deriving (23) (using equations (17) and (20) in (12) to eliminate \( x_0 \) and \( y_0 \)), to give

\[ (\dot{z}_0)^2 = -2z_0^3 - 2C_{13} z_0^2 + 2C_{23} z_0 + 2C_{13} C_{23}. \]  
(24)

Equation (22) can be solved analytically, its solution is a **Jacobi elliptic function**

\[ x_0 = A \text{sn}(t|m). \]

The other components can also be determined,

\[ y_0 = A \text{cn}(t|m) \text{dn}(t|m) \]
\[ z_0 = \text{dn}^2(t|m) m \text{cn}^2(t|m) \]  
(26)
(where $m = -A^2 / 4$) so the system is well behaved (not chaotic). The first order equations (13) - (15) are linear so they cannot possibly lead to chaotic solutions. Thus we have shown that while the actual system can be chaotic, the perturbation solutions may not be

Figure 6 shows a phase portrait of the solution of the full system (6) - (8), the zero order system (10) - (12), and the perturbation solution to first order (i.e., with $\varepsilon$ times the solution of (13) - (15) added to the zero order solution). The perturbation solution tracks the nonlinear solution for a short while then it moves off in a different direction. The perturbation solution also rapidly grows to order one, so that the expansion (9) is valid for only a limited time.

![Phase portrait of solution](image)

Figure 6. Solutions to the Lorenz equations for large Rayleigh number ratio (equations (6)-(8)). The solid line is the solution to the nonlinear (chaotic) system. The long dashed line is the solution to the zero order perturbation experiment. The short dashed line is the perturbation solution to first order.

2. **Practical problems in estimating chaotic parameters from actual data**

Most methods developed for quantifying chaos (e.g., the Grassberger-Procaccia (1983) method) require very long data sets in order to converge with a reasonable uncertainty. Such lengthy data sets do not exist in oceanography, so methods that work with short data sets (see for example, Ellner (1988), Havstad and Ehlers (1989) or Abraham et al., 1986) must be used. Also the presence of noise (either due to measurement errors or to small scale oceanic process) complicates the calculations. In addition, the ill-considered use of filters applied to the data can make things worse, not better.
2.1 The effect of noise

Random errors in the observations of a system can complicate the estimation of the fractal dimension of a system. It has the effect of increasing the apparent dimension of the system. This is unfortunate since estimation methods have data requirements that grow exponentially with the dimension of the system.

In addition, while truly random processes ought to be infinitely dimensional, biases in commonly used dimension algorithms indicate finite dimension when presented with random data.

For colored noise, the correlations between nearby points can produce effects that mimic a finite correlation dimension (Theiler, 1991). Osborne and Provenzale (1989) provide an example of this effect. Kennel and Isabelle (1992) have investigated the possibility of distinguishing colored noise effects from chaos.

2.2 The effect of filtering the observations

One traditional way to deal with noise in the observations is to apply a filter in an attempt to remove the frequencies that are attributed to the noise. With chaotic systems, the effect of the filter is to potentially increase the apparent dimension of the system (Badia et al., 1988).

Consider a physical system $\dot{u}(t) = -F(u)$ and an ideal lowpass filter, which can be described as a differential equation that adds to the original system:

$$\dot{z}(t) = -\eta z(t) + X(t)$$

where $z(t)$ is the filter output, and $\eta$ is the filter cutoff frequency.

With this filter present, the Lyapunov exponents of the system consist of the original Lyapunov exponents plus a new one $\lambda_f = -\eta$ resulting from the filter.

From the Kaplan-Yorke equation for the Lyapunov dimension

$$D_L = j + \sum_{k=1}^{j} \frac{\lambda_k}{|\lambda_{j+1}|}$$

The dimension, $D_L$ of the system will remain unchanged as long as

$$\eta \geq |\lambda_{j+1}|$$
Otherwise the dimension of the filtered system will **increase**. In fact, depending upon the size of \( \eta \) compared to the other Lyapunov exponents, \( D_f \) can increase as much as 1. There has been some work (e.g., Chennaoui et al., 1990) to remove this effect of filtering on chaotic time series by (at least in a topological sense) unfiltering the time series.

3 **Methods from systems dynamics**

Even if it turns out that the ocean is not chaotic, certain techniques developed for analyzing chaotic systems may prove useful. For many of these methods the fact that a nonlinear system is a chaotic one is not essential for the analysis method to be usable.

3.1 **Mutual information and dynamical connections**

The mutual information of two (discrete scalar) messages \( S \) and \( Q \) is (Fraser and Swinney, 1986)

\[
I(Q,S) = H(Q) + H(S) - H(Q,S)
\]

where

\[
H(Q) = -\sum_i P_q(q_i) \log(P_q(q_i))
\]

(and similarly for \( S \))

\[
H(Q,S) = -\sum_i P_{qs}(q_i,s_j) \log(P_{qs}(q_i,s_j)).
\]

When \( Q \) is a set of time delayed measurements \( (q(t + \tau)) \) then the **first** minimum of \( I \) as a function of \( \tau \) is a good choice of the lag time in the higher dimensional reconstruction (Fraser, 1986).

By taking the appropriate limits, we can calculate an **information dimension** from the mutual information

\[
D_I = D_q + D_s - D_{qs}.
\]

\( D_I \) is nonnegative and has the following properties:

- \( D_I = D_q \) when \( q = s \)
- \( D_I < D_q \) when \( q \) and \( s \) are time shifted versions of each other or when they are dynamically related (and have the same dimension)
- \( D_I = 0 \) when \( q \) and \( s \) are dynamically independent.
Hence we have a test for synchronizability and for dynamical relatedness. This could be exploited to determine if two different time series (say one from a model and another from actual observations) are controlled by the same dynamics or not.

3.2 A theorem on dynamic dependence

The dimensions and entropies of series can also be used to determine whether two systems are dynamically independent or not. The following theorem is due to Hartt and Kahn, 1990.

Consider a composite system

$$x^{-}_{ab} = \begin{bmatrix} x^{-}_{a,i} \\ x^{-}_{b,j} \end{bmatrix}$$

where

$$x^{-}_{a,i} = [Y(t_i), Y(t_i + \tau), \ldots, Y(t_i + (d - f - 1) \tau)]^T$$

$$x^{-}_{b,j} = [Z(t_j), Z(t_j + \tau), \ldots, Z(t_j + (f - 1) \tau)]^T$$

with combined dimension of $d, ((d - f) + (f))$. We investigate the effects of the dependence and independence of these subsystems. The supremum norm gives

$$\rho_{ab}(i, j) = \text{dist}(x^{-}_{ab,i}, x^{-}_{ab,j}) = \max_{k=0, d-1} |X_{ab,i,k} - X_{ab,j,k}|$$

where $k$ represents a component. It follows that

$$\rho_{ab}(i, j) = \max(\rho_{a}(i, j), \rho_{b}(i, j))$$

The simplest way to obtain dimensions and entropies is to evaluate the generalized correlation integrals

$$C_q^{j}(\ell) = \left[ \frac{1}{N_r} \sum_i \left( \frac{1}{N_s} \sum_{j=1}^{\ell} \theta(\ell - \rho_{a}(i, j)) \right)^{q-1} \right]^{\frac{1}{q-1}}$$

where $N_r =$ number of reference points and $N_s =$ number of sample points in the vector time series. Then

$$\theta(\ell - p_{ab}(i, j)) = \theta(\ell - p_{a}(i, j)) \theta(\ell - p_{b}(i, j))$$

There are two important special cases:

- Identical subsystems
- Independent subsystems
3.2.1 Identical subsystem

In this case $p_a = p_b$, and $(\frac{d}{2} = d - f)$. Here, $\theta(\ell - p_a(i, j))\theta(\ell - p_b(i, j)) = \theta(\ell - p_a(i, j))$ and $C_{ab,d}^q(\ell) = C_{a,\frac{d}{2}}^q(\ell)C_{ab,d}^q(\ell) = C_{a,\frac{d}{2}}^q(\ell)$ for all $q$ and $\ell$. Asymptotically for $\ell \to 0$,

$$C_{ab,d}^q(\ell) \sim \ln \ell^\nu \exp(-d\tau K_{ab,d}^q(d, \tau))$$ (41)

and similarly for $C_{ab,d}^q(\ell)$. Then

$$\ln C_{ab,d}^q(\ell) = v_{ab} \ln \ell - d\tau K_{ab,d}^q(d, \tau)$$  
$$= v_a \ln \ell - \frac{d}{2} \tau K_{a,d}^q(\frac{d}{2})$$ (42)

from which we arrive at

$$v_{ab} = v_a$$ (43)

and

$$K_{ab,d}^q(d) = K_{a,d}^q(\frac{d}{2})$$ (44)

3.2.2 Dynamically independent subsystems

Here, $p_b(i, j)$ takes values that are independent of $p_a(i, j)$. Then $\theta(\ell - p_b(i, j))$ can be replaced by its average value over the entire series. The cases $q = 1$ and $q = 2$ are especially important. In both of these cases it follows

$$C_{ab,d}^{1,2}(\ell) = C_{a,d-f}^{1,2}(\ell)C_{b,f}^{1,2}(\ell)$$ (45)

from which asymptotically,

$$v_{ab}^{1,2} = v_a^{1,2} + v_b^{1,2}$$ (46)

and in the case $\frac{d}{2} = f = d - f$,

$$K_{ab}^{1,2} = \frac{1}{2} \left[ K_a^{1,2} + K_b^{1,2} \right]$$ (47)

Clearly, $C(i, j) < C(i)$. 
4 Summary

- Several oceanic phenomena, El Niño and drifter trajectories in particular, are suggestive of chaos. For El Niño, the presence of chaos is inconclusive. Drifter trajectories, on the other hand, are probably not chaotic.
- Limitations on the quantities of data have prevented a definitive conclusion on the existence of chaos in the ocean.
- The existence of chaos means that special care must be used when dealing with both the equations and the data.
- The properties of dynamically connected chaotic systems may be useful in identifying the dynamical system.

References


Lorenz, E.N., 1964; The problem of deducing the climate from the governing equations, *Tellus*, XVI, 1–11.


