Multiple Equilibria, Periodic, and Aperiodic Solutions in a Wind-Driven, Double-Gyre, Shallow-Water Model

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ABSTRACT

A reduced-gravity shallow-water (SW) model is used to study the nonlinear behavior of western boundary currents (WBCs), with particular emphasis on multiple equilibria and low-frequency variations. When the meridionally symmetric wind stress is sufficiently strong, two steady solutions—nearly antisymmetric about the x axis—are achieved from different initial states. These results imply that 1) the inertial WBCs could overshoot either southward or northward along the western boundary, depending on their initial states; and thus, 2) the WBC separation and eastward jet could occur either north or south of the maximum wind stress line. The two equilibria arise via a perturbed pitchfork bifurcation, as the wind stress increases. A low-order, double-gyre, quasigeostrophic (QG) model is studied analytically to provide further insight into the physical nature of this bifurcation. In this model, the basic state is exactly antisymmetric when the wind stress is symmetric. The perturbations destroying the symmetry of the pitchfork bifurcation can arise, therefore, in the QG model only from the asymmetric components of the wind stress. In the SW model, the antisymmetry of the system's basic response to the symmetric forcing is destroyed already at arbitrarily low wind stress. The pitchfork bifurcation from this basic state to more complex states at high wind stress is accordingly perturbed in the absence of any forcing asymmetry.

Periodic solutions arise by Hopf bifurcation from either steady-state branch of the SW model. A purely periodic solution is studied in detail. The subtropical and subpolar recirculations, separation, and eastward jet exhibit a perfectly periodic oscillation with a period of about 2.8 years. Outside the recirculation zones, the solutions are nearly steady. The alternating anomalies of the upper-layer thickness are periodically generated adjacent to the ridge of the first and strongest downstream meander and are then propagated and advected into the two WBC zones, by Rossby waves and the recirculating currents, respectively. These anomalies periodically change the pressure gradient field near the WBCs and maintain the periodic oscillation. Aperiodic solutions are also studied by either increasing wind forcing or decreasing the viscosity.

1. Introduction and motivation

The seasonal variability of western boundary currents (WBCs) is well documented (e.g., various descriptions in Warren and Wunsch 1981; Robinson 1983). Their interannual variability has also started to attract greater attention. Yoshida (1961) already observed two persisting configurations of the Kuroshio's mean path. The alternation of the Kuroshio path between large- and small-meander states has a period of several years (Taft 1972). More recent investigations of interannual variability in the western North Pacific have been reported by Mizuno and White (1983) and Qiu and Joyce (1992). Price and Magaña (1980) combined all available hydrographic data over the North Pacific and found evidence for long baroclinic Rossby waves with a period of 2–10 years. Spillane et al. (1987) and White and Saur (1983) have suggested that these waves are primarily generated by wind fluctuations or by the propagation of tropical wind-driven disturbances into the midlatitudinal ocean interior via Kelvin waves along the eastern boundary.

Using expendable bathythermograph profiles, Roemmich and Cornuelle (1990) investigated the interannual variability of the subtropical gyre in the South Pacific Ocean. They found two persistent gyral circulations with a reversal in velocity at the sea surface south of Fiji. These two nearly steady states were maintained for up to 2 years, and the transitions from one state to the other were rather rapid.

Over the 5 years from 1980 to 1985, Auer (1987) found a statistically significant northward shift in the Gulf Stream of more than half a degree latitude but no explanation for it. From satellite imagery, Brown

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and Evans (1987) plotted a 4-year-long time series of monthly averaged Gulf Stream location. The series clearly indicated an interannual variability of the Gulf Stream’s mean position, which the authors attributed to eddy–current interaction. Hanson (1991) presented the interannual sea surface temperature variability of the western subtropical convergence zone in the North Atlantic during the Frontal Air–Sea Interaction Experiment (FASINEX). His time series exhibited variations with a period of 2 to 5 years and no obvious correlations with other global-scale variability, such as El Niño. Interpentadal variability of steric sea level and geopotential thickness of the North Atlantic Ocean was discussed by Levitus (1990).

Olson et al. (1988) explored low-frequency variability of separation latitudes of the Brazil and Malvinas Currents, using satellite and drifter data. Significant interannual bimodality in the Brazil Current was observed, with an abrupt transition between two rather steady positions of the separation during the middle of 1986.

On the other hand, theoretical studies of interannual variability in WBCs seem to be rather limited so far. Veronis (1963) was the first to analyze multiple steady states and transitions between them in the subtropical wind-driven gyre. He concluded that multiple solutions exist for a low-order truncated vorticity equation when the wind forcing is sufficiently strong. Holland and Haidvogel (1981) have obtained oscillation in short-period oscillations due to a long-period modulation of a larger-scale flow in more highly resolved quasigeostrophic (QG) models. Chao (1984) has numerically verified the alternation of the Kuroshio path between large and small meander states. He concluded that the lateral and bottom topographies play an important role in this alternation, which Charney and Flierl (1981) explained theoretically in terms of form drag instability. Gyre-scale undulations with a period of approximately 4 years, identified as first baroclinic Rossby waves, have been presented in a primitive equation, eddy-resolving numerical model (Cox 1987). Seidov (1989) used a simple box model to mimic the energy balance evolution in eddy-resolving models and provide a minimal model for the interaction between large-scale currents and synoptic eddies. He found a low-frequency, self-sustained oscillation with period doubling in the current-eddy system.

In many eddy-resolving numerical ocean experiments (e.g., Holland 1978), it has been common to force the model ocean with a symmetric wind stress pattern. This forcing yields an idealized subtropical gyre in the southern half of a rectangular domain and a subpolar gyre in the northern half, often referred to as a double-gyre system (McWilliams et al. 1990). By introducing an asymmetric double-gyre wind-forcing configuration, Moro (1990) has demonstrated Hopf bifurcation from stable steady flow to a periodically oscillating one in a rectangular basin model governed by the barotropic vorticity equation. These studies, however, give little or no information on multiple equilibria and low-frequency oscillations in the double-gyre system that arise purely from the system’s nonlinearity, rather than being due to topography or other external effects. The intrinsic nonlinearity of the double-gyre system and nonlinear interactions between the subtropical and subpolar gyres deserve further study.

The simplest useful model that still captures a wealth of dynamical WBC behavior is of the reduced-gravity type (e.g., Veronis 1973; Hurlburt and Thompson 1980; Cushman-Roisin 1986). In this model, a dynamically active upper layer lies above an infinitely deep and motionless layer; the interface can then be thought of as a model for the permanent thermocline. In other words, the first baroclinic mode of the system is modeled as if mathematically equivalent to the barotropic mode. The present paper is devoted to the nonlinear dynamics of such a simple model, for there is still much to be learned despite its simplicity. Baroclinic instability, an outcropping thermocline, and thermodynamics are likely to play an important role in WBC behavior, but lie beyond the scope of this restricted study.

Our results thus far are reported in this article as follows. In section 2, the shallow-water (SW) model equations, forcing, choice of parameters, boundary conditions, and corresponding numerical schemes are described briefly. Section 3 analyzes and interprets the numerical results, exploring the nonlinear behavior of the system including a perturbed pitchfork bifurcation and multiple Hopf bifurcations. It further discusses a possible mechanism for low-frequency fluctuation of the eastward jet and its associated confluence point. Model sensitivity to the viscous boundary condition (see appendix A) and to domain size is also tested. In section 4, a low-order, double-gyre QG model is analyzed to facilitate our understanding of the multiple equilibria and perturbed pitchfork bifurcation that appear in the numerical experiments with the full SW model; the low-order QG model is derived in appendix B. Finally, in section 5 we summarize our results, identify deficiencies in the present approach, and suggest necessary future work. This work is already proceeding, with more exhaustive numerical experimentation and a more complete analysis of the results.

2. Upper-ocean model

A reduced-gravity, SW model in transport form is used to study the nonlinear behavior of WBCs in mid-latitudes. In this model the ocean is assumed to consist of a single active layer of fluid of constant density $\rho$ and variable thickness $h(x, y, t)$, overlying a deep and motionless layer of density $\rho + \Delta \rho$; the motion of the upper layer represents the gravest baroclinic mode. All thermodynamic effects are neglected. The interface between the two fluid layers is thus a material surface,
which represents the permanent thermocline. The model domain is confined to a rectangular basin given by \(0 \leq x \leq L\) and \(0 \leq y \leq D\), with equilibrium depth \(H\). The governing equations are

\[
\frac{\partial U}{\partial t} + \nabla \cdot (vU) = -g' \frac{\partial h}{\partial x} + fV + \alpha_A A \nabla^2 U - RU + \alpha_r \frac{\tau^x}{\rho}, \quad (2.1a)
\]

\[
\frac{\partial V}{\partial t} + \nabla \cdot (vV) = -g' \frac{\partial h}{\partial y} - fU + \alpha_A A \nabla^2 V - RV, \quad (2.1b)
\]

\[
\frac{\partial h}{\partial t} = -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}; \quad (2.1c)
\]

here

\[
U i + V j = \nu = h(u i + v j) \quad (2.1d)
\]
is the upper-layer mass flux vector, while \(u\) and \(v\) represent the eastward \((x)\) and northward \((y)\) components of velocity in the upper layer, respectively, with \(i\) and \(j\) the appropriate unit vector. The Coriolis parameter is given by \(f = f_0 + \beta y, g' = (\Delta \rho / \rho) g\) is reduced gravity, and \(g\) the acceleration of gravity.

The upper layer of the ocean is driven by a zonal wind stress denoted by \(\tau^x\), which is constant in time but varies with latitude in a simple sinusoidal pattern,

\[
\tau^x = -\tau_0 \cos(2\pi y/D), \quad (2.2)
\]

where \(\tau_0\) is the amplitude. This leads to the formation of a double gyre: an anticyclonic subtropical gyre and a cyclonic subpolar one.

The system is closed by some hypothetical dissipation given here by a Rayleigh-type bottom friction scaled by \(R\) (Stommel 1948) and a Laplace-type lateral viscosity scaled by \(A\) (Munk 1950). The latter should be interpreted as subgrid-scale viscosity, since the model itself is nearly eddy resolving (Holland and Lin 1975). In general, the subgrid-scale viscosity is one or two orders of magnitude smaller than the eddy type (cf. Bryan 1963). The parameters for the experiments performed here are listed in Table 1. In Table 1, the wind stress amplitude and the lateral viscosity are similar to those in Holland and Lin (1975). We chose a smaller basin size, similar to that of Holland (1978) and Marshall (1985), to maximize resolution while preserving computational efficiency. This size does not affect substantially the system’s nonlinear dynamics, since the nonlinear term itself is almost negligible outside the boundary current regions. The choice \(H = 500\) m is dictated by the desire to have the model’s barotropic Rossby wave propagation speed similar to the baroclinic speed observed in the midlatitude ocean.

Note that the amplitude of the fluid motion and hence the strength of the system’s nonlinearity is directly proportional to the wind stress and inversely proportional to the viscosity. Thus, nonlinear advection effects can be studied by varying the two parameters \(\alpha_A\) and \(\alpha_r\) of the system (2.1); their values will be allowed to range between 0 and 1.8 (equivalent to taking values of viscosity and wind stress from normally low to abnormally high).

No normal flow is imposed at the horizontal boundaries. The tangential boundary condition is a linear combination of tangential velocity and stress:

\[
\gamma v + (1 - \gamma)L_D \frac{\partial v}{\partial x} = 0 \quad \text{at} \quad x = 0, L, \quad (2.3a)
\]

\[
\gamma u + (1 - \gamma)L_D \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0, D, \quad (2.3b)
\]

where the parameter \(\gamma\) has the limiting values of 0 for the free-slip (no stress) and 1 for the no-slip condition, and \(L_D\) is the viscous-dissipation length (see appendix A). Haidvogel et al. (1992) argued that an appropriate lateral boundary condition should lie somewhere between the no-slip and the free-slip limits; that is, \(0 < \gamma < 1\). For simplicity, the no-slip condition is applied in the basic experiments reported here. Other values of \(\gamma\) are used in the sensitivity studies of section 3f.

A convenient description of the globally oscillatory behavior of the system can be given in terms of the total energy. The total energy equation for the vertically integrated motion is

\[
\frac{\partial (PE + KE)}{\partial t} = \mathcal{L} + \mathcal{R} + \mathcal{W}. \quad (2.4)
\]

The available potential energy \(PE\), kinetic energy \(KE\), and rate of work done by the wind stress \(\mathcal{W}\), lateral friction \(\mathcal{L}\), and bottom friction \(\mathcal{R}\) are defined by

\[
PE = \left(\frac{\rho}{2}\right)\langle (h - H)^2 \rangle \quad (2.5a)
\]

\[
KE = \left(\frac{\rho}{2}\right)\langle u^2 + v^2 \rangle \quad (2.5b)
\]

\[
\mathcal{L} = \langle \rho A (u^2 \nabla U + v^2 \nabla V) \rangle \quad (2.5c)
\]

\[
\mathcal{R} = \langle -\rho R (u^2 + v^2) \rangle \quad (2.5d)
\]

\[
\mathcal{W} = \langle u \tau^x \rangle. \quad (2.5e)
\]
the angle brackets denote a horizontal average over the entire basin.

Time and space differencing are performed using a space-centered Euler backward (Matsuno) scheme and a staggered C grid (Arakawa and Lamb 1977), respectively. For the nonlinear advection term a variant of the Lilly (1965) scheme (i.e., Scheme C in Grammeltvedt 1969) is adopted. This scheme conserves the system’s total energy in the absence of time truncation errors. The more advanced and demanding Arakawa Jacobian generalized for this system, with quasi conservation of mass, kinetic energy, and enstrophy (Arakawa and Lamb 1977), is not used here because of its high computational expense. Jiang (1994) provides further numerical considerations, as well as a more complete derivation and discussion of the model (2.1).

3. Numerical results

Our aim is to explore the possible existence of multiple steady states in a symmetrically forced model and the transition between steady, periodic, and chaotic states, that is, the bifurcation tree of the model. We also wish to investigate the physical causes for low-frequency oscillation of the WBCs and their associated confluence point (C point): in particular, how advection and propagation of pressure anomalies change the pressure gradient field near the western boundary. Here, the C point is defined as the merging point of the two separated WBCs; these in turn are defined by the visually sharp discontinuity between tight h contours—inside the WBCs—and much more widely spaced ones—in the curvilinear triangle between the two separation points, southward and northward, and the C point (see Figs. 1 and 2).

The classical definition of the separation point is where the southward and northward WBCs meet on the western boundary (Veronis 1973; Agra and Nof 1993). However, careful analyses of observational data show that the separation points of the southward and northward WBCs and their C point do not usually coincide but lie close to one another. An example given by Olson et al. (1988) indicates that the separation points of the Malvinas and Brazil Currents, each defined by the intersection of a strong thermal front with the coast, have water of intermediate temperature up to 300 km wide between them.

a. Linear and nonlinear steady solutions

The model is spun up subject to a sustained wind stress [Eq. (2.2)], allowing for the formation of a double gyre, until a steady state is reached. The spinup is carried out both with and without the nonlinear term. Figure 1 shows the linear (panels a and c) and nonlinear (panels b and d) steady states of the h pattern with \( \alpha_A = 1.3 \) and \( \alpha_E = 0.95 \). The solid (dashed) lines represent h deeper (shallower) than the equilibrium depth \( H = 500 \) m.

For the linear case, the double-gyre circulation is nearly antisymmetric about the x axis. Western boundary intensification occurs due to the \( \beta \) effect, and the interior flow follows the Sverdrup approximation. No eastward jet is generated between the two gyres. The latitude of maximum eastward flow, \( y = 1000 \) km, marks the boundary between the large-scale gyres, that is, the boundary between poleward mass divergence and equatorward convergence due to Ekman drift. This reference latitude will be called the R line. Note that the C point of the WBCs is located a small distance eastward of the western boundary’s midpoint, along the R line, where the wind curl vanishes (Fig. 1c). This finding coincides with linear theory (Stommel 1948; Munk 1950).

Nonlinear processes distort the antisymmetry between the gyres present in the linear solution (Fig. 1b). Secondary recirculations clearly appear in both gyres, along with a sharp pressure gradient in the basin’s interior, which is known as the eastward jet. The cyclonic and anticyclonic eddies east of the recirculating features produce positive and negative concentrations of vorticity. The Sverdrup interior remains confined to the northeastern and southeastern corners of the domain. Comparison with the linear solution shows that these subbasin-scale recirculations and the associated eastward jet are not directly wind driven and thus must arise from the system’s nonlinearity. In other words, the transport by these recirculations is far too large to be directly driven by the wind system prescribed in Eq. (2.2) through the Sverdrup relation. As expected, the nonlinear term allows the penetration of inertial WBCs across the R line, known as overshooting, and consequently shifts the C point away from the R line. In this particular case, the WBC of the northern gyre overshoots southward and the steady C point lies to the south of the R line (Fig. 1d).

The semipermanent meanders in the Gulf Stream or Kuroshio are well known (Tait 1972; Watts 1983). While the existence of a free-jet solution depends on the forcing geometry of the lateral boundary (Cessi and Thompson 1990), meandering of a free jet is usually ascribed to the bottom topography, such as the New England Seamounts or the Izu Ridge (Warren 1963). The bottom topography may either maintain a steady meander or cause instability of the jet. When the wind stress pattern is asymmetric, Moro (1988) and Verron and Le Provost (1991) pointed out that bottom topography is not needed for the existence of semipermanent meanders in eastward jets. In their formulation, a more intense secondary recirculation can be induced in the region of the basin with stronger forcing; it then “winds up” the associated jet around itself. Figure 1b shows that the subtropical recirculation is substantially stronger than the subpolar one due to nonlinear processes that enhance the asymmetric response of the system to the symmetric forcing. The separation of the WBC southeastward (Fig. 1b) can still be attrib-
ute here to the stronger recirculation south of the R line wrapping the jet around it, although the cause of the difference in recirculation strength is not any asymmetry in the forcing.

**b. Multiple equilibria**

For linear dynamics, the asymptotic solution of the system subject to sustained constant wind forcing is uniquely determined. However, when nonlinear processes come into play, multiple steady solutions satisfying identical boundary conditions can arise. Transition between such steady states of the wind-driven circulation can provide a mechanism of natural climate variations on interannual and interdecadal timescales (Ghil and Vautard 1991; Speich and Ghil 1994).

To obtain the multiple steady states suggested by the symmetry breaking in Fig. 1b, the fully nonlinear system is integrated from two different initial states. Figures 2a,c and 2b,d show two equilibrium states for such an experiment with $\alpha_x = 1.3$ and $\alpha_y = 0.9$. The initial state for Fig. 2a is a state of rest, while that for Fig. 2b is identical to Fig. 1b. Interestingly, the two steady solutions have patterns that are nearly symmetric with respect to each other about the R line. For Fig. 2a, we can see that the recirculation in the northern gyre is
more intense than the one in the southern gyre; the opposite happens in Fig. 2b. Consequently, the confluent jet in Figs. 2a,c extends in the northeast direction, while the one in Figs. 2b,d extends southeastward. Accordingly, the C point in Fig. 2c lies north of the R line, whereas that in Fig. 2d stays south of it.

c. Bifurcation diagrams

The clear separation between the respective C points of these two steady solutions motivates us to construct a bifurcation diagram based on the position of the C point, with respect to the R line, and the strength of the wind stress $\alpha_r$. Figure 3 provides such a diagram for $\alpha_d = 1.3$, with positive values on the ordinate indicating that the equilibrium C point lies north of the R line.

For the northern branch (positive), the C point shifts monotonically northward with increasing wind stress $\alpha_r$ (heavy solid curve). For the southern branch (negative), equilibrium values of the C point are found only for $\alpha_r > 0.72$ and then migrate farther south as $\alpha_r$ increases (heavy solid). If $\alpha_r$ is below 0.72, no equilibrium C point exists to the south of the R line. Unstable steady solutions cannot be determined numerically by the present forward integration method.
qualitative appearance of the stable branches and the inherent symmetry of the linear problem (Fig. 1a) lead us to conjecture (cf. Ghil and Childress 1987; Zhao and Ghil 1991) that it is a perturbed pitchfork bifurcation. An analytical study of this bifurcation in a low-order QG model with the same geometry is given in section 4 and confirms this conjecture. The appropriate unstable solution branch (heavy dashed) is therefore drawn in Fig. 3 as very plausible for the present SW model as well. It is clear that the multiple equilibrium solutions coexist in a range of \( \alpha \), from 0.72 to 0.98.

Above 0.98, the steady southern branch loses its stability to a periodic solution. Our numerical experiments indicate that the Hopf bifurcation at \( \alpha = 0.98 \) is supercritical. In Fig. 3, the light solid lines outline the range of C-point positions for the periodic solution; this range grows as \( \alpha \) increases. A similar phenomenon also occurs on the northern branch, suggesting its supercritical Hopf bifurcation at \( \alpha = 1.12 \). Furthermore, as the parameter \( \alpha \) crosses 1.1, the periodic solution along the southern branch loses its stability and aperiodic solutions arise instead. The total range of C-point latitudes for these solutions is larger (light dashed) than that of the limit cycle from which they bifurcate. As \( \alpha \) continues to increase, the system becomes more and more chaotic. Along the northern branch, a similar bifurcation from the periodic to aperiodic solutions appears at \( \alpha = 1.25 \). The oscillation periods for the northern branch are much shorter than for the southern one (not shown; cf. also S. Speich, H. Dijkstra, and M. Ghil 1994, personal communication).

A regime diagram in the two parameters \( \alpha_d \) and \( \alpha \), exhibits a cusp catastrophe (e.g., Guckenheimer and Holmes 1983). Computational expense permits us to perform only two more experiments similar to that discussed above, choosing \( \alpha_d = 1.0 \) and \( \alpha_d = 1.8 \). Spline interpolation among the curves computed for \( \alpha_d = 1.8 \), \( \alpha_d = 1.3 \), and \( \alpha_d = 1.0 \) yields the catastrophe surface. Extrapolation beyond these \( \alpha \) values can be conjectured in a straightforward way. Figure 4 outlines the resulting cusp diagram. The solid, dotted, and dashed curves represent the perturbed pitchfork bifurcation and Hopf bifurcations along the southern and northern branch, respectively. To relate Fig. 4 directly to Fig. 3, circles

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**Fig. 3.** Bifurcation diagram for the position of the C point, as a function of wind stress (\( \alpha_d = 1.3 \)).

**Fig. 4.** Catastrophe diagram in terms of the viscosity parameter \( \alpha_d \) and the wind-forcing parameter \( \alpha \).
1, 2, and 3 denote these bifurcations for the experiment with \( \alpha_4 = 1.3 \). The occurrence of multiple equilibrium states must fall into the zone between the pitchfork bifurcation line (solid) and the Hopf bifurcation line that lies closer to it. Based on the extrapolation of Fig. 4 (not drawn), it can be presumed that the system loses its multiple equilibria when \( \alpha_4 < 0.8 \) or \( \alpha_4 \) becomes large (larger than 10). The wind forcing \( \alpha \) giving rise to the northern Hopf bifurcation does not have to be larger than that for the southern Hopf bifurcation at all \( \alpha_4 \) values. In fact, the former \( \alpha \) can be smaller at either lower (\( \alpha_4 < 1.0 \)) or higher \( \alpha_4 \) (\( \alpha_4 > 2.0 \)). In general, the solutions become more complex as \( \alpha \) increases and as \( \alpha_4 \) decreases.

d. Periodic solutions

To follow the oscillating behavior of model solutions in response to constant external forcing, we first spin up the model, with \( \alpha_4 \) and \( \alpha_4 \) chosen to be 1.0 and 0.8, respectively, until the flow becomes exactly periodic. The results are shown for 48 years in Fig. 5 as the variation of \( h \) along \( x = 80 \) km, where the C point is approximately situated, extending from \( y = 0 \) to \( y = 2000 \) km. After a spinup of about 20 model years, the eastward jet and associated C point clearly exhibit a purely periodic oscillation with a period of about 2.8 years. Outside the recirculation zones, the solutions are nearly steady. The C point shifts northward more abruptly than southward.

The oscillating behavior of the system can also be observed by plotting, in Fig. 6, the energetics, as described by Eqs. (2.4) and (2.5). Panel a shows the components of available potential (heavy solid) and kinetic (light solid) energy, and panel b the rates at which the various terms on the right-hand side of (2.5) do work. After 20 years of integration, a regular oscillation with a period of about 2.8 years is clearly seen. About 98% of the energy fluctuation is generated around the recirculation zone.

Figure 7 shows eight snapshots of \( h \), given every one-seventh period (0.4 years) of the 2.8-year interval from year 26.2 to 29.0 (cf. Fig. 5). At year 26.2, the C point is located at its southernmost position (Fig. 7a). The subtropical recirculation is much stronger than the subpolar one, and it wraps the eastward jet in a southeast direction. Accordingly, the (standing) meander to the east of the recirculating gyres reaches its maximum amplitude. As time goes on (Fig. 7b), the extent of both recirculation gyres shrinks. The subtropical gyre is weakened and shifts northward while the subpolar gyre recedes and intensifies. The subsidence of the meander gradually changes the orientation of the jet toward a zonal direction. By year 27.0, the C point has reached its northernmost position, and the subtropical recirculation has continuously developed (Fig. 7c). The meander subsides fully; both secondary-recirculation gyres and the associated jet stretch out to the east (Fig. 7d). Next, the meander starts to grow again (Figs. 7e–f). At year 28.6, both fully stretched recirculations have begun migrating southward (Fig. 7g). The subtropical recirculation has intensified, whereas the subpolar one has weakened. By year 29.0, an \( h \) pattern very similar to Fig. 7a reappears (Fig. 7h); the slight difference arises because the period is not an exact multiple of the time step.

Shown in Fig. 8 are the instantaneous anomalies of \( h \), corresponding to the difference between the \( h \) fields shown in Fig. 7 and the average of the \( h \) field over one full period. The anomalies resemble an elongated wave pattern, confined mostly to the secondary-recirculation region of the domain. Examining these wavelike patterns leads to the following remarks. 1) When the C point reaches its southernmost position, at model year 26.2, the anomaly waves attain their maximum amplitude (Fig. 8a); the troughs and ridges of these waves line up along the NW–SE direction. Then, the wave amplitude decays continuously while the orientation of the nodal lines changes from NW–SE toward W–E (Fig. 8b). When the C point moves up to its northernmost position, the anomaly waves have minimal amplitude; the troughs and ridges are approximately aligned with the parallels (Fig. 8c). In Figs. 8d–g, the anomaly waves grow gradually with their alignment leaning from the zonal to the NW–SE direction. The

![Fig. 5](image-url)

**Fig. 5.** The periodic variation of \( h \) along \( x = 80 \) km for 48 years. The parameter values are \( \alpha_4 = 0.8 \) and \( \alpha_4 = 1.0 \). Solid and dashed lines as in Fig. 1; contour interval is 20 m.
anomaly pattern in Fig. 8h is almost identical to that of Fig. 8a. 2) The anomalies, generated alternately around the ridge of the downstream meander (at $x \approx 400$ km and $y \approx 1400$ km), grow and propagate roughly southwestward. Meanwhile, the advection effect is quite evident. First, the anomaly patterns are elongated along the eastward current's direction of flow. Second, the southwestern propagation of anomaly patches with alternating sign is distorted by the jet and recirculating gyral advection. The striking example for the latter effect is that, having crossed the eastward jet, each anomaly patch begins spinning around the southern recirculation.

Since the C point is so closely associated with the two separation points on the western boundary, it is likely that these separation points might oscillate in a manner similar to that of the C point in the present study. Although the mechanisms controlling WBC separation and its position have not yet been fully established (Ierley 1990; Haidvogel et al. 1992), the adverse pressure gradient seems to be important (Stern and Whitehead 1990). More precisely, it is the ageostrophic component of the pressure gradient that is involved in the WBC separation (Haidvogel et al. 1992; J. C. McWilliams 1993, personal communication). Plots of the ageostrophic part of $\nabla h$ (not shown) exhibit periodic variations with the largest amplitude near the western boundary and the $R$ line. Haidvogel et al. (1992) have shown that the ageostrophic pressure-gradient term is balanced essentially by the $\beta$ term and the (hyper) viscous friction term.

In the present periodic case, the climate mean of the $h$ field determines the mean position of the separation points. The periodically varying patterns of the $h$ anomalies around the western boundary region change these three terms there periodically, thus dominating the periodic oscillation of the C point and the associated separation points.

e. Aperiodic solutions

Based on the results in Figs. 3 and 4, the aperiodic solutions can be obtained by either increasing $\alpha$, or decreasing $\alpha_A$ with respect to the values that yield periodic solutions, while smaller $\alpha$, or larger $\alpha_A$, lead to steady states. Thus, two experiments were studied in detail: one with larger $\alpha$, and the other with smaller $\alpha_A$. Figures 9a–d show, in panels a,c, the globally averaged kinetic (light solid) and available potential (heavy solid) energies and, in panels b,d, the rate of work done by the wind stress (heavy solid), the bottom friction (dotted), and the viscosity (light solid) as a
function of time for the experiment with $\alpha_r = 0.95$ and $\alpha_A = 1.0$ (panels a,b) and for the experiment with $\alpha_r = 0.8$ and $\alpha_A = 0.75$ (panels c,d). The time series of $h$ along the meridional section at $x = 80$ km are illustrated in Figs. 10a and 10b for each of the two experiments, respectively.

Figures 9a,b and 10a still exhibit strong interannual near-periodicity with a fluctuating "period" of 3.2 to 3.8 years and irregular amplitudes. On the other hand, variability at two distinct interannual periods is clearly present in Figs. 9c,d and 10b. In this case, the longer period, of about 6 years, and the shorter one, of about 3 years, are associated with stronger and weaker amplitudes, respectively. Since the two periods are in an approximate 2:1 ratio, this systematic effect might be related to period doubling. Note that more irregular eddy timescale fluctuations, of several months, are evident in both Figs. 10a and 10b. As in the periodic results, the dominant oscillations occur around the eastward jet; the solutions are nearly steady around the northern and southern edges of the basin, where the dynamics is approximately linear; the C point quickly shifts northward and then migrates southward slowly; each sudden northward jump of the eastward jet and the associated C point is accompanied by vigorous eddy activity in both gyres. These systematic features of the time-dependent solutions recall relaxation oscillations and require further study.

f. Model sensitivity to $\gamma$ and $L$

In general, properties of the oscillatory solutions are sensitive to the boundary conditions, such as the parameter $\gamma$ in Eq. (2.3), and to the domain size, $L \times D$. It is therefore important to know how model solutions are affected by the choice of these parameters. For consistency, we use here the same values of $\alpha_r$ and $\alpha_A$ as in the periodic case (i.e., $\alpha_r = 0.8$ and $\alpha_A = 1.0$) and
vary $\gamma$ or $L$ only. Recall that up till now $\gamma = 1$ (no slip) and $L = 1000$ km.

1) Sensitivity to $\gamma$. When $\gamma = 0.4$, the time variations of $h$ (Fig. 10c) and energetics (Figs. 11a,b) are still very nearly periodic. However, in comparison with the periodic solution (Figs. 5 and 6), the period is much shorter and the fluctuation of the eastward jet and confluence point occurs to the north of the R line rather than to the south. This rather different oscillatory behavior is already observed for $\gamma < 0.7$ (not shown). When $\gamma = 0.3$, the oscillations become strongly aperiodic; this is clearly apparent from the energetics in Figs. 11c,d, as well as from the $h$ plots (not shown). For even smaller values of $\gamma$, the aperiodic character of the solutions becomes even more pronounced. It is not difficult to understand that the model ocean becomes more and more energetic, and thus aperiodic, as boundary adherence is less and less constraining.

2) Sensitivity to $L$. For simplicity, we carried out one experiment with a slightly larger $L$, equal to 1200 km. Shown in Figs. 12a–c are the time evolutions of $h$ and energetics. The oscillations are clearly aperiodic, and the model ocean is quite energetic. As we know already, the energy of the model ocean is imparted by the external wind forcing and dissipated in the western boundary layer. When the dissipative parameters $A$ and $R$ are fixed, the width of the frictional boundary layer (Munk 1950; Veronis 1966a) remains unchanged. As a result, the model ocean becomes more energetic as $L$ increases. Hence, model solutions become more aperiodic when the domain size is enlarged. The character of these solutions resembles more that of Figs. 9c,d and 10b than that of Figs. 9a,b and 10a. In particular, the mean period is longer and close to 5 years.

By and large, the aperiodicity of model solutions increases as $\gamma$ decreases or $L$ increases. The mean pe-
Fig. 9. As in Fig. 6 but with (a, b) $\alpha_r = 0.95$ and $\alpha_d = 1.0$; (c, d) $\alpha_r = 0.8$ and $\alpha_d = 0.75$. 
Fig. 10. As in Fig. 5 but with (a) $\alpha_r = 0.95$ and $\alpha_d = 1.0$; (b) $\alpha_r = 0.8$ and $\alpha_d = 0.75$; and (c) $\gamma = 0.4$ (vs $\gamma = 1$, with $\alpha_r = 0.8$, and $\alpha_d = 1.0$).

4. Analytic results

In this section, we study a low-order, QG approximation of the model equations to elucidate further the nature of the perturbed pitchfork bifurcation in the numerical results of section 3. This approximate model yields to a weakly nonlinear analysis that sheds light on the physical mechanism by which the multiple equilibria arise in the full model of section 2.

a. Truncated QG model

To make the problem more tractable, we reduce first the system (2.1) to

$$\frac{\partial u}{\partial t} + v \cdot \nabla u = -g \frac{\partial h}{\partial x} + fu - Ru + \frac{\tau^x}{hp} \quad (4.1a)$$

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -g \frac{\partial h}{\partial y} - fu - Rv \quad (4.1b)$$

$$\frac{\partial h}{\partial t} = -\nabla \cdot (hv), \quad (4.1c)$$

by setting $A = 0$ and $\alpha_r = 1$. The simplified equations (4.1) accommodate only the single boundary condition of no normal flow, while in the SW model both non-normal flow and no-slip boundary conditions are imposed. The dimensionless QG equation corresponding to Eqs. (4.1a–c) (cf. Hendershot 1986; Veronis 1966a) is
Fig. 11. As in Fig. 6 (α, = 0.8, αd = 1.0, and γ = 1) but with (a, b) γ = 0.4; (c, d) γ = 0.3.
\[
\frac{\partial}{\partial t} (\nabla^2 \psi) + \text{Ro} [J, (\nabla^2 - \lambda^2)\psi] + \frac{\partial \psi}{\partial x} = -\epsilon \nabla^2 \psi + w_f. \quad (4.2)
\]

Here
\[
w_f = -\frac{\partial}{\partial y} \left( \frac{x}{y} \right), \quad (4.3)
\]

\(\psi\) is a streamfunction, and \(J\) denotes the Jacobian. The variables \((t, x, y, h, \tau^x, \psi)\) in (4.2) and (4.3) have been scaled by \((\beta^{-1}L^{-1}, L, L, H, W, W^{-1}\beta^{-1}H^{-1})\); here \(L\) and \(H\) denote horizontal and vertical scales, while \(W\) is the scale of the wind stress. The nondimensional parameters are defined as

\[
\text{Ro} = \frac{W}{\rho H \beta^2 L^3}, \quad \epsilon = \frac{R}{\beta L}, \quad \lambda = \frac{L}{L_R}, \quad (4.4)
\]

where \(L_R = \sqrt{g H / f_0}\) is an internal Rossby radius of deformation. The Rossby number, Ro, and the frictional parameter \(\epsilon\) measure the effects of nonlinearity and friction, respectively [compare also Ghil and Childress (1987) section 6.4 and Legras and Ghil (1985) for a similar atmospheric model]; \(\lambda^2\) is the rotational Froude number. The boundary condition of no-normal flow requires that \(\psi\) vanish on all sidewalls; that is,

\[
\psi = 0 \begin{cases} \text{on} & x = 0, \pi \\ \text{on} & y = 0, \pi \end{cases} \quad (4.5)
\]
Veronis (1963) expanded the streamfunction for a single gyre system into a double Fourier series in the $x$ and $y$ directions and retained a very limited number of terms. However, as pointed out by Veronis himself, the $\beta$ term cannot be well represented by such a highly truncated Fourier series, nor can the full zonally asymmetric structure of the double gyre solutions here, including westward intensification (linear and nonlinear) and nonlinear recirculation. Therefore, we retain the Veronis sine expansion in the $y$ direction, but include a decaying exponential in the $x$ direction to account for the zonally asymmetric structure. The limited set of basis functions used yields the expansions

$$\psi = A(t)F(x) \sin y + B(t)F(x) \sin 2y,$$  \hspace{1cm} (4.6a)

$$F(x) = e^{-ax} \sin x,$$  \hspace{1cm} (4.6b)

$$w_y = -w_1(x) \sin y - w_2(x) \sin 2y;$$  \hspace{1cm} (4.7)

in appendix B we show that $\alpha$ is chosen by a first-order resemblance of $F(x)$ to the zonal structure of the nonlinear solution in Fig. 1b. The curl of the wind forcing $w_y$ consists of a symmetric pattern (the first term) and a symmetric one (the second term), each of which generates a meridionally antisymmetric and a symmetric component of the $\psi$ field accordingly [see Eq. (4.6a) here and McWilliams et al. (1990)]. By substituting Eqs. (4.6) and (4.7) into Eq. (4.2) and projecting only on the $y$ modes that appear in Eq. (4.6a), we have (cf. the appendix)

$$\frac{dA}{dt} - \mu AB + \nu A = \eta_1,$$  \hspace{1cm} (4.8a)

$$\frac{dB}{dt} + \mu A^2 + \nu B = \eta_2,$$  \hspace{1cm} (4.8b)

where

$$\mu = \frac{2aR_0}{\pi \lambda^2} \frac{1 + e^{-ax}}{1 + a^2}, \quad \nu = \frac{2a}{\pi \lambda^2},$$  \hspace{1cm} (4.9a,b)

$$\eta_1 = \frac{2\mathcal{F}_1}{\pi \lambda^2}, \quad \eta_2 = \frac{2\mathcal{F}_2}{\pi \lambda^2},$$  \hspace{1cm} (4.9c,d)

$$\mathcal{F}_1 = \int_0^x w_1 e^{ax} \sin x dx, \quad \mathcal{F}_2 = \int_0^x w_2 e^{ax} \sin x dx,$$  \hspace{1cm} (4.9e,f)

so that $\mu$ and $\nu$ are positive constants. Eqs. (4.8a,b) yield the steady states

$$\mu AB - \nu A = -\eta_1,$$  \hspace{1cm} (4.10a)

$$\mu A^2 + \nu B = \eta_2.$$  \hspace{1cm} (4.10b)

By eliminating $B$ from (4.10), we derive a cubic equation for $A$, which is most conveniently written as

$$\mu^2 A^3 + (\nu^2 - \mu \eta_2)A - \nu \eta_1 = 0.$$  \hspace{1cm} (4.11)

We analyze this pitchfork bifurcation equation (e.g., Guckenheimer and Holmes 1983) in the following subsection.

b. Bifurcation diagrams

When the wind stress $w_y$ is dominated by the symmetric part $w_2$, that is,

$$|\eta_2| \gg |\eta_1|,$$  \hspace{1cm} (4.12)

a double-gyre wind-driven circulation prevails in the domain. There are two cases:

1) PURELY SYMMETRIC WIND STRESS CURL

In this case, $w_1 = 0$ and $\eta_1 = 0$. The steady-state solutions are

$$\begin{cases}
A_1 = 0 & \text{for all } \eta_2 \\
A_{2,3} = \pm \sqrt{\frac{\mu \eta_2 - \nu^2}{\mu}} & \text{for } \eta_2 \gg \nu^2/\mu.
\end{cases}$$  \hspace{1cm} (4.13)

Stability analysis of these three steady solutions shows that $A_1$ is stable for $\eta_2 \ll \nu^2/\mu$ and unstable for $\eta_2 > \nu^2/\mu$; the solutions $A_{2,3}$ are stable for $\eta_2 \gg \nu^2/\mu$. Clearly, the point

$$\eta_a = \nu^2/\mu$$  \hspace{1cm} (4.14)

is a bifurcation point. Thus, the nonlinearity results in symmetry breaking and gives a pure pitchfork bifurcation (Fig. 13a).

2) NEARLY SYMMETRIC WIND STRESS CURL

In this case, $w_1 \neq 0 \neq \eta_1$. Hence, the nature of the solutions of Eq. (4.11) depends on whether the discriminant

$$\Delta = \left(\frac{\nu \eta_1}{2 \mu}\right)^2 + \left(\frac{\nu^2 - \mu \eta_2}{3 \mu^2}\right)^3$$

is larger or less than zero. If $\Delta < 0$, that is,

$$\eta_2 > \eta_b = \eta_a + r \eta_a^{1/3} \eta_1^{2/3}, \quad r \approx 1.9,$$  \hspace{1cm} (4.15a,b)

three unequal real roots are obtained. If $\eta_2 < \eta_b$, one root is real and the two other roots are complex conjugate. The point $\eta_b$ is a saddle-node bifurcation point (Fig. 13b) and reduces to the previous one ($\eta_a$ in Fig. 13a) when $\eta_1 = 0$.

A perturbed pitchfork bifurcation is thus obtained when an antisymmetric portion $w_1$ is added to the symmetric double-gyre wind stress $w_2$ (Fig. 13b). This analytic result seems at first to contradict the numerical results of section 3c, since the wind stress in Eq. (2.2) is symmetric about the R line. Nevertheless, the mean $h$ of the northern gyre is shallower than the southern one (Fig. 1b) for any nonzero wind stress. In other words, the meridional structure of the domain's $h$ field is asymmetric at arbitrarily low values of the forcing.
This leads to an asymmetric response of the fully nonlinear SW model ocean to the symmetric wind forcing ($w_1 = 0$), while a symmetric response of the truncated QG model to the symmetric $w_f$ is maintained below the bifurcation point $\eta_a$.

It remains to examine the bifurcation diagram with respect to the antisymmetric component $\eta_1$. A saddle-node bifurcation might exist provided that (4.15) holds (Fig. 13c). The corresponding criterion for existence of multiple equilibria is

$$|\eta_1| < \eta_c = 0.385 \frac{\sqrt{\mu}}{\nu} |\eta_2 - \eta_a|^{3/2},$$  (4.16)

implying that $\eta_1$ must be small enough.

When the wind stress is dominated by its antisymmetric part,

$$|\eta_1| \gg |\eta_2|,$$  (4.17)

a single-gyre wind-driven circulation is prevalent. In this case, Eq. (4.16) is not satisfied. Thus, only one real solution exists. The relation between the unique real solution of (4.11) and $\eta_1$ is monotonic as shown in Fig. 13d. To summarize, multiple equilibrium solutions of this low-order system are related to symmetry breaking of the double-gyre wind-driven circulation.

5. Discussion and conclusions

We have presented multiple equilibria, as well as periodic and aperiodic behavior of western boundary currents and their associated features—secondary recirculations, eastward jet, and confluence—in a very idealized wind-driven, double-gyre, reduced-gravity, shallow-water model. The domain size of our model is similar to that of Holland (1978) and Marshall (1985), so as to maximize resolution and to represent the nonlinear dynamics in a satisfactory manner, while keeping the total number of discrete variables within reason. The flow is nearly Sverdrupian in the northeast and southeast corners of the domain.

Classical linear theory of the mean ocean circulation, as developed by Stommel (1948) and Munk (1950), holds that a WBC does not overshoot. Thus, separation and also confluence occur at the latitude of maximum wind stress (R line), as shown in Fig. 1a. Based on the analysis of observed wind stress, Leetmaa and Bunker (1978) found that the time-mean Gulf Stream separation indeed lies near the maximum wind stress. Nevertheless, Ierley (1990) pointed out that this concidence with linear theory is fortuitous for two reasons. 1) Observed separations of other WBCs are certainly not located at the latitude predicted by simple linear theory: for example, the separation of the Brazil Current lies on average about 1300 km north of the maximum wind forcing (Olson et al. 1988) and 2) substantial meridional variation of the separation with time is observed in all WBCs: for instance, the East Australia Current has a point of detachment that can vary by as much as 500 km (Boland and Church 1981).
These two observational facts provoke interest in how the system’s intrinsic nonlinearity affects the WBCs. A nonlinear steady solution (Fig. 1b) exhibits the observed meridional asymmetry of the double-gyre circulation pattern and southward overshooting of the southward inertial WBC across the R line, despite the symmetric wind driving. This result resembles some aspects of the nonlinear behavior of the WBCs explored by Bryan (1963), Veronis (1966a,b), and others. Observed semipermanent meanders in WBCs in the central Pacific have been attributed to topographic influences (e.g., Chao 1984) or to asymmetric wind forcing (e.g., Moro 1988). Figure 1b shows that sufficient nonlinearity could also produce such a pattern, even when the wind stress is symmetric and the bottom is flat. Nonlinear processes can thus substantially enhance asymmetric response to symmetric forcing.

The mechanism controlling WBC separation and its position has not been firmly established. Besides the existence of a line of maximum wind stress, potential mechanisms proposed so far in the literature include 1) outcropping of the thermocline (Parsons 1969; Veronis 1973), 2) potential vorticity crisis (Greenspan 1962), 3) adverse pressure gradient (Stern and Whitehead 1990; Haidvogel et al. 1992), and 4) interaction of several physical factors, such as boundary conditions, the coastal geometry, and wind stress patterns (Deng 1993). The position and alongcoast shift of the detachment point has also been studied recently, emphasizing momentum balance, by Agra and Nof (1993). We do not intend to discriminate among these or additional mechanisms, since their coexistence is very likely. The nonuniqueness of a steady-state solution suggests, in fact, that the intrinsically unstable and nonlinear dynamics play its own role in determining the time-varying WBC paths and their separation from the coast.

Our present results indicate that multiple equilibria arise provided that the wind stress is sufficiently strong (e.g., $\alpha > 0.72$ is required in Fig. 3 when $\alpha_d = 1.3$). To illustrate this further, we have carried out integrations from two distinct initial states, for identical model parameter values. Two different equilibria have been obtained (Figs. 2a,b). This result has two important implications for:

1) The direction in which the inertial WBCs tend to overshoot. Figure 2a shows the northward overshooting of the southern WBC, whereas Fig. 2b shows the reversed situation. In other words, when the nonlinearity is sufficiently strong, the inertial WBCs could overshoot the maximum wind stress line (R line) either southward or northward along the western boundary, depending on their initial states.

2) Where the separation and the eastward jet tend to occur. In Figure 2a, the nonlinear processes cause the further extension of the northward WBC and thus the appearance of a tighter subpolar recirculation. This leads to the separation lying to the north of the R line and the northeastward jet being wrapped around the stronger subpolar recirculation. In Fig. 2b, a nearly reversed pattern is shown.

The strength of the system’s nonlinearity increases with increasing wind forcing $\alpha, r_0$ and decreasing viscosity $\alpha_d A$, where $r_0$ and $A$ are fixed constants close to those usually chosen in eddy-resolving models (e.g., Holland and Lin 1975). Nonlinear advection effects can thus be studied by altering $\alpha$ and $\alpha_d$. We have constructed a bifurcation diagram in terms of the meridional position of the confluence of the two separated WBCs (the C point) with respect to the R line and the strength of the wind stress $\alpha$, (Fig. 3). It has been shown that multiple equilibria coexist in the range $0.72 < \alpha < 0.98$ when $\alpha_d = 1.3$. Interestingly, the pitchfork bifurcation at $\alpha = 0.72$ is asymmetric, although the wind stress is symmetric. Weakly nonlinear analysis of a low-order, double-gyre QG system indicates that the pitchfork bifurcations, given symmetric wind forcing, is symmetric when the unique model equilibrium is symmetric at low wind stress (Fig. 13a). In the QG system with small imposed asymmetry, multiple solutions are still possible, but one of the asymmetric solutions is favored over the other. Indeed, as shown in Fig. 13b, the pitchfork bifurcation has been perturbed by the antisymmetric component of the wind stress. The numerical results for the SW model indicate that, at low wind stress, the time-mean structure of the upper-layer thickness $h$ is meridionally asymmetric with the higher (lower) $h$ in the subtropical (subpolar) gyre. This asymmetry in the basic $h$ is present already in the linear solution for the double-gyre system, due to the meridional Ekman transport across the eastward jet and the zonal flows along the north and south walls. It leads to the system’s asymmetric response to an increase in the symmetric forcing and therefore to a perturbed pitchfork bifurcation.

Multiple WBC equilibria may be important in studying very long term changes in ocean climate (Ruddiman and McIntyre 1981). Interannual variability of model solutions is more interesting for understanding the recent instrumental record of climate change. We have obtained purely periodic solutions for a range of parameter values and have studied in detail one such solution for $\alpha = 0.8$ and $\alpha_d = 1.0$ (Figs. 5–8). Though the southward overshooting of the “cold” WBC prevails, this solution is characterized by 1) periodic interaction between the subtropical and subpolar recirculations and, thus, 2) the periodic fluctuation of the C point and eastward jet in space and time. Analysis of the $h$ anomalies (Fig. 8) shows that anomalies of alternating sign are generated in the vicinity of the downstream meander ridge. These anomalies are then propagated and advected into the WBC zone by Rossby waves and the mean recirculating currents, respectively. Periodic invasion of the WBC zone
by the anomalies may change the ageostrophic pressure gradient field there and help produce the periodic oscillation of the separation and confluence points in space and time. Indeed, Haidvogel et al. (1992) showed that an adverse pressure gradient is important in separating the WBC. The exact physical causes for the separation here are still unclear. A more complete analysis and numerical experiments are proceeding to better understand the oscillatory instability, which leads to the transition from steady to periodic solution.

The character of the periodic solutions changes as $\alpha_i$ and $\alpha_d$ change (Figs. 9 and 10), and the solution tends to become aperiodic eventually, as the wind stress is increased or the friction is decreased. An experiment with a slightly larger $\alpha_i$ ($\alpha_i = 0.95$ vs 0.8) presents a nearly periodic oscillation but with variable amplitude (Figs. 9a,b). In contrast, variability with two distinct interannual mean periods is clearly shown (Figs. 9c,d) in an experiment with a smaller $\alpha_d$ ($\alpha_d = 0.75$ vs 1.0). Both experiments exhibit interesting interaction between the interannual periods and the vigorous eddy timescale activity (Figs. 10a and 10b). As mentioned in the introduction, interannual variability has been observed in various WBCs. Topography is considered to be an important mechanism in producing this variability. Clearly, the nonlinear dynamics also affects the WBCs’ interannual variability. Just how much the intrinsic nonlinearity contributes to such variability is still an open question.

Another interesting problem is the sudden northward jump of the eastward jet and the associated C point found in periodic and aperiodic solutions (Figs. 5 and 10). Such abrupt transitions have already been observed for the Gulf Stream (e.g., Brown and Evans 1987) and the Brazil Current (e.g., Olson et al. 1988). Further observational validation of and physical explanation for this phenomenon would be very valuable.

The oscillatory properties of the model solutions depend on the choice of the viscous boundary-condition parameter $\gamma$ (cf. appendix A) and the domain size $L$. In general, oscillating solutions become more aperiodic as $\gamma$ decreases or $L$ increases.

Our initial results are encouraging, but considerable future work is still needed. To be examined further are problems associated with changing vertical and horizontal resolution (How do our $1^{1/2}$-layer results differ from those in multilayer models involving the baroclinicity?), model dynamics (What happens to the model solutions when the deep western boundary current is included?), and model--data comparisons [Is the interannual variability of model solutions relevant to the observations of Brown and Evans (1987), Olson et al. (1988), and others?]. Again, we do not discount the possibility that interannual WBC variability could be described more realistically by including 1) ventilation of the thermocline (Veronis 1981; Luyten et al. 1983), 2) topography (Charney and Flierl 1981; Chao 1984), 3) slope water (Zheng et al. 1984), and 4) surface heat flux and vertical mixing (Ezer and Mellor 1992). All these processes appear of interest for a successful WBC theory. However, the main issues discussed in this paper—multiple equilibria, periodic and aperiodic solutions—are relevant to all nonlinear physical systems, no matter what the details of their dynamics might be. We hope that our results will stimulate a renewed interest in the nonlinear variability of the WBCs.

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APPENDIX A

The Partial-Slip Boundary Condition

For simplicity, let us consider the partial-slip boundary condition on the northern wall, that is,

$$u + l_d \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = D, \quad (A.1)$$

where $l_d = \mu / R$ is the viscosity length; $R$ and $\mu$ are the Rayleigh friction and the subgrid-scale viscosity coefficients. Physically, the proportionality (A.1) between tangential velocity and tangential stress describes the balance between the Rayleigh friction on the lateral boundary ($Ru$) and the Newtonian viscous stress ($-\mu \partial u / \partial y$). Mathematically, (A.1) is a homogeneous boundary condition of the third kind or generalized Robin boundary condition (Farlow 1982; Garabedian 1964). We further introduce a nondimensional parameter $\gamma$ so that

$$l_d = \frac{1 - \gamma}{\gamma} L_D, \quad (A.2)$$

where $L_D$ is a fixed viscosity length while $\gamma$ can be varied from 0 to 1 as $l_d$ changes from $\infty$ to 0.

Equation (A.1) can be rewritten as

$$\gamma u + (1 - \gamma) L_D \frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = D, \quad (A.3)$$

which is identical to Eq. (2.3b). When $\gamma = 1$ (i.e., $R \gg \mu$), (A.3) reduces to the no-slip boundary condition (Dirichlet or first kind); in the opposite case of $\gamma = 0$, it becomes the free-slip boundary condition (Neumann

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1 Appendix added in proof, including additional references.
or second kind). Therefore, the parameter $\gamma$ measures the partial slip along a lateral boundary (Haidvogel et al. 1992).

For the continuous case, we integrate (A.3) from the interior $I$ to the boundary $B$. This yields

$$u_B = u_I \exp \left( -\gamma \frac{d}{L_D} \right),$$

(A.4)

where $u_I$ and $u_B$ are typical velocities in the interior and at the boundary, and $d$ is the distance between $u_I$ and $u_B$. Clearly

$$u_B = \begin{cases} 
  u_I & \text{when } \gamma = 0 \\
  0 & \text{when } \gamma = 1.
\end{cases}$$

(A.5)

For the discrete case, the second-order finite-difference discretization of (A.3) gives

$$\gamma u_j + (1 - \gamma) \eta (u_{j+1} - u_{j-1}) = 0,$$

(A.6)

where $\eta = L_D/2d$; $u_j$ is the boundary velocity, and $u_{j+1}$ and $u_{j-1}$ are velocities at a distance $d$ outside and inside of the boundary. Substitution of $u_j = (u_{j+1} + u_{j-1})/2$ into (A.6) gives

$$u_j = \frac{2\eta(1 - \gamma)}{\gamma + 2\eta(1 - \gamma)} u_{j-1}.$$

(A.7)

For $L_D = d$, one has $\eta = 1/2$, and Eq. (A.7) becomes

$$u_j = (1 - \gamma) u_{j-1},$$

(A.8)

with $1 \gg \gamma \gg 0$. The numerical experiments described in section 3f were carried out with $d = \Delta y = 20$ km (cf. Table 1) for (2.3b) and $d = \Delta x = \Delta y$ for (2.3a).

The discrete boundary condition (A.8) is simple and has the following major advantages. 1) Currently, either free-slip or no-slip boundary conditions are used in most ocean models (neither is realistic: boundary condition (A.8) lies between these two limits and thus is more appropriate). 2) Since many oceanic flows are highly dependent on the prescribed boundary conditions, the parameter $\gamma$ can probably be estimated from observed information on the interior flow, using variational or sequential estimation techniques (Ghil and Malanotte-Rizzoli 1991; Hao 1994; Jiang and Ghil 1993). 3) Once $\gamma$ is determined, there is no need to specify the boundary value $u_I$ for externally forced ocean models [the discretized system is closed by (A.8)]. 4) The free-slip or no-slip boundary condition can be easily obtained by setting $\gamma = 0$ or 1 in the numerical code if necessary.

APPENDIX B

Derivation of Low-Order QG System

For the reader's convenience, we sketch the derivation of the low-order QG Eqs. (4.8) and (4.9) in this appendix. Substituting Eqs. (4.6) and (4.7) into Eq. (4.2) and keeping only two terms in $\sin y$ and $\sin 2y$ yields

$$c_1 \frac{dA}{dt} + g_1 AB + d_1 A = -w_1$$

(B.1a)

$$c_2 \frac{dB}{dt} + g_2 A^2 + d_2 B = -w_2,$$

(B.1b)

where

$$c_1 = F'' - (1 + \lambda^2)F$$

(B.2a)

$$c_2 = F'' - (4 + \lambda^2)F$$

(B.2b)

$$g_1 = \frac{Ro}{2} (9FF' - F'F'' + FF''')$$

(B.2c)

$$g_2 = \frac{Ro}{2} (F'F'' - FF''')$$

(B.2d)

$$d_1 = F' + \epsilon(F'' - F)$$

(B.2e)

$$d_2 = F' + \epsilon(F'' - 4F),$$

(B.2f)

and the primes denote the derivatives with respect to $x$.

Introducing the weighted inner product

$$\langle f, g \rangle = \int_0^\pi f \, ge^{2\alpha x} \, dx,$$

(B.3)

we have

$$\langle F, F \rangle = \frac{\pi}{2}$$

(B.4a)

$$\langle F', F \rangle = -\frac{\alpha \pi}{2}$$

(B.4b)

$$\langle F'', F \rangle = \frac{(a^2 - 1)\pi}{2}$$

(B.4c)

$$\langle F'''', F \rangle = \frac{a(3 - a^2)\pi}{2}.$$  

(B.4d)

Multiplying Eqs. (B.1a,b) by $Fe^{2\alpha x}$ and then integrating them from 0 to $\pi$, one obtains the system

$$\mathcal{C}_1 \frac{dA}{dt} + \mathcal{G}_1 AB + \mathcal{D}_1 A = -\mathcal{F}_1,$$

(B.5a)

$$\mathcal{C}_2 \frac{dB}{dt} + \mathcal{G}_2 A^2 + \mathcal{D}_2 B = -\mathcal{F}_2,$$

(B.5b)

where

$$\mathcal{C}_1 = \langle c_1, F \rangle = \frac{\pi}{2} (a^2 - \lambda^2 - 2)$$

(B.6a)

$$\mathcal{C}_2 = \langle c_2, F \rangle = \frac{\pi}{2} (a^2 - \lambda^2 - 5)$$

(B.6b)
\[ g_1 = \langle g_1, F \rangle = -Ro \frac{a^2 - 9a + ae^{-ax}}{a^2 + 9} \quad (B.6c) \]

\[ g_2 = \langle g_2, F \rangle = -Ro \frac{a + ae^{-ax}}{a^2 + 1} \quad (B.6d) \]

\[ D_1 = \langle d_1, F \rangle = \frac{\pi}{2} (ea^2 - 2 \epsilon - a) \quad (B.6e) \]

\[ D_2 = \langle d_2, F \rangle = \frac{\pi}{2} (ea^2 - 5 \epsilon - a) \quad (B.6f) \]

\[ \mathcal{F}_1 = \langle w_1, F \rangle \quad (B.6g) \]

\[ \mathcal{F}_2 = \langle w_2, F \rangle. \quad (B.6h) \]

The use of the following values for the dimensional parameters: \( L \sim 200 \text{ km}, L_R \sim 40 \text{ km}, H \sim 500 \text{ m}, \rho \sim 1022 \text{ kg m}^{-3}, W \sim 0.1 \text{ N m}^{-2}, \beta \sim 2 \times 10^{-11} \text{ m}^{-1} \text{s}^{-1}, \) and \( R = (200 \text{ day})^{-1} \) leads to

\[ Ro \sim 0.0625, \quad \lambda \sim 5, \quad \epsilon \sim 0.014. \quad (B.7) \]

The value of \( a \) is chosen to give the best fit to the non-linear solution of the full SW model. Figure 14a shows the zonal structure of the meridionally averaged \( h \) field in the subtropical gyre of Fig. 1b; that is,

\[ \hat{h}(x_i) = \frac{1}{N} \sum_{j=1}^{N} h_{ij}, \]

where \( i \) and \( j \) are gridpoint indices in the \( x \) and \( y \) directions and \( h_{ij} \geq 500 \text{ m}. \) By trial and error, we found that when

\[ a = 1.3, \quad (B.8) \]

\( F(x) \) fits the model’s \( \hat{h}(x) \) quite well (see Fig. 14b).

Substituting (B.7) and (B.8) into (B.6a–f), one can rewrite (B.5a,b) as

\[ \frac{dA}{dt} - \mu AB + vA = \eta_1 \quad (B.9a) \]

\[ \frac{dB}{dt} + \mu A^2 + \nu B = \eta_2, \quad (B.9b) \]

where

\[ \mu = \frac{2a}{\pi \lambda^2} \left( 1 + e^{-ax} \right) \left( \sim 7.8 \times 10^{-4} \right) \quad (B.10a) \]

\[ \nu = \frac{2a}{\pi \lambda^2} \left( \sim 3.3 \times 10^{-2} \right) \quad (B.10b) \]

\[ \eta_1 = \frac{2}{\pi \lambda^2} \left( \sim 2.5 \times 10^{-2} \mathcal{F}_1 \right) \quad (B.10c) \]

\[ \eta_2 = \frac{2}{\pi \lambda^2} \left( \sim 2.5 \times 10^{-2} \mathcal{F}_2 \right) \quad (B.10d) \]

the values in parentheses are calculated using (B.7) and (B.8).

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**REFERENCES**


