Most observed time-series can be thought of as being the sum of two parts

- The signal, \( u(t) \) that we want to measure
- Noise \( n(t) \)

The measured signal is therefore corrupted by the noise. We will call this observed signal \( c(t) \). In addition, the measuring process may not be able to record all frequencies; hence the true signal \( u(t) \) is “blurred” or smeared which can be described as a convolution of \( u(t) \) with the known instrument response \( r(t) \). The smeared signal is then

\[
s(t) = r(t) * u(t) = \int_{-\infty}^{\infty} r(t-\tau) u(\tau) d\tau \quad \leftrightarrow \quad S(\omega) = R(\omega) \cdot U(\omega)
\]

(16.1)

Add in the noise and we have

\[
c(t) = s(t) + n(t)
\]

(16.2)

In the absence of noise we can easily invert (or deconvolve) by solving \( U(\omega) = S(\omega)/R(\omega) \). However, with noise a different method must be applied. We want to find the optimal filter, \( \phi(t) \) [or \( \Phi(\omega) \)] which, when applied to the measured signal \( c(t) \) [or \( C(\omega) \)], and then deconvolved by \( r(t) \) [or \( R(\omega) \)], produces a signal \( \hat{u}(t) \) [or \( \hat{U}(\omega) \)] that is as close as possible to the uncorrupted signal \( u(t) \) [or \( U(\omega) \)]. In other words, we will estimate the true signal by

\[
\hat{U}(\omega) = \frac{C(\omega) \cdot \Phi(\omega)}{R(\omega)}
\]

(16.3)

What do we mean by being “close” to the true signal? We mean in the least square sense:

\[
\min \int_{-\infty}^{\infty} (\hat{u}(t) - u(t))^2 dt \leftrightarrow \min \int_{-\infty}^{\infty} (\hat{U}(\omega) - U(\omega))^2 d\omega
\]

(16.4)

With the transform of the noise given as \( N(\omega) \), we substitute in (16.2-3) and find

\[
\min \int_{-\infty}^{\infty} \left\{ \left( S(\omega) + N(\omega) \right) \cdot \Phi(\omega) \left[ \frac{S(\omega)}{R(\omega)} \right]^2 \right\} d\omega
\]

Carrying out the square we obtain
\[
\min \int \frac{1}{R^2(\omega)} \left\{ \left( S^2(\omega) + 2S(\omega)N(\omega) + N^2(\omega) \right) \Phi^2(\omega) - 2S(\omega) \left[ \Phi(\omega) + S(\omega) \right] \Phi(\omega) + S^2(\omega) \right\} d\omega
\]

which simplifies to

\[
\min \int \frac{1}{R^2(\omega)} \left\{ S^2(\omega) \left[ 1 - \Phi(\omega) \right]^2 + N^2(\omega) \Phi^2(\omega) \right\} d\omega
\]

(16.5)

because we assume \(S\) and \(N\) are uncorrelated, hence their product integrated over all frequencies equal zero.

Obviously, (16.5) is only minimized if the integrand is minimized for all \(\omega\). Thus we can find what the best choice for \(\Phi(\omega)\) is by setting the derivative of the integrand with respect to \(\Phi\) to zero. This gives

\[-2S^2(\omega)(1 - \Phi(\omega)) + 2N^2(\omega)\Phi(\omega) = 0\]

Since this applies to all frequencies we may solve for the optimal filter:

\[
\Phi(\omega) = \frac{S^2(\omega)}{S^2(\omega) + N^2(\omega)} = \frac{S^2(\omega)}{C^2(\omega)}
\]

(16.6)

This is the optimal filter known as the Wiener filter. Note it only involves the power of \(S\), the smeared signal, and the power of the noise \(N\). (16.6) does not contain \(U\), the true signal. This simplifies life: We can determine \(\Phi\) independently of \(R(\omega)\). However, we need to separate out \(S^2\) and \(N^2\) from \(C^2\). There is no way to do that unless we have some extra information. Luckily, a way out is often presented by looking at the spectrum \(C^2\). Often it will be clear what shape the signal and noise spectra must have. Consider the following case:

\[
\log |C|^2
\]

\[
S^2 (\text{derived})
\]

\[
N^2 (\text{estimated})
\]

\[
C^2
\]

It appears that the noise spectrum is slightly tilted. We simply extrapolate this for all frequencies and subtract to get \(S^2\). We can now form the filter \(\Phi(\omega)\) from \(S^2\) and \(N^2\). As you can see, the filter will be 1 where the noise is minimal, and drop smoothly to zero when \(N^2\) dominates. Simple, but very powerful, and not very sensitive to errors in separating \(S^2\) and \(N^2\). In fact, a crude separation by eye based on a power spectrum is usually adequate.

Hi-pass, Low-pass, and Band-pass filters
These are filters that, as their names imply, seek to keep a limited part of the frequency content in the signal. Sometimes hi-pass is called low-cut and low-pass is called hi-cut, for obvious reasons. In the frequency domain, low-pass and high-pass filters may look like the ones below:

It is straightforward to design the complementary hi-pass filter since we can simply write

\[ H(\omega) = 1 - L(\omega) \]

In the time-domain, we find the corresponding filter as the inverse transform of \( H(s) \):

\[ H(\omega) = 1 - L(\omega) \Leftrightarrow h(t) = \delta(t) - l(t) \]

Thus high-pass filtering the data \( x(t) \) in the time domain gives

\[ y(t) = x(t) * h(t) = x(t) * [\delta(t) - l(t)] = x(t) * \delta(t) - x(t) * l(t) = x(t) - x(t) * l(t) \]

This simply says that you high-pass-filter a data set by using a (complementary) low-pass filter and subtract the output from the original signal.

Band-pass filters are simply a linear computation of hi- and low-pass filters. You can either design \( B(\omega) \) directly or multiply data by \( H(\omega) \) then by \( L(\omega) \) (or 2 convolutions in the time domain) or let \( B(\omega) = L(\omega) \cdot H(\omega) \).

The use of complex notation simplifies much of the algebra associated with Fourier analysis and is therefore mathematically convenient to use. We will employ Euler’s relation for complex values to our expressions of the sine and cosine transform.

### Euler’s relation

Using a Taylor series expansion we can write

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (16.1a)
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (16.1b)
\]

and
\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \] (16.2)

Now let

\[ x = i\theta = \sqrt{-1} \theta \]

We get

\[
e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots
\]

\[
\left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \right) + i \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \right)
\]

\[ \cos \theta + i \sin \theta \]

For \( x = -i\theta \) we get

\[
e^{-i\theta} = \cos \theta - i \sin \theta
\]

\[
e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots
\]

\[
\left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \right) + i \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5} - \cdots \right) = \cos \theta + i \sin \theta
\]

\( e^{i\theta} \) is a unit vector rotated an angle \( \theta \) counterclockwise from the \( x \)-axis. For the frequencies in the Fourier series (where \( \theta = \theta(t) = \omega(t) \)), the Euler relation is

\[ e^{i\omega t} = \cos \omega t + i \sin \omega t \]

(16.4)

and the inverse relationships are:

\[ \cos \omega t = \frac{1}{2} \left( e^{i\omega t} + e^{-i\omega t} \right) \]

\[ \sin \omega t = \frac{1}{2i} \left( e^{i\omega t} - e^{-i\omega t} \right) \]

(16.5)

The complex conjugate (denoted by \( * \)) of a function \( f(x) \) is given by \( f^*(x) \) where:

\[
\begin{align*}
\hat{f}(\lambda) &= R(\lambda) + iI(\lambda) \\
\hat{f}^*(\lambda) &= R(\lambda) - iI(\lambda)
\end{align*}
\]

(16.6)

where \( R(x) \) is the real part of \( f(x) \) and \( I(x) \) is the imaginary part of \( f(x) \). The magnitude or amplitude of a complex number is given by:

\[
|\hat{f}(\lambda)| = \sqrt{\hat{f}(\lambda) \cdot \hat{f}^*(\lambda)} = \sqrt{R^2(\lambda) + I^2(\lambda)}
\]

(16.7)

The orthogonality relations now become:

\[
\sum_{\substack{n=1 \atop j \neq k}}^{n} e^{i\omega n \cdot \lambda} \cdot e^{-i\omega k \cdot \lambda} = \begin{cases} 
  n, & j = k \\
  0, & \text{otherwise}
\end{cases}
\]

(16.8)

It follows that
Any real-valued function can be represented as a complex-valued function with a zero imaginary part. Using the inverse Euler relations, the Fourier series can be rewritten in complex form as follows:

\[ y = \sum_{j=0}^{\frac{n}{2}} \left[ a_j \cos \omega_j t + b_j \sin \omega_j t \right] = \sum_{j=0}^{\frac{n}{2}} \left[ \frac{1}{2} \left( a_j + b_j \right) e^{i \omega_j t} + \frac{1}{2} \left( a_j - b_j \right) e^{-i \omega_j t} \right] \]

\[ = \sum_{j=0}^{\frac{n}{2}} \left[ \frac{1}{2} \left( a_j + b_j \right) e^{i \omega_j t} + \frac{1}{2} \left( a_j - b_j \right) e^{-i \omega_j t} \right] \]

\[ = \frac{1}{2} \sum_{j=0}^{\frac{n}{2}} \left[ (a_j - ib_j) e^{i \omega_j t} + (a_j + ib_j) e^{-i \omega_j t} \right] \]

\[ \text{(16.10)} \]

Since the second term contains \( e^{-i \omega_j t} \), we should consider the effects of negative frequencies to simplify this expression. In general

\[ (a_j - ib_j) e^{i \omega_j t} = (a_j + ib_j) e^{-i \omega_j t} \]

or

\[ J_{-j} = J_j^* \]

\[ \text{(16.12)} \]

because:

\[ a_j = \frac{2}{\pi} \sum_{t=1}^{\frac{n}{2}} y \cos \omega_j t = \frac{2}{\pi} \sum_{t=1}^{\frac{n}{2}} y \cos \frac{2\pi j}{T} t = \frac{2}{\pi} \sum_{t=1}^{\frac{n}{2}} y \cos \frac{2\pi j}{T} t = a_j \]

\[ b_j = \frac{2}{\pi} \sum_{t=1}^{\frac{n}{2}} y \sin \omega_j t = \frac{2}{\pi} \sum_{t=1}^{\frac{n}{2}} y \sin \frac{2\pi j}{T} t = \frac{2}{\pi} \sum_{t=1}^{\frac{n}{2}} y \sin \frac{2\pi j}{T} t = -b_j \]

so \( a_j \) is an even function, \( b_j \) an odd function, and (obviously)

\[ e^{i \omega_j t} = e^{-i \omega_j t} \]

So, the second term of Equation (16.10) can be dropped if we merely write the sum over \(-n/2 < j \leq n/2\):

\[ y = \frac{1}{2} \sum_{j=-\frac{n}{2}}^{\frac{n}{2}} (a_j - ib_j) e^{i \omega_j t} = \sum_{j=-\frac{n}{2}}^{\frac{n}{2}} J e^{i \omega_j t} \]

\[ \text{(16.13)} \]

(notice that the complex form has twice as many coefficients as the real form reflecting that each value of \( y \) contains a real and imaginary part - so there are twice as many \( y \) values).

Equation (16.13) represents the general complex form of the Fourier series, where
\[ J_j = \frac{1}{2} (a_j - ib_j) \]  
(16.14)

By substituting the expression (14.6) and (14.7) for \(a_j\) and \(b_j\) into the expression above (for \(J_j\)) or by multiplying Equation (16.13) by \(e^{-i\omega_j t}\), summing over \(t_i\), and using the orthogonality relations, we can derive the general complex form for \(J_k\):

\[
\sum_{j=1}^{n} \chi_j e^{-i\omega_j t} = \sum_{j=-n}^{n} J_k \sum_{j=1}^{n} e^{i\omega_j} \chi_j e^{-i\omega_j t}
\]

\[ J_k = \frac{1}{n} \sum_{j=1}^{n} \chi_j e^{-i\omega_j t}, \quad -\frac{n}{2} < k \leq \frac{n}{2} \]  
(16.15)

Equation (16.15) is the discrete Fourier transform of the series \(y_l\) and equation (16.13) is the discrete inverse Fourier transform. Since \(J_{-n/2} = J_{n/2}\) we need not compute it for each, hence \(-n/2 < j \leq n/2\); this eliminates need to define \(a_0\) and \(a_{n/2}\) as 1/2 their values.

When \(y_l\) is a real-valued series, then \(\chi_j = \chi^*_j\) because the imaginary part is zero, so:

\[
J_j = \frac{1}{n} \sum_{j=1}^{n} \chi_j e^{-i\omega_j t} = \frac{1}{n} \sum_{j=1}^{n} \chi_j \left( \cos \omega t_i - i \sin \omega t_i \right)
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \chi_j \cos \omega t_i - i \frac{1}{n} \sum_{j=1}^{n} \chi_j \sin \omega t_i
\]

and

\[
J_{-j} = J^*_j
\]

So, the transform \((J_j)\) is completely determined by the positive values of \(j\), thus, as developed earlier, they are completely determined by the \(n\) values of \(a_j\) and \(b_j\).

For continuous \(y\), then as before:

\[
f(t) = \sum_{j=-\infty}^{\infty} J_j e^{i\omega_j t}
\]

(16.16a)

\[
J_j = \frac{1}{T} \int_{-T}^{T} f(t) e^{-i\omega_j t} dt, \quad -\infty < j < \infty
\]

(16.16b)

These two equations represent special cases of the general Fourier transform and its inverse. The same expressions result if we compute a least squares fit of a Fourier series (with less than \(n\) terms) at the Fourier frequencies.