A Canonical View of Fluid Dynamics

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Abstract. The Hamiltonian descriptions of fluid dynamics are reviewed. These descriptions have been developed over the last fifty years. They only apply to perfect fluids. The Lagrangian equations of motion are canonical and can be derived from a Lagrangian density by using Hamilton’s principle, in complete analogy to Hamilton’s principle for discrete particles. The Eulerian equations of motion are non-canonical. They represent a reduced description. The symmetry that allows the reduction is a particle relabeling symmetry. The associated conservation law is the material conservation of potential vorticity. Eulerian forms of Hamilton’s principle can be derived by introducing auxiliary variables. Hamiltonian descriptions of fluid dynamics are useful in that they allow the utilization of results from other fields of physics. They also allow abridgements of fluid flows and approximations that automatically insure covariances and symmetries.

1 Introduction

Fluid dynamics studies the motion of fluids (gases or liquids) in response to internal and external forces. It is the continuous limit of discrete particle mechanics. Its basic results are given in volume 6 “Hydrodynamics” of Landau-Lifshitz’s Course of Theoretical Physics. If one looks through the index of this volume, one searches in vain for “Lagrangian,” “Hamiltonian,” or “canonical variables,” which are the basic concepts in other fields of physics. Why is this so? There are two reasons:

The first reason is that dissipative processes such as friction, heat conduction, and diffusion are important in actual fluid flows. The equations describing such dissipative motions are non-Hamiltonian in a physical sense. Of course, these equations can be made the Euler-Lagrange equations of a variational principle by introducing auxiliary variables. This approach has indeed been proposed but is ignored here, as it has been by Landau and Lifshitz.

The second reason is that the continuous limit of discrete particle physics leads to the Lagrangian description in which the fluid motion is given by the position of fluid particles as a function of a continuous label and time. Fluid dynamics, instead, uses the Eulerian description in which the field variables such as velocity and density are given as a function of space and time. The Eulerian
description is a true field description. The Eulerian equations are much simpler than their Lagrangian counterparts because they can be solved (in principle) without knowing the particle trajectories. The price to be paid for this simplification is that these Eulerian equations are non-canonical, even for perfect fluids.

Nevertheless, there are canonical Hamiltonian descriptions of perfect fluids, and they are the subject of this review paper. First we consider Lagrangian forms of Hamilton's principle for perfect fluids. These are straightforward generalizations of Hamilton's principle for discrete particles. This section also recalls some of the basic definitions and concepts of Hamiltonian dynamics. In fluid dynamics, the particle labels are a three-dimensional continuous manifold and can be subjected to infinitesimal transformations which according to Noether's theorem can give rise to conservation laws. These new conservation laws, which have no counterparts in discrete particle physics, are discussed in section 3. Section 4 gives and discusses some of the basic properties of the Euler equations, the Eulerian field equations for perfect fluids. The transformation from the Lagrangian equations to the Eulerian equations is a non-canonical transformation. For a one-component fluid it reduces the number of variables from six to five. In section 5 we identify the symmetry that allows this reduction and the conservation law that is associated with it. We then consider Eulerian forms of Hamilton's principle. These must involve auxiliary non-physical variables. These principles can actually be derived, as opposed to being guessed, from the Lagrangian forms. This derivation also gives meaning to the auxiliary variables. In the discussion section, we list some of the areas where Hamiltonian formulations of fluid dynamics have proved to be useful, rather than just being aesthetically and physically pleasing.

For further details and references, the papers by Salmon (1988) and Müller (1995) should be consulted.

2 The Lagrangian description

In the Lagrangian description, the fluid flow is described by the position \( x \) of a fluid particle as a function of its label \( s \) and time \( \tau \)

\[
\mathbf{x} = \mathbf{x}(s, \tau)
\]  

(2.1)

The label coordinates \( s = (s_1, s_2, s_3) \) are assumed to form a continuous three-dimensional manifold. Often the initial position \( x_0 := x(s, \tau = 0) \) is used as a label. The Lagrangian description (2.1) represents a time-dependent mapping from label to position space.

For a perfect fluid in a gravitational potential the equations of motion are given by

\[
\dot{\mathbf{x}} = -\frac{1}{\rho} \nabla p - \nabla \phi_g
\]

(2.2)

where \( \rho \) is the mass density, \( p \) the pressure, and \( \phi_g(\mathbf{x}, \tau) \) the given gravitational potential. The "dots" denote partial differentiation with respect to time and
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\( \nabla := (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3) \) is the gradient operator in physical space. The density can be inferred from the conservation of mass. It is convenient to choose labels such that the mass density is uniform in label space. Mass conservation then states

\[
\rho(s, \tau) = \frac{1}{K} \tag{2.3}
\]

where

\[
K := \frac{\partial(x)}{\partial(s)} \tag{2.4}
\]

is the Jacobian of the mapping from label into position space. The pressure is given by the equation of state

\[
p = p(\rho, \eta) \tag{2.5}
\]

where we have assumed a one-component fluid and chosen the specific entropy \( \eta \) as the second independent thermodynamic state variable. For a perfect fluid the specific entropy of a fluid particle does not change. Therefore

\[
\eta(s, \tau) = \eta_0(s) \tag{2.6}
\]

where \( \eta_0(s) \) is the initial distribution. The equations of motion (2.2) together with (2.3), (2.5), and (2.6) provide a complete dynamical description for the motion of a perfect one-component fluid.

The equations of motion (2.2) can be derived from a Lagrangian density using Hamilton’s variational principle. The derivation was first given by Herivel (1955) for an incompressible fluid and by Serrin (1959) and Eckart (1960) for a compressible non-homentropic flow. Hamilton’s principle states that

\[
\delta \int d\tau \int \int d^3s \mathcal{L}(x, \dot{x}, \mathcal{D}; s, \tau) = 0 \tag{2.7}
\]

for variations \( \delta x(s, \tau) \) that vanish on the boundary. The action is an extremum for actual fluid motions. The Lagrangian density \( \mathcal{L} \) is a function of position \( x(s, \tau) \), velocity \( \dot{x}(s, \tau) \), displacement gradient tensor \( \mathcal{D}(s, \tau) \) with components \( D_{ij} = \partial x_i / \partial s_j = x_{i,j} \) and the independent variables \( s \) and \( \tau \). The variational principle gives the Euler-Lagrange equations

\[
\frac{\partial}{\partial \tau} \left( \frac{\delta \mathcal{L}}{\delta \dot{x}_i} \right) + \frac{\partial}{\partial s_j} \left( \frac{\delta \mathcal{L}}{\delta x_{i,j}} \right) - \frac{\delta \mathcal{L}}{\delta x_i} = 0 \tag{2.8}
\]

where we have used index notation. The symbol “\( \delta \)” denotes differentiation when \( \mathcal{L} \) is viewed as function of \( (x, \dot{x}, \mathcal{D}; s, \tau) \) and the symbol “\( \partial \)” differentiation when \( \mathcal{L} \) is viewed as a function of \( s \) and \( \tau \).

The specific equation of motion (2.2) can be derived from the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \ddot{x} \cdot \ddot{x} - e(v, \eta) - \phi_g \tag{2.9}
\]

where \( e(v, \eta) \) is the specific internal energy and \( v = \rho^{-1} \) the specific volume. The pressure \( p \) enters through the thermodynamic relation \( p = -(\partial e / \partial v)_{\eta} \). Since
\( v = K \) and \( \eta(s, \tau) = \eta_0(s) \) it follows that \( e = e(K, \eta_0(s)) \). The position vector \( \mathbf{x} \) thus enters the potential energy term in undifferentiated form only through the gravitational potential \( \phi_g \) and in differentiated form only through the Jacobian \( K \) in \( e \).

The Lagrangian density (2.9) can be generalized or abridged in various ways. For a multi-component fluid the specific internal energy becomes

\[
e = e(v, \eta, c_2, \ldots, c_N) \tag{2.10}
\]

where \( c_i (i = 2, \ldots, N) \) are the concentrations of the additional components. For a perfect fluid these concentrations satisfy \( c_i(s, \tau) = c_i^0(s) \). Thus, the way in which the position vector enters the Lagrangian density and hence the equations of motion is not changed. For a homentropic fluid the specific entropy is constant. Its dynamic is obtained by simply dropping \( \eta \) as a variable in the specific internal energy, \( e = e(v) \). The special case of an incompressible fluid is more complicated. Incompressibility is a singular thermodynamic limit. The pressure ceases to be a thermodynamic variable. Instead, it has to be determined "dynamically" from the incompressibility constraint \( K = v_0(s) \). This constraint must be added to the Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} - \phi_g + \lambda[K - v_0(s)] \tag{2.11}
\]

where the Lagrangian multiplier \( \lambda \) must be identified with the fluid pressure.

In geophysical applications it is often useful to transform to a frame of reference that moves with constant angular velocity \( \Omega \). This transformation is simply achieved by the substitution

\[
\dot{\mathbf{x}} = \dot{\mathbf{x}}_r + \mathbf{U} \tag{2.12}
\]

where \( \dot{\mathbf{x}}_r \) is the velocity relative to the rotating frame and \( \mathbf{U} = \Omega \times \mathbf{x} \) the velocity of the frame. Two new terms appear in the Lagrangian density: \( \dot{\mathbf{x}}_r \cdot \mathbf{U} \) which gives rise to the Coriolis acceleration in the equations of motion, and the centrifugal potential \( \phi_c = -\frac{1}{2} \mathbf{U} \cdot \mathbf{U} \) which gives rise to the centrifugal acceleration. Usually, the centrifugal and gravitational potentials are combined into the geopotential \( \phi = \phi_g + \phi_c \).

Canonical Hamiltonian formulations can be obtained by introducing the momentum density

\[
\mathbf{p} = \frac{\delta \mathcal{L}}{\delta \dot{\mathbf{x}}} \tag{2.13}
\]

and forming the Hamiltonian density

\[
\mathcal{H}(\mathbf{x}, \mathbf{p}, \mathcal{D}; s, \tau) = \mathbf{p} \cdot \dot{\mathbf{x}} - \mathcal{L} \tag{2.14}
\]

Independent variations \( \delta \mathbf{x}(s, \tau) \) and \( \delta \mathbf{p}(s, \tau) \) in

\[
\delta \int d\tau \int \int d^3s (\mathbf{p} \cdot \dot{\mathbf{x}} - \mathcal{H}) = 0 \tag{2.15}
\]
then lead to Hamilton’s equations

\[ \dot{x}_i = \frac{\delta H}{\delta p_i} \]  

\[ \dot{p}_i = -\frac{\delta H}{\delta x_i} + \frac{\partial}{\partial s_j}(\frac{\delta H}{\delta x_{i,j}}) \]  

(2.16a)  
(2.16b)

These equations might not look very satisfying to the physicist’s eyes. They can be made more symmetric by adding the term \(-\frac{\partial}{\partial s_j}(\frac{\delta H}{\delta p_{i,j}})\) to the right hand side of (2.16a). The Hamiltonian density for a perfect one-component fluid is

\[ H = \frac{1}{2} p \cdot \dot{p} + e(v, \eta) + \phi_g \]  

(2.17)

Once the Hamiltonian density and Hamilton’s equations are given, one can define Poisson brackets and develop the theory further in close analogy to discrete particle mechanics. One of the few novel aspects arises from the application of Noether’s theorem given in the next section.

3 Pseudomomentum

An important theorem that governs variational principles is due to Noether (1918). It states: if the Lagrangian density is invariant under infinitesimal transformations \( \delta s, \delta \tau \) and \( \delta x \), then there exists a conservation law of the form

\[ \frac{\partial}{\partial \tau}(\mathcal{L}\delta \tau + \frac{\delta \mathcal{L}}{\delta \dot{x}_i} \Delta x_i) + \frac{\partial}{\partial s_j}(\mathcal{L}\delta s_j + \frac{\delta \mathcal{L}}{\delta x_{i,j}} \Delta x_j) = 0 \]  

(3.1a)

or

\[ \frac{\partial}{\partial \tau} \int \int \int d\mathcal{V}(\mathcal{L}\delta \tau + \frac{\delta \mathcal{L}}{\delta \dot{x}_i} \Delta x_i) = 0 \]  

(3.1b)

for suitable volumes or boundary conditions. Here

\[ \Delta x_i := \delta x_i - \dot{x}_i \delta \tau - x_{i,j} \delta s_j \]  

(3.2)

is the variation of \( \mathbf{x} \) at fixed \( s \) and \( \tau \). Noether’s theorem gives the conservation of momentum

\[ \frac{\partial}{\partial \tau}(\frac{\delta \mathcal{L}}{\delta \dot{x}_i}) + \frac{\partial}{\partial s_j}(\frac{\delta \mathcal{L}}{\delta x_{i,j}}) = 0 \]  

(3.3)

if the Lagrangian density is invariant under a uniform infinitesimal translation \( \Delta \mathbf{x} = \text{const} \) \( (\delta \tau = 0, \delta s = 0) \). It implies the conservation of energy

\[ \frac{\partial}{\partial \tau}(\mathcal{L} + \frac{\delta \mathcal{L}}{\delta \dot{x}_i} \dot{x}_i) + \frac{\partial}{\partial s_j}(\frac{\delta \mathcal{L}}{\delta x_{i,j}} \dot{x}_i) = 0 \]  

(3.4)

if the Lagrangian density is invariant under a uniform infinitesimal translation \( \delta \tau = \text{const} \) \( (\delta s = 0, \delta \mathbf{x} = 0) \).
For a continuous system the Lagrangian density can in addition be invariant under a uniform translation \( \delta s = \text{const} \) in label space \( (\delta \tau = 0, \delta \mathbf{x} = 0) \). The associated conservation law is

\[
\frac{\partial}{\partial \tau} B_i + \frac{\partial}{\partial s_j} (\mathcal{L} \delta_{ij} - \frac{\delta \mathcal{L}}{\delta x_{k,j}} x_{k,i}) = 0 \tag{3.5}
\]

where

\[
B_i := -\frac{\delta \mathcal{L}}{\delta x_j} x_{j,i} \tag{3.6}
\]

The vector \( \mathbf{B} \) is called the pseudomomentum. Its relation to the momentum can be seen by writing (3.6) as

\[
\mathbf{B} = -p_1 \hat{\nabla} x_1 - p_2 \hat{\nabla} x_2 - p_3 \hat{\nabla} x_3 \tag{3.7}
\]

where \( \hat{\nabla} = (\partial/\partial s_1, \partial/\partial s_2, \partial/\partial s_3) \) is the gradient operator in label space. The components of the momentum vector are the components of the pseudomomentum vector in a basis given by the three curvilinear vectors \( -\nabla x_i \) (\( i = 1, 2, 3 \)).

For a one-component system \( e = e(K, \eta_0) \). If one chooses \( \eta_0 \) as one of the labeling coordinates, say \( s_3 = \eta_0 \), then the Lagrangian density is independent of \( s_1 \) and \( s_2 \), and the pseudomomentum components \( B_1 \) and \( B_2 \) are conserved. For a two-component fluid \( e = e(K, \eta_0, c_0) \) and the choice \( s_3 = \eta_0 \) and \( s_2 = c_0 \) implies the conservation of \( B_1 \). For a homentropic fluid \( e = e(K) \) and all three components of the pseudomomentum vector are conserved.

The conservation of pseudomomentum has to be distinguished from the conservation of momentum. One follows from invariance under translation in label space, the other from invariance under translation in physical space. Not making this distinction has confused early applications of Hamiltonian fluid dynamics to wave-mean flow interactions. Note that the conservation of pseudomomentum has no analogue in discrete particle mechanics since Noether's theorem requires continuous infinitesimal transformations.

4 Euler's equations

In the Eulerian description of fluid dynamics the position \( \mathbf{x} \) and time \( t \) are introduced as independent variables and the fluid velocity

\[
\mathbf{u}(\mathbf{x}, t) = \dot{x}(s, \tau) \tag{4.1}
\]

as a dependent variable. To avoid ambiguities time is denoted by \( t \) in the Eulerian frame and by \( \tau \) in the Lagrangian frame. Lagrangian and Eulerian time derivatives are related by

\[
\frac{\partial}{\partial \tau} f(s, \tau) = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla) f(\mathbf{x}, t) =: \frac{D}{Dt} f(\mathbf{x}, t) \tag{4.2}
\]
where \( \frac{D}{Dt} \) is called either the advective, convective, Lagrangian or material derivative. The momentum, mass and entropy equations (2.2), (2.3) and (2.6) are thus transformed to

\[
\frac{D}{Dt} \mathbf{u} = - \frac{1}{\rho} \nabla p - \nabla \phi_a \tag{4.3a}
\]

\[
\frac{D}{Dt} \rho = - \rho \nabla \cdot \mathbf{u} \tag{4.3b}
\]

\[
\frac{D}{Dt} \eta = 0 \tag{4.3c}
\]

These equations have again to be augmented by the equation of state \( p = p(\rho, \eta) \).

The equations (4.3) are true field equations. The independent variables are position and time. These equations were established by Euler in 1755 in his treatise “Principes généraux du mouvement des fluides” and carry his name. Augmented by terms describing dissipative processes, these equations form the basis of fluid dynamics as, e.g., covered in volume 6 of Landau and Lifshitz’s “Course of Theoretical Physics.”

The field equations (4.3) are written in a form that contains the material derivative \( \frac{D}{Dt} \). Any quantity \( \psi(x, t) \) that satisfies

\[
\frac{D}{Dt} \psi = 0 \tag{4.4}
\]

is said to be materially conserved. Its value does not change along a fluid trajectory. The field equations can also be written in flux form

\[
\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = - \nabla p - \rho \nabla \phi_a \tag{4.5a}
\]

\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{4.5b}
\]

\[
\partial_t (\rho \eta) + \nabla \cdot (\rho \eta \mathbf{u}) = 0 \tag{4.5c}
\]

If a quantity \( a \) satisfies an equation of the form

\[
\partial_t (\rho a) + \nabla \cdot (\rho \mathbf{F}) = 0 \tag{4.6}
\]

then the substance \( A \) of which \( a \) is the concentration is said to be globally conserved since

\[
\frac{d}{dt} \int \int \int_V d^3 x \rho a = 0 \tag{4.7}
\]

for any volume that moves with velocity \( \mathbf{F}/\rho \). Global conservation is less stringent than, and is implied by, material conservation.

Nearly two hundred years after Euler established the field equations (4.3), Ertel (1942) derived from them his “new hydrodynamic vorticity theorem” which states that

\[
\frac{D}{Dt} \frac{(\nabla \times \mathbf{u} \cdot \nabla \psi)}{\rho} = - \frac{1}{\rho^3} \frac{\partial}{\partial(x_1, x_2, x_3)} \left( \frac{\partial(p, \rho, \psi)}{\partial(x_1, x_2, x_3)} \right) \tag{4.8}
\]
for any materially conserved tracer $\psi$. Any known vorticity theorem in fluid dynamics can be derived from (4.8) or its generalization to dissipative fluids by specifying $\psi$. Specifically, $\psi = \eta$ implies that the quantity

$$q = \frac{(\nabla \times \mathbf{u}) \cdot \nabla \eta}{\rho}$$

(4.9)

is materially conserved for perfect one-component fluids since the Jacobian on the right hand side of (4.8) vanishes because of $p = p(\rho, \eta)$. The quantity $q$ is now called Ertel’s potential vorticity and its conservation is the most powerful theorem in geophysical fluid dynamics (Müller, 1995). The material conservation of $q$ for perfect fluids, of course, implies that the substance of which $q$ is the amount per unit mass is globally conserved. It took nearly another fifty years after Ertel before Haynes and McIntyre (1987, 1990) realized that this substance is globally conserved, even if friction, heating, and other forces are added to the equations. Furthermore, this substance cannot cross surfaces of constant entropy.

The Eulerian fluid equations have become the equations of choice because they are simpler than the Lagrangian fluid equations. The Eulerian fluid equations can in principle be solved without finding the trajectory of every fluid particle. After the Eulerian velocity field has been determined, one can obtain the Lagrangian trajectories by solving the ordinary differential equations

$$\frac{d}{dt} \mathbf{x}(t) = \mathbf{u}(\mathbf{x}(t), t)$$

(4.10)

subject to the initial conditions $\mathbf{x}(t = 0) = \mathbf{x}_0$, if one so wishes. The Eulerian description is a reduction of the dynamics. This is reflected in the fact that the equations (4.3) or (4.5) or the Hamiltonian density (now in physical space)

$$\mathcal{H} = \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \rho \mathbf{e}(\rho, \eta) + \rho \phi$$

(4.11)

only contain five prognostic variables, namely $u_1, u_2, u_3, \rho$ and $\eta$. The price to be paid for this reduction is that these variables are non-canonical. Questions are: Why is the reduction possible and can a canonical Hamiltonian description of the Eulerian equations be constructed by introducing auxiliary variables?

5 Particle relabeling symmetry

A reduced dynamical description is possible if the original Lagrangian density has symmetry properties that permit the reduction. To identify this symmetry property, inspect the Lagrangian density (2.9). An infinitesimal particle relabeling transformation

$$s \rightarrow s' = s + \delta s$$

(5.1)

only affects the Jacobian $K$ and the initial specific entropy $\eta_0$ in the internal energy term. The condition $\delta K = 0$ implies $\nabla \cdot \delta \mathbf{s} = 0$ where $\nabla$ is the gradient operator in label space. This condition is satisfied by $\delta \mathbf{s} = \nabla \times \delta \mathbf{A}(s)$ where $\delta \mathbf{A}(s)$ is an arbitrary infinitesimal vector field. The condition $\delta \eta_0 = 0$ requires
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$\delta s \cdot \mathbf{\nabla} \eta_0 = 0$ which is satisfied by $\delta A = \delta a(s) \mathbf{\nabla} \eta_0$ where $\delta a(s)$ is an arbitrary infinitesimal scalar field. The internal energy and the Lagrangian density are hence invariant under the infinitesimal transformation

$$\delta s = \mathbf{\nabla} \times [\delta a(s) \mathbf{\nabla} \eta_0] \tag{5.2}$$

This invariance represents the fact that the thermodynamic state of a one-component fluid is completely determined by two scalar variables (the density $\rho$ and the specific entropy $\eta$) whereas the label $s = (s_1, s_2, s_3)$ represents a three-dimensional manifold. The one-dimensional relabeling transformation (5.2) can be applied without changing the thermodynamic state of the fluid. For a two-component fluid the thermodynamic state is completely determined by three scalar variables ($\rho$, $\eta$ and $c$, the concentration of the second component) and no such relabeling transformation exists. For a homentropic fluid $e = e(K)$ and the Lagrangian density is invariant under the infinitesimal transformation

$$\delta s = \mathbf{\nabla} \times \delta A(s) \tag{5.3}$$

According to Noether’s theorem, these invariances imply conservation laws. Substituting (5.2) into the integral form (3.1b) of Noether’s theorem one finds

$$\frac{d}{dt} \int \int \int d^3 s \mathbf{B} \cdot [\mathbf{\nabla} \times (\delta a \mathbf{\nabla} \eta_0)] = 0 \tag{5.4a}$$

or

$$\int \int \int d^3 s \frac{\partial}{\partial \tau} [(\mathbf{\nabla} \times \mathbf{B}) \cdot (\mathbf{\nabla} \eta_0)] \delta a(s) = 0 \tag{5.4b}$$

where $B_i = -\frac{\delta c}{\delta x_i}, x_{j,i}$ are the components of the pseudomomentum vector discussed in section 3. Equation (5.4) is a global conservation law. However, since $\delta a(s)$ is an arbitrary function of $s$, equation (5.4b) implies the material conservation law

$$\frac{\partial}{\partial \tau} [(\mathbf{\nabla} \times \mathbf{B}) \cdot \mathbf{\nabla} \eta_0] = 0 \tag{5.5a}$$

or

$$\frac{D}{Dt} \frac{(\mathbf{\nabla} \times \mathbf{u})}{\rho} \cdot \mathbf{\nabla} \eta = 0 \tag{5.5b}$$

when transformed from label to physical space. The conservation law associated with the particle relabeling symmetry (5.2) is the material conservation of potential vorticity. This connection has been recognized by Bretherton (1970), Ripa (1981), Salmon (1982), Henyey (1983) and others for specific circumstances. The above general derivation was first given by Salmon (1988).

For a homentropic fluid, the invariance (5.3) implies

$$\frac{\partial}{\partial \tau} \Pi = 0 \tag{5.6}$$

where

$$\Pi := \mathbf{\nabla} \times \mathbf{B} \tag{5.7}$$
Now the vector $\Pi$ is materially conserved. In physical space $\Pi$ is given by

$$\Pi = (\nabla \times B) \cdot \mathbf{v} s = -\frac{(\nabla \times \mathbf{u}) \cdot \nabla s}{\rho}$$ (5.8)

and cannot be expressed in terms of the Eulerian field variables $\mathbf{u}$ and $\rho$ since it involves the particle label $s$.

The symmetry that allows the reduction from the Lagrangian to the Eulerian description is a particle relabeling symmetry. For a one-component fluid the thermodynamic state is determined by two variables, $\rho$ and $\eta$, and one-dimensional relabeling transformations exist that do not affect the thermodynamic state. For a homentropic fluid the thermodynamic state is determined by one variable, $\rho$, and two-dimensional relabeling transformations exist. These relabeling transformations are local transformations, i.e., they depend on $s$, and therefore imply local, i.e., material conservation laws. These relabeling transformations must be distinguished from the global transformations $\delta s = \text{const}$, which lead to the global conservation of pseudomomentum, as discussed in section 3. It is also important to recognize that Noether's theorem requires infinitesimal transformations. Potential vorticity and pseudomomentum conservation do not have analogues in discrete particle mechanics. The global conservation of potential vorticity for dissipative fluids, mentioned in the last section, is not implied by any symmetry. It is simply a consequence of the fact that

$$\rho \eta = (\nabla \times \mathbf{u}) \cdot \nabla \eta = \nabla \cdot (\mathbf{u} \times \nabla \eta)$$ (5.9)

can be written as a divergence and therefore

$$\partial_t (\rho \eta) = \nabla \cdot (\partial_t \mathbf{u} \times \nabla \eta + \mathbf{u} \times \nabla \partial_t \eta)$$ (5.10)

Of course, (5.10) becomes a non-trivial conservation law only if $\partial_t \mathbf{u}$ and $\partial_t \eta$ are substituted from the equations of motion, including friction, heating and other forces.

6 Eulerian forms of Hamilton's principle

There have been various attempts to formulate Eulerian forms of Hamilton's principle dating back to Clebsch (1859). The first general versions were given by Lin (1963) and Seliger and Whitham (1968) and were arrived at in an ad hoc fashion. The first derivation is due to Salmon (1988). He did the obvious, in hindcast, and simply changed the independent variables from $(s, t)$ to $(x, t)$ and introduced the inverse mapping $s = s(x, t)$ instead of the forward mapping $x = x(s, t)$. Hamilton's principle (2.7) then becomes

$$\delta \int dt \int \int d^3 x M \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - e(v, \eta) - \phi_\eta \right) = 0$$ (6.1)
where $M := \frac{\partial \mathbf{s}}{\partial \mathbf{x}}$ is the Jacobian of the inverse mapping. The velocity $\mathbf{u}$ in (6.1) has to be determined from the “Lin” constraint
\[
\frac{D}{Dt} \mathbf{s} = 0
\]  
which gives
\[
u_m = G_{mi}^{-1} \partial_s s_i
\]  
where $G$ is the deformation tensor of the inverse mapping with components \( G_{ij} = \frac{\partial s_i}{\partial x_j} \). The specific volume $v$ and specific entropy $\eta$ in the internal energy term are given by $v = M^{-1}$ and $\eta(\mathbf{x}, t) = \eta_0(\mathbf{s})$. When $M$, $\mathbf{u}$, $v$, and $\eta$ are substituted then variations $\delta \mathbf{s}(\mathbf{x}, t)$ yield the Euler equation (4.3a).

The more common form of Seliger and Whitham (1968) is obtained by adding the “Lin” constraint (6.2) to the variational principle with Lagrangian multiplier $MB$. Then
\[
\delta \int dt \int \int d^3x \left[ \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - e(v, \eta) - \phi_g + B \cdot \frac{D \mathbf{s}}{Dt} \right] = 0
\]  
where $\mathbf{s}$, $\mathbf{u}$, and $B$ are to be varied independently. Variations $\delta \mathbf{u}$ yield the velocity representation
\[
\mathbf{u} = -B_1 \nabla s_1 - B_2 \nabla s_2 - B_3 \nabla s_3
\]  
The components of the Lagrange multiplier $B$ are the components of the velocity vector $\mathbf{u}$ in a basis given by the three curvilinear vectors $\nabla s_i$ $i = 1, 2, 3$. The vector $\mathbf{B}$ is the pseudomomentum vector as can be seen by inverting (6.5).

Substitution of (6.5) into Hamilton’s principle yields
\[
\delta \int dt \int \int d^3x [MB \cdot \frac{\partial \mathbf{s}}{\partial t} - M(\frac{1}{2} \mathbf{u} \cdot \mathbf{u} + e(v, \eta) + \phi_g)] = 0
\]  
where $\mathbf{s}$ and $MB$ are canonical conjugate variables and are to be varied independently. The velocity vector $\mathbf{u}$ is just the abbreviation (6.5). Hamilton’s equations are
\[
\frac{\partial}{\partial t} (MB_i) + \frac{\partial}{\partial x_j} (MB_i u_j - \frac{\partial M}{\partial s_i} c) = -MT \frac{\partial \eta_0}{\partial s_i}
\]  
\[
\frac{Ds}{Dt} = 0
\]  
where
\[
\Gamma := \frac{1}{2} \mathbf{u} \cdot \mathbf{u} + e + \phi_g + \frac{p}{M}
\]  
is a Bernoulli function, $p = -\left( \frac{\partial \mathbf{u}}{\partial x} \right) \eta$, the pressure and $T = -\left( \frac{\partial \mathbf{u}}{\partial x} \right) \eta$ the temperature.

Another form of the variational principle is obtained by adding mass conservation as a constraint with associated Lagrange multiplier $\phi$. Then
\[
\delta \int dt \int \int d^3x \left[ \rho \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - e(\rho, \eta) - \phi_g + C \cdot \frac{D \mathbf{s}}{Dt} \right) - \phi [\partial_x \rho + \nabla \cdot (\rho \mathbf{u})] \right] = 0
\]
where s, C, u, ρ and ϕ are to be varied independently. Variations δu yield the velocity representation

\[ \mathbf{u} = -C_1 \nabla s_1 - C_2 \nabla s_2 - C_3 \nabla s_3 - \nabla \varphi \]  

(6.10)

which shows that the Lagrange multiplier ϕ is a velocity potential. When (6.10) is substituted into (6.9) one obtains

\[ \delta \int dt \int \int d^3x [\rho C \cdot \frac{\partial \mathbf{s}}{\partial t} + \rho \frac{\partial \varphi}{\partial t} - \rho (\frac{1}{2} \mathbf{u} \cdot \mathbf{u} + e(\rho, \eta) + \phi_y)] = 0 \]  

(6.11)

where s and ρC, and ϕ and ρ are canonical conjugate pairs of variables that are to be varied independently. Hamilton’s equations are

\[ \frac{DC}{Dt} C = -T \nabla \eta_0 \]  

(6.12a)

\[ \frac{DS}{Dt} s = 0 \]  

(6.12b)

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \]  

(6.12c)

\[ \partial_t \varphi + C \cdot \partial_t \mathbf{s} = \Gamma \]  

(6.12d)

We have now arrived at eight variables. This number ought to be reducible. Indeed, if we choose \( s_3 = \eta_0 \) then (6.12a) implies \( DC_i/Dt = 0 \) for \( i = 1, 2 \). Furthermore, any velocity field can be represented in the form (6.5). If \( \varphi \) is chosen such that \( \frac{\partial \varphi}{\partial s_1} = B_1 \), then the representation (6.5) reduces to the representation (6.10) with \( C_1 = 0, C_2 = B_2 - \frac{\partial \varphi}{\partial s_2}, C_3 = B_3 - \frac{\partial \varphi}{\partial s_3} \) (Boozer, 1985). Thus, if \( C_1 = 0 \) initially it remains so. Without loss of generality \( C_1 \) can be set to zero and we arrive at the variational principle

\[ \delta \int dt \int \int d^3x [\rho C_2 \partial_t s_2 + \rho C_3 \partial_t \eta + \rho \partial_t \varphi - \rho (\frac{1}{2} \mathbf{u} \cdot \mathbf{u} + e(\rho, \eta) + \phi_y)] = 0 \]  

(6.13)

with

\[ \mathbf{u} = -C_2 \nabla s_2 - C_3 \nabla \eta - \nabla \varphi \]  

(6.14)

This is the most commonly used Eulerian form of Hamilton’s principle for one-component perfect fluids.

If \( C_2 \) is set to zero, then the velocity field is given by the representation

\[ \mathbf{u} = -C_3 \nabla \eta - \nabla \varphi \]  

(6.15)

and the variational principle describes flows for which

\[ (\nabla \times \mathbf{u}) \cdot \nabla \eta = -(\nabla C_3 \times \nabla \eta) \cdot \nabla \eta = 0, \]

(6.16)

i.e., flows for which the potential vorticity is zero.

For a homentropic fluid \( \eta = \text{const} \) and one might simply use this fact to reduce the variational principle (6.13) to

\[ \delta \int dt \int \int d^3x [\rho C_2 \partial_t s_2 + \rho \partial_t \varphi - \rho (\frac{1}{2} \mathbf{u} \cdot \mathbf{u} + e(\eta) + \phi_y)] = 0 \]  

(6.17)
with the Clebsch (1859) representation

\[ u = -C_2 \nabla s_2 - \nabla \varphi \]  \hspace{1cm} (6.18)

This supposes that any velocity field \( u \) can be represented in the form (6.18) with three scalar functions \( C_2, s_2 \) and \( \varphi \). However, this is only true locally in the neighborhood of points where the vorticity does not vanish (Lamb, 1932). If this representation is assumed to hold globally it restricts the possible velocity fields. To see this, consider the helicity

\[ H := \int \int \int_V d^3x (\nabla \times u) \cdot u \]  \hspace{1cm} (6.19)

in a volume \( V \) made up of closed vortex lines. The helicity can be unequal to zero. It describes the degree of knottedness of the vortex lines (Moffatt, 1969). The helicity is a constant of motion for perfect homentropic fluids where vortex lines are material lines. If we substitute the Clebsch representation (6.18) and assume it to be globally valid then

\[ H = \int \int \int_V d^3x \nabla \cdot [\varphi (\nabla \times u)] = \int \int d^3x \varphi \text{e} \cdot (\nabla \times u) \]  \hspace{1cm} (6.20)

which is zero for volumes made up of closed vortex lines. Thus assuming a global Clebsch representation in the variational principle restricts the solutions to solutions with zero helicity (Bretherton, 1970). For more general solutions the Clebsch coefficients \( C_2, s_2 \) and \( \varphi \) are multi-valued functions with singularities. This does not affect the equations of motion which are a local statement of the dynamics, but the variational principle which is a global statement. This example shows some of the subtleties involved in the application of Hamilton’s principle to fluid flows. If additionally \( C_2 \) is set to zero, then the velocity field is given by

\[ u = -\nabla \varphi \]  \hspace{1cm} (6.21)

and the variational principle describes irrotational flows.

For an incompressible fluid one starts from the variational principle (2.11) with Lagrange multiplier \( \lambda = \theta \) for the incompressibility constraint. Repeating the same steps as for the one-component fluid one arrives at the variational principle

\[ \delta \int dt \int \int d^3x \left[ \rho C_2 \partial_t s_2 + \rho \gamma \partial_t \rho - \rho \left( \frac{1}{2} u \cdot u + \phi \right) \right] = 0 \]  \hspace{1cm} (6.22)

where

\[ u = -C_2 \nabla s_2 - \gamma \nabla \rho - \frac{1}{\rho} \nabla \theta \]  \hspace{1cm} (6.23)
Now \((s_2, C_2)\) and \((\rho, \gamma)\) are pairs of conjugate canonical variables. Hamilton’s equations are

\[
\begin{align*}
\frac{D}{Dt} C_2 &= 0 \\
\frac{D}{Dt} s_2 &= 0 \\
\frac{D}{Dt} \rho &= 0 \\
\frac{D}{Dt} \gamma &= \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - \phi_g
\end{align*}
\] (6.24a-d)

Variation with respect to the Lagrangian multiplier \(\theta\) gives the incompressibility constraint

\[
\nabla \cdot \mathbf{u} = 0 
\] (6.25)

Since this is a kinematic constraint, no conjugate variable exists for \(\theta\). Rather, one should use the kinematic constraint to determine \(\theta\) from

\[
\nabla \cdot \left( \frac{1}{\rho} \nabla \theta \right) = -\nabla \cdot (C_2 \nabla s_2) - \nabla \cdot (\gamma \nabla \rho) 
\] (6.26)

and substitute it into Hamilton’s principle. The pressure \(p\) is given by

\[
p = \frac{D}{Dt} \theta + \rho \frac{1}{2} \mathbf{u} \cdot \mathbf{u} - \rho \phi_g 
\] (6.27)

which shows that the pressure is determined by the velocity field in incompressible fluids.

7 Discussion and conclusion

Hamiltonian descriptions of fluid dynamics have been developed during the last fifty years. They are a continuous limit of discrete classical particle mechanics. They have been formulated both for the Lagrangian and Eulerian descriptions of fluid motions. In the Eulerian description, they require auxiliary variables.

Hamiltonian methods only describe the dynamics of perfect fluids. Most problems in fluid dynamics involve dissipative processes and the methods are not applicable. Nevertheless, Hamiltonian methods offer certain advantages, especially for the theoretical fluid dynamists.

First of all, Hamiltonian fluid dynamics can make use of all the results, theorems, concepts and insights developed in classical mechanics. It thus related the material conservation of potential vorticity to a particle relabeling symmetry and gave a rational classification of all vorticity theorems. It is also currently utilizing the powerful nonlinear stability theorems of classical mechanics.

A second advantage of Hamiltonian formulations is that they easily allow abridgements of the dynamics. Thus, assuming a Clebsch representation for the velocity field in a one-component fluid restricts the solutions to those that have zero potential vorticity.

Thirdly, approximations can be made directly to the Hamiltonian density and it is easier to respect the symmetries and covariances of the physical problem.
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References