OBSERVATIONS OF THE VERTICAL GRADIENT OF ISOPYCNAL VERTICAL DISPLACEMENT

Robert Pinkel, Jeffrey Sherman, and Steven Anderson
Marine Physical Laboratory of the Scripps Institution of Oceanography,
University of California, San Diego, La Jolla, CA 92093

ABSTRACT

The dynamics of the fine scale motion field (1-50 m vertical scales) in the sea remains the subject of conjecture (Gregg, 1987; Munk, 1981). Efforts to investigate these motions experimentally have centered on measurements of the vertical profile of temperature gradient (Gregg, 1977) and shear (Gargett et al., 1981). The vertical wavenumber spectra obtained in these efforts have served as the basis for subsequent theoretical speculation. In this work, a repeated profiling CTD is used to monitor the fine scale density field in both time and depth. Here, the vertical separation between successive isopycnal surfaces is tracked. The quantity obtained is related to the vertical derivative of vertical displacement and will be referred to as the strain. The purpose of this work is to present a simple picture of the fine scale strain field as it evolves in time as well as depth. It is hoped that the observed time evolution of the field will provide additional clues to the underlying dynamics.

When viewed in isopycnal following coordinates, the qualitative nature of the strain field depends strongly on the characteristic vertical scale over which it is estimated. The "20 m strain" field has a strongly wave-like character, dominated by inertial and semi-diurnal tidal motion. The "2 m strain" more closely resembles the classical picture of fine structure. Lenses of low density gradient fluid are separated by sheets of higher gradient water. The lenses are seen to persist from several hours to a fraction of a day. They can propagate with respect to the density field over tens of meters. The low gradient regions evolve into regions of high gradient and vice versa, as a consequence of both spatial and temporal variability in the density field. Histograms of observed isopycnal separation are log-normal, with the skewness of the distributions increasing as the mean separation is decreased. Since all scalar fields in the sea are strained by the same velocity field, fluctuations in the fine scale vertical gradients of a variety of quantities are correlated. This may be of consequence to aspects of large scale modeling.

INTRODUCTION

The early oceanographic discussions of the fine scale field were based on the assumption that the motions are small scale, perhaps non-linear, internal waves (Garrett and Munk, 1975, 1979; Munk, 1981). Relationships between the shear and strain (the vertical derivative of vertical displacement) fields were derived using internal wave scaling relationships (Munk, 1981; Desaubies and Smith, 1982). More recently, Holloway (1983), Müller (1984) and others have suggested that the small-scale component of so-called quasi geostrophic two-dimensional turbulence is indeed responsible for the variance observed in the ocean. The relationship between strain and shear for this class of motions is unclear.
Interestingly, power spectra of both shear and strain are found to have the same general form, as a function of vertical wavenumber. Spectral density is independent of vertical wavenumber, for wavenumbers less than 0.1 cpm. Between wavenumbers of 0.1 and 1 cpm the spectra have a $k_z^{-1}$ dependence (Fig. 1).

At present there are a number of arguments used to "explain" the form of the fine scale spectrum of strain and shear over the wavenumber band 0.1-1 cpm. Gargett et al. (1981) use dimensional arguments to set the spectral form and level in this region. They suggest that the rate of turbulent kinetic energy dissipation, $\varepsilon$, and the local Vaisala frequency, $N$, are the sole parameters of importance. Holloway (1983) cites studies in atmospheric dynamics, which suggest that the fine scale field is the result of the non-linear cascade of enstrophy from large to small horizontal scales. Internal waves play no role here. Finally, Fritts (this volume) presents a model for the fine scale spectrum based entirely on an internal wave breaking criterion. Atmospheric radar observations are used to support his argument.

In this work, the objectives are more modest than the resolution of the above uncertainty. Here, we attempt to add insight by extending the fine scale observations into the time as well as depth domains. Repeated profiling CTD (conductivity-temperature-depth) instruments mounted on the research platform FLIP are used to observe the time evolution of the density field in the upper 500 m of the sea. The instantaneous density profiles measured by the CTD are assumed to be displaced and strained versions of some monotonic mean density profile, described equivalently by the functions $\rho(z)$ or $\tilde{z}(\rho)$. The vertical displacement of an isopycnal, $\eta$, from its mean depth, $\bar{z}(\rho)$, is related to its instantaneous depth, $z$, by

$$\eta(t, \rho) = \eta(t, \bar{z}(\rho)) = z(t, \rho) - \bar{z}(\rho)$$

Using this notation one can express the isopycnal depth difference, $\Delta z$, and its normalized equivalent, the finite difference strain, $\gamma$, as

$$\Delta z(t, \bar{z}; \Delta \bar{z}) = z(t, \rho_1) - z(t, \rho_2)$$

and

$$\gamma(t, \bar{z}; \Delta \bar{z}) = (\eta(t, \bar{z}) - \eta(t, \bar{z}_2)) / \Delta \bar{z} = (\Delta z - \bar{z}) / \bar{z}$$

Here, $\bar{z} = (\bar{z}(\rho_1) + \bar{z}(\rho_2)) / 2$ is the mean depth of the isopycnal pair and $\Delta \bar{z} = \bar{z}(\rho_1) - \bar{z}(\rho_2)$ is the mean separation. It is these quantities, $\Delta z$ and $\gamma$, which are the subject of this investigation.

The first section of this paper describes the measurement approach. Observations of the fine scale strain field are then presented in the depth-time domain. A simple statistical view of the data is given, in which histograms of the strain are seen to be clearly non-Gaussian. A brief discussion of some implications of this non-Gaussianity concludes the work.

**EXPERIMENT DESCRIPTION**

The data are derived from sets of density profiles obtained from a profiling CTD system on the Research Platform FLIP. Approximately six thousand profiles, from the surface
Fig. 1. A schematic representation of the spectrum of shear as a function of vertical wavenumber. The strain spectrum is found to have similar form, although units and the spectral level differ and scale differently with depth. (From Gargett et al., 1981). Note that approximately two thirds of the variance associated with non-overturning events (scales greater than 1m) is found in the 0.1-1 cpm band.

to 320 m, were collected over a 12 day period in the 1983 experiment, MILDEX. Ten thousand profiles were collected from the surface to 560 m in the 1986 PATCHEX Experiment. Both operations commenced at 34°N, 127°W, about 500 km west of Point Conception, California. Water depth at the site is 4 km. In MILDEX, FLIP drifted in a counterclockwise trajectory (Pinkel, Plueddemann and Williams, 1987), covering over 100 km. In the 1986 cruise, FLIP was placed in a two-point taut moor, restricting lateral motion to an area of several hundred meters square.

The CTD's used were Seabird Instruments model SBE-9, modified for high speed operation. The modifications include the construction of an open cylindrical frame to protect and support the instrument, and the addition of a weighted (25 kg lead shot) nose piece to increase the fall rate. Also a static pressure port is interfaced to the digi-quartz pressure sensor, to reduce the magnitude of turbulent pressure fluctuations associated with the high fall rate.

In MILDEX, a single instrument profiled from the surface to 320 m. In PATCHEX, a pair of instruments covered the depth ranges 0-300 m and 260-560 m. The CTD's were cycled every three minutes, corresponding to a drop rate of 3.8 m/s. The Nyquist
frequency, 10 cph, is not sufficient to prevent aliasing of internal wave motions in the upper thermocline (80-150 m). The aliasing of low mode internal wave motions is the dominant high frequency noise in these measurements.

The instruments were sampled at 12 Hz, corresponding to a sampling distance of 32 cm. Given the high fall rate, it was not necessary to pump the conductivity cell to achieve reasonable spatial resolution. The relative phase and amplitude response of the temperature and conductivity sensors was estimated by comparison of temperature and conductivity gradient cross spectra using data from nearly isohaline regions of the water column. The processing method is discussed in greater detail in Sherman (1989). Potential temperature, salinity and potential density profiles were obtained from the response corrected CTD information. Cruise-averaged profiles were computed for both cruises. A set of reference densities, whose mean depths were separated by 1 m, was then chosen. The encounter depth of each of these reference densities was computed for each profile using linear interpolation.

OBSERVATIONS

In this section the CTD information is first used to describe the large scale isopycnal vertical displacement field. Subsequent plots display strain calculated on progressively finer depth and time scales. The objective here is to contrast qualitatively the nature of the motions at the various scales.

In Fig. 2, we see the isopycnal displacement field of the 1986 PATCHEX experiment. A set of 100 isopycnals is tracked. The mean separation between isopycnals over the 20 day record is 5 m. A low-pass filter with a two hour rectangular window is used to reduce the time variability of the data. During the period between days 4.2 and 4.5, the system was not operating.

Clearly apparent in the motion field is the semi-diurnal baroclinic tide. This has vertical wavelength long compared to the 400 m observation depth and vertical displacements as great as 40 m, crest-to-trough. Some evidence of a spring-neap fluctuation in amplitude is seen, particularly at depth.

The fine vertical scale motions of interest in this study are largely obscured by the tide and the other higher frequency constituents of the wavefield which are so coherent with depth. To view the smaller scale field, it is convenient to form plots of isopycnal depth-difference, \( \Delta z(t) \). These can be plotted against either the instantaneous mean depth of the isopycnal pair or against the cruise averaged mean depth of the pair, \( \bar{z} \). In the latter case the time evolution of the field is seen in a "semi-Lagrangian" frame, with \( \bar{z} \) effectively serving as a label for the isopycnal pair. In Fig. 3, the depth-time series of 20 m strain, \( \Delta z(t, \bar{z}; 20) \), is plotted. This is a record of the instantaneous difference in depth between an isopycnal whose mean depth is 10 m above the depth signified in the ordinate and one whose mean depth is 10 m below. The two hour smoothing nearly obscures the period on day 4 when the no data were taken. Surprisingly, the dominant signals apparent in the 20 m strain are associated with both inertial waves and semi-diurnal motions. The vertical coherence of the tidal strain signal is greatly reduced when compared to the displacement signal. This is to be expected, as the strain is effectively the derivative of displacement with respect to depth. The vertical wave number dependence of the strain spectrum is "less red" than that of displacement by the factor \( k_z^2 \). While the strain field has a lumpy and irregular aspect in general, there are
ISOPYCNAL VERTICAL DISPLACEMENT
MEAN SEPARATION 5 m

SMOOTHED 2 HOURS IN TIME
START DAY 279-5, 1986
(NOON 6 OCT., pst)

Fig. 2. The vertical displacement field observed in the 1986 experiment, PATCHEX. One hundred isopycnal surfaces, each separated by 5 m in the mean are tracked. The record starts at noon local time, 6 October and is smoothed by a two hour running mean filter. The filtering partially obscures a 6 hour period (days 4.2–4.5) when data were not collected.

...periods where both upward (day 3, ~350–450 m, diurnal frequency) and downward (days 15–20, 100–300 m, semi-diurnal frequency) phase propagation are seen.

In some sense it is surprising to see a strong inertial signal in the strain field. Linear inertial motions are not expected to produce a vertical displacement signature. Nevertheless the signal is present. In the 1983 experiment, MILDEX, power spectra of strain (Fig. 10F in Pinkel et al., 1987) also reveal weak inertial as well as tidal peaks. When contrasted with other Southern California observations (Pinkel, 1984) the near inertial fields in MILDEX and PATCHEX are relatively weak. It is likely that the near inertial signal is a ubiquitous feature of the strain field.

The dominant motions in Fig. 3 have vertical scales of tens of meters. Most of the strain variance, as described spectrally in Fig. 1, occurs at far smaller scale. We can focus on the smaller scale motions by considering the depth difference between isopycnals whose mean separation is 5 m rather than 20 m.
\[
\frac{\partial n}{\partial z} \quad \text{ISOPYCNAL 20 m STRAIN}
\]

Smoothered 2 hours in time
Start day 279-5, 1986
(Noon 6 Oct., pst)

Fig. 3. Twenty-m isopycnal strain or depth difference time series. Vertical excursions of these lines are proportional to the separation between isopycnals whose mean separation is 20 m. Data are smoothed by two hours in time. Successive strain series are offset in depth, in proportion to the mean depth of the isopycnal pair. This removes the effect of the gross vertical advection of the internal wavefield.

In Fig. 4 the logarithm of 5 m isopycnal separation is plotted over a 40 hour period. Here data from the MILDEX experiment are presented. The displacement of the profiles to the right in Fig. 4 corresponds to large isopycnal separation—a layer of depressed local Vaisala frequency. Displacement to the left corresponds to small isopycnal separation—a sheet of enhanced stability. In this display of 15 min averaged strain profiles, there is little evidence of the large scale semi-diurnal signals which are prominent in the preceding figure. Here one sees motions appear as isolated lumps, slowly propagating vertically across isopycnal surfaces. The maximum rate of phase propagation is of order 4 m/hr ~ 10^{-1} \text{ cm/s}. Note the downward phase propagation between hours 16 and 19 at a depth of 180 m. It appears that the event reverses direction beyond hour 20, propagating upward. A linear internal wave group with characteristic vertical wavenumber 0.1 cpm and intrinsic frequency midway between the inertial and Vaisala frequencies would have a characteristic horizontal phase speed of
Fig. 4. Five-m strain: the logarithm of instantaneous separation between isopycnals whose mean separation is 5 m. Depth differences are averaged over 15 min. prior to forming these profiles. Data are plotted as a function of isopycnal mean depth. Excursions to the right correspond to large isopycnal separations, or layers. Excursions to the left correspond to small isopycnal separation. The signal consists predominately of isolated events which persist for a fraction of a day, and can propagate significantly with respect to the density field.

\[ c = \frac{\omega}{k_H} - \frac{N}{k_z} = 1 \text{ cm/s} \]

for \( N = 3.6 \text{ cph} \).

Here, \( c \) is the phase speed, \( N \) is the local Vaisala frequency and \( k_H \), \( k_z \) are horizontal and vertical wavenumbers, respectively. As typical horizontal advection speeds are larger than 1 cm/s, it is likely that spatial as well as temporal variation is being seen.

In Fig. 5, the logarithm of 2 m isopycnal separation is presented. Profiles are presented every 3 minutes for a single day of the PATCHEX experiment. As in Fig. 4, layers or lenses of nearly constant density appear as fine, nearly horizontal lines. Cross isopycnal propagation of these lenses is often seen (0200 at 300 m or 0800 at 320 m, for example). The layers persist for periods of one to five hours. The evolution of these layers takes place against a fine-scale background that is remarkably repeatable in detail, except in isolated depth-time regions of rapid and irregular change. The region above 200 m is typically quiescent, with the smallest details in the strain profiles repeatable over periods of an hour or more.
Fig. 5. The logarithm of 2-m strain. Depth difference profiles are presented every 3 min. for a single day from the PATCHEX experiment.

The regions of rapid variation (300-380 m at 0100 for example) stand out strongly in Fig. 5. The physical phenomenon which causes this effect is not clear. It was originally suspected that problems associated with estimating density from the conductivity and temperature sensors on the CTD were responsible for the periods of irregularity. However analogous plots using only temperature data show similar behavior. It remains to investigate the correlation of occurrence of these active patches with the horizontal velocity and shear fields measured using the Doppler sonars on Flip. If patches tend to occur during periods of high horizontal velocity, the advection of small horizontal scale perturbations in the density field must be considered. If they occur during periods of low velocity, instrument wake might be the cause.

STRAIN STATISTICS

The essential character of the strain field changes markedly as progressively smaller scales are considered. We have seen this in a qualitative sense above. Simple statistics are now presented to emphasize the effect of changing scale. In Fig. 6, the isopycnal depth difference variance is plotted as a function of mean vertical separation, Δz. The square of the mean separation is removed from the variance estimates prior to plotting, in order to emphasize the contribution of the fluctuating components of the signal. Patterns from four depth zones are presented. At mean separations greater than 10 m, the variance is seen to increase roughly linearly with mean separation. This is consistent with a strain spectrum which is essentially white in vertical wavenumber (Appendix) at vertical wavenumbers less than 0.1 cm. The straight lines, if extrapolated to small separation, would not indicate zero variance at zero mean separation. The actual curves indeed roll down at less than 10 m, to provide a physically sensible result. The roll down is a consequence of the finite variance of the strain spectrum (Appendix).
Fig. 6. The variance of isopycnal displacement difference $<\Delta \eta^2>$ as a function of mean separation $\overline{\Delta z}$ from PATCHEX.

At very large mean separations, (not shown in Fig. 6), where the participating isopycnals are uncorrelated, the variance should approach a constant value, equal to twice the variance of the individual isopycnal displacements.

The mean separation and variance are, in fact, not adequate to describe the statistics of the strain measurements. This is seen in Figs. 7, 8, and 9 where representative histograms of isopycnal separation are presented. The "events" recorded in these histograms represent individual encounters of the isopycnal pairs during the first 9000 profiles of the PATCHEX experiment. The events are thus equally sampled in time but not necessarily in depth. Each histogram records the distribution of nearly two million events. However, adjacent events in time are highly correlated. The profiles are repeated 20 times per hour and the correlation time of the strain field is several hours or longer. Thus, the number of independent strain events represented by these histograms is probably a factor of fifty less than the total number of events recorded. The issue of statistical stability is not of great import here, however, as the histograms have clearly converged to stable forms.

One sees at large separation (Fig. 7) near Gaussian forms. The deep data have broader histograms than the shallow, at similar mean vertical separation. This corresponds to the general increase in strain variance with depth, seen in Fig. 6. As mean separation is decreased (Figs. 8, 9), the distributions become progressively more skewed. Also, the difference between the upper and lower depth ranges decreases.
In the algorithm used to calculate isopycnal position, density inversions are removed. Hence, there is no possibility of observing "crossed isopycnals" in this data set. Isopycnals separated by $\Delta z$ meters, on average, will never be observed closer together than $-\Delta z$ meters. Thus the distributions are forced into a truncated form at negative separation. Also, the occurrence of closely spaced isopycnals implies the existence of large density gradients. The finite resolution of the CTD significantly smooths the high gradient regions, artificially separating the isopycnals. As a consequence of this effect, a precutoff increase in probability is often seen just preceding the truncation at maximum negative separation. Instrument resolution has little effect on other aspects of histogram form.
Fig. 8. Probability of isopycnal separation, as in Fig. 7, except for smaller separations. The PDF of 1m separation has been truncated.

DISCUSSION

These histograms closely approximate a log-normal family of distributions. The log-normality is a reflection of the fact that isopycnals which are closer together than their mean separation tend to remain close together for a relatively long time, as they experience the same motion field. Isopycnals which are farther apart than their mean separation, experience a greater difference in advecting velocities. The relative separation between such isopycnals can be expected to change more rapidly. The sampling format of this experiment, with profiles spaced at equal increments of time, necessarily results in skewed isopycnal separation distributions.
Fig. 9. Probability of isopycnal separation, as in Figs. 7 and 8, for very small mean separations. Note the change in vertical scale relative to the previous figures. The histograms representing the upper and lower depth ranges become more nearly identical as the mean separation is decreased.
The assertion that these skewed distributions are members of a log-normal family can be easily rationalized. Consider a Gaussian vertical velocity field over the time interval \((-T, T)\) described by the Fourier series

\[
  w(z, t) = \sum_{n=-\infty}^{\infty} a_n(z) e^{i\omega_n t}
\]

where \(\omega_n = \frac{n\pi}{T}\)

The separation between neighboring isopycnals, each advected by the local velocity field, can be expressed in terms of a Taylor expansion about the mid depth, \(\bar{z}\).

\[
  \frac{\partial(\Delta z)}{\partial t} = \frac{\partial w}{\partial z} \bigg|_{z = \bar{z}} \Delta z
\]

Here \(\Delta z\) is the instantaneous separation.

Thus

\[
  \frac{\partial(\log(\Delta z/\Delta \bar{z}))}{\partial t} = \frac{\partial w}{\partial z} \bigg|_{z = \bar{z}}
\]

If the rate of strain component \(\partial w/\partial z\) has Gaussian statistics, the time derivative of log separation will be Gaussian as well. Integrating with respect to time, it is seen that isopycnal separation will be log normal provided the vertical gradient of the vertical "progressive vector"

\[
  \dot{\eta} = \int_{0}^{t} w dt'
\]

has Gaussian statistics. This indeed appears to be the case.

The log-normality of isopycnal separation has significant consequences for oceanography, in general. In particular, it is easily demonstrated that if a quantity, such as isopycnal separation has a log-normal probability distribution, then so does its inverse, \(\Delta z^{-1}\).

It is usually the case that properties in the stratified regions of the ocean vary more rapidly in the cross-isopycnal direction than in an isopycnal plane. If one describes the instantaneous vertical gradient of such properties in density co-ordinates,

\[
  \frac{\partial \Theta}{\partial z}(t, \bar{z}; \Delta z) = \frac{\Theta(p_1) - \Theta(p_2)}{\Delta z(t)}
\]

\[
  = \frac{\Theta(p_1) - \Theta(p_2)}{\rho_2} \frac{\Delta z}{\Delta z(t)}
\]
The instantaneous value of the gradient equals the product of the mean value and the normalized inverse separation. It follows that fluctuations of the vertical gradients of a variety of passive quantities such as temperature or density itself, will have log-normal distributions. Knowledge of the inherent log normality of vertical gradient fluctuations allows one to predict fluctuation statistics in a manner far simpler and more appropriate than did Desaubies and Gregg (1981). There is no need to linearize about small values of the strain.

While these gradient fluctuations may appear as "sub grid scale" effects to large scale modelers, not all aspects of the gradient modulation problem disappear with spatial averaging. In particular, since the vertical gradients of many quantities are modulated by the identical strain field, averages of the products of gradients can be expected to differ significantly from products of the averages.

We note that there is no inference that the layers result from "lateral intrusions" which are small scale manifestations of large horizontal scale inhomogeneity. The lenses can come and go as a consequence of the (perhaps Gaussian) fields of horizontal and vertical velocity. They might represent local constructive interference in the orbital motion of the internal wavefield. The lenses can appear to propagate across isopycnal surfaces, although their persistence time/distance (as with any constructive interference) will be shorter than the characteristic times of the constituent motions which interfere. The lenses will, of course, disappear as a result of lateral divergence of the underlying current field. This divergence need not be driven by gravitational collapse, as discussed by Stommel and Federov (1967) and others. To the extent that the fine structure is truly reversible, there is no implication of dissipative processes.

Finally, it must be emphasized that the log normal nature of these small scale strain observations is very much a consequence of the sampling method employed, i.e. isopycnal following. By definition, one passes through more iso-surfaces per unit distance in high gradient regions than in low. Averages taken along isopycnal surfaces will be biased toward the high gradient regions, to an extent determined by the log normality of the iso-surfaces. In terms of the simple sheet and layer point of view, the waters of the sea are found in the layers, while the isopycnal surfaces congregate in the sheets.

Consistent with this viewpoint, data obtained from horizontal tows (space-series) could also be expected to have skewed distributions of vertical isopycnal separation, at the fine scale. However isopycnal separation data obtained from Lagrangian floats should be more nearly Gaussian. As isopycnals congregate to form high gradient sheets, the floats will tend to be exhausted laterally. Only the isopycnals converge in high gradient regions; the water does not, being incompressible.

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APPENDIX

In this work, a finite difference approximation is used to describe the strain field in a semi-Lagrangian frame. The relation between a finite difference and a derivative is well known. It is reviewed here for the sake of completeness. If the isopycnal vertical displacement field, \( \eta(t, \bar{z}) \), is described over the depth interval \((0, Z)\) by a Fourier series,

\[
\eta(t, \bar{z}) = \sum_k A(t, k)e^{ik\bar{z}}
\]

then the associated strain field is given by

\[
\frac{\partial \eta}{\partial \bar{z}} = i \sum_k kAe^{ik\bar{z}}
\]

The strain variance, for a stationary, homogeneous process, is

\[
\langle (\frac{\partial \eta}{\partial \bar{z}})^2 \rangle = \sum_k k^2 \langle AA^* \rangle
\]

\[
= \sum_k k^2 S(k) \Delta k
\]

\[
= \sum_k \Gamma(k) \Delta k
\]

Here, the brackets refer to either ensemble or time averaging, \( S(k) \) is the power spectral estimate of displacement, \( \Gamma(k) \) is the power spectral estimate of strain and \( \Delta k = 2\pi/Z \) is the resolution bandwidth of the spectral estimates.

The isopycnal depth-difference time series, discussed in the text, is related to

\[
\Delta \eta(t, \bar{z}; \Delta z) = \eta(t, \bar{z}_1) - \eta(t, \bar{z}_2)
\]

\[
= \sum_k A(k, t)(e^{ik\bar{z}_1} - e^{ik\bar{z}_2})
\]

\[
= 2 \sum_k Ae^{ik\bar{z}} \sin \frac{k\Delta z}{2}
\]

where \( \bar{z} = (\bar{z}_1 + \bar{z}_2)/2 \)

and \( \Delta z = (\bar{z}_1 - \bar{z}_2) \).

The depth-difference variance is

\[
\langle \Delta \eta^2 \rangle = 4 \sum_k \langle AA^* \rangle \sin^2 \frac{k\Delta z}{2}
\]

\[
= \Delta z^2 \sum_k \Gamma(k) \sin^2 \frac{k\bar{z}}{2\pi} \Delta k
\]
where \( \text{sinc}(x) = \sin(\pi x) / (\pi x) \)

The sinc^2 function has unit value when \( k \Delta z \) is small compared to 2\( \pi \). It falls to a local zero when \( k \Delta z = 2\pi \). The sinc^2 thus serves to low-pass filter the strain spectrum. If the strain spectrum is band limited, i.e., \( \Gamma(k) = 0 \) for \( k > k_o \), then one can always determine a differencing interval \( \Delta z_o \ll 2\pi/k_o \) such that the finite difference strain is a good approximation to the total strain field. In general, as one reduces the differencing interval \( \Delta z \), the filter broadens and less of the spectral variance is filtered out. If the strain spectrum is independent of wavenumber, the observed strain variance will vary like \( \Delta z^{-1} \) as the differencing interval is altered. The isopycnic depth-difference variance, \( \langle \Delta \eta^2 \rangle = \langle (\partial \eta / \partial z)^2 \rangle \Delta z^2 \) will thus increase linearly with \( \Delta z \), as is seen in Fig. 5.

REFERENCES


