Intraseasonal Oscillations in the Extratropics: Hopf Bifurcation and Topographic Instabilities

F.-F. Jin and M. Ghil

Climate Dynamics Center, Department of Atmospheric Sciences, and Institute of Geophysics and Planetary Physics, University of California, Los Angeles, California

(Manuscript received 6 November 1989, in final form 1 June 1990)

ABSTRACT

A potential vorticity model in a \( \beta \)-channel is used to analyze the resonant response of equivalent-barotropic flow to topography in the presence of a forced zonal jet with arbitrary meridional structure. The nonlinear dynamics near different resonances is studied considering both wave-wave and wave-zonal flow interactions. It is shown that Hopf bifurcations from stationary to periodic flows are possible due to the nonlinear instability of nonzonal, topographically forced flow. Low-frequency, finite-amplitude oscillations arise due to a combination of two factors: (i) nonlinear wave-wave interactions, which tend to reduce the Rossby wave frequency; and (ii) wave-zonal flow interactions, which reflect the importance of wave momentum transport in shifting the westerly jet and of the topographic form drag. The physical mechanism of atmospherically realistic Hopf bifurcations depends crucially on the meridional profile of the mean zonal flow giving rise to a dipole-shaped resonance. The bifurcation phenomena studied here might give some insight into the inherent dynamics of intraseasonal oscillations in the Northern Hemisphere extratropics.

1. Introduction and motivation

During the last decade, considerable research has been done on low-frequency variability (LFV) of the atmosphere in general, and on the so-called 30–60 day oscillation discovered by Madden and Julian (1971) in the tropics in particular. Somewhat more limited attention has been devoted to extratropical oscillations in the same frequency band. Anderson and Rosen (1983) showed that there is a significant 40–50 day oscillation in zonal angular momentum over both the tropics and the Northern Hemisphere (NH) extratropics. Weickmann et al. (1985) and Lau and Phillips (1986) described characteristic features of a NH intraseasonal oscillation in the 250 mb streamfunction field and 500 mb geopotential height field, respectively. More recently, Ghil and Mo (1990) analyzed in considerable detail intraseasonal oscillations in the NH 700 mb geopotential field.

Several theoretical explanations of 30–60 day oscillations in the tropics and NH extratropics have been put forward in the literature over the past few years. Equivalent-barotropic Rossby-wave propagation between the tropics and extratropics gives an essentially linear view of the linkage between oscillating phenomena in the tropics and middle latitudes (Hoskins and Karoly 1981; Hsu et al. 1990; Hoskins and Jin 1990). The linearly unstable low-frequency modes discovered by Simmons et al. (1983) for a nonzonal, climatological basic flow provide a different explanation for the origin of the extratropical oscillation. Ghil (1987) pointed out that nonlinear instability in the interaction of nonzonal westerly flow with topography could provide yet another explanation for these intraseasonal oscillations in the NH extratropics.

A systematic attack on atmospheric LFV using the concepts of nonlinear dynamics started with the discovery of multiple equilibria for a low-order truncated version of the barotropic vorticity equation with idealized forcing and topography by Charney and DeVore (1979; CDV hereafter). Some aspects of the low-order truncation results of CDV and of several subsequent papers—e.g., the actual existence of multiple equilibria in the atmosphere—have been questioned since (Tung and Rosenthal 1985). More highly resolved models (Legras and Ghil 1985; Keppenne 1989; Tribbia and Ghil 1990), however, produce more complex aspects of LFV than equilibria, such as periodic and aperiodic solutions with a broad peak in the 30–60 day band; some aspects of these solutions exhibit remarkable resemblance with observations (Ghil and Mo 1990) and with General Circulation Model (GCM) results (Marcus et al. 1990) in this band.

The two models of Legras and Ghil (1985) and of Tribbia and Ghil (1990), like CDV, are based on equivalent-barotropic nonlinear dynamics, but include more highly resolved versions of the flow field and of

Corresponding author address: Dr. Michael Ghil, Department of Atmospheric Sciences, University of California, Los Angeles, CA 90024-1565.

© 1990 American Meteorological Society
the topography than CDV, with 25 and 460 modes, respectively. Keppenne's (1989) model is a two-layer primitive-equation model on the sphere with variable triangular truncation in spherical harmonics, up to T15; i.e., up to 1520 spectral coefficients. The results of all three models suggest that an oscillatory form of topographic instability could generate the 30–60 day oscillation in the NH midlatitudes.

Topographic instability was first found by CDV, from their analysis of a three-component model, and then by Pedlosky (1981), by weakly nonlinear analysis of barotropic and baroclinic flow over topography, as a purely exponential, nonoscillatory instability. Such an instability gives rise to the possibility of multiple equilibria in a nonlinear model through saddle-node bifurcation (Ghil and Childress 1987; Guckenheimer and Holmes 1983). This instability depends crucially on resonance between a linear wave and topography. Pedlosky (1981) found that the instability could be either subresonant or superresonant, depending on the spatial structure of the resonating wave.

Legras and Ghil (1985; LG hereafter) found that one resonance in their model is equivalent to CDV resonance, but corresponds to an unrealistically strong zonal flow. The LG model, however, exhibited also another resonance, with a much more realistic circulation. The latter resonance occurred due to a more complex meridional structure of the mean zonal flow than in CDV. Different Hopf bifurcations, leading from stationary to oscillatory solutions, arise near both resonances in the LG model. The dynamical processes, however, that control these bifurcations were not clarified sufficiently by LG.

In this paper, an asymptotic analysis of the Hopf bifurcation near different resonances is presented. The dynamical interpretation of the corresponding bifurcation phenomena will add, we hope, to the understanding of low-frequency extratropical oscillations in the earlier models and in the atmosphere. The approach here is similar to the one used by Pedlosky (1981); it exploits systematically the near-resonant character of the nonlinear finite-amplitude phenomena investigated. The approach is extended to a more general basic flow, in the presence of external forcing, and to Hopf bifurcations. Nonlinear dynamics near different resonances, with wave–wave as well as wave–zonal flow interactions, is studied by focusing on the generation of low-frequency oscillations.

Resonances induced by the presence of topography play therewith a crucial role in our analysis. It has been suggested (e.g., Tung and Rosenthal 1985) that finite-amplitude resonances only arise by wave trapping in a β-channel (CDV) or in a spherical model with equatorial symmetry (LG). But the T21 equivalent-barotropic model of Tribbia and Ghil (1990) and the T15 baroclinic model of Keppenne (1989) both use realistic earth topography at the given resolution and obtain the same type of intraseasonal oscillation as LG. Tribbia and Ghil (1990) have, in fact, verified the predominantly absorbing character of the equatorial critical layer in their model.

The model studied in this paper is governed by the equivalent-barotropic form of the potential vorticity equation in a β-plane channel, with simplified topography, a forced midlatitude zonal jet and Ekman-type dissipation. In section 2, we describe the model. In section 3, we present a linear resonance analysis in the presence of zonal jets with arbitrary meridional structure. The meridional structures of resonant waves are the eigenfunctions of the corresponding eigenvalue problem; they are more complicated than simple Fourier harmonics.

In section 4, the multiple-scale perturbation method is applied in order to derive systematically a simplified system describing the weakly nonlinear dynamics near the different resonances. Both wave–wave and wave–zonal flow interactions are included. The corresponding mechanisms leading to low-frequency Hopf bifurcation are analyzed in sections 5 and 6, respectively. The physical interpretation and relevance to intraseasonal oscillations in the NH extratropics are discussed in section 7. A brief description of Hopf bifurcation and of the CDV topographic instability is given in appendix A, and a general procedure for the weakly nonlinear analysis of section 4 is outlined in appendix B.

2. Model

The model is governed by the equivalent-barotropic form of the equation for the conservation of potential vorticity in a β-channel (Ghil and Childress 1987; Pedlosky 1987) in nondimensional form:

$$\frac{\partial}{\partial t} (\zeta - \lambda^{-2}\psi) + J(\psi, \zeta) + \beta \frac{\partial \psi}{\partial x} + J(\psi, h_B) = -\gamma \nabla^2 (\psi - \Psi^*) \tag{2.1}$$

here $\Psi(x, y, t)$ is the stream function, $h_B(x, y)$ is topographic height, $\lambda$ is the Rossby radius of deformation, $x$ and $y$ are Cartesian coordinates pointing east and north, respectively; $\zeta = \nabla^2 \psi$ is the relative vorticity, while $\nabla^2$ and $J$ are the Laplacian and Jacobian operators, respectively. The right-hand side induces a forced relaxation towards a zonal jet $\Psi^*(y)$ with a characteristic time scale $\gamma^{-1}$. This model is the same as the one used by LG except that spherical geometry was considered there; in the Cartesian geometry used here, $\beta$ is the meridional gradient of the Coriolis parameter.

The flow is contained within a channel on whose boundaries

$$\frac{\partial \psi}{\partial x} = 0, \quad y = 0, \pi, \tag{2.2}$$

while the zonally averaged part of the streamfunction, \( \bar{\psi}(y, t) \) must satisfy
\[
\frac{\partial^2 \bar{\psi}}{\partial t \partial y} = 0, \quad y = 0, \pi. \tag{2.3}
\]

In the next two sections, we are going to use slightly simplified versions of Eq. (2.1) in order to analyze
linear and nonlinear resonance over wavy topography in a generalized basic flow.

3. Linear resonance

For the inviscid case, \( \gamma = 0 \), and a prescribed basic flow \( \bar{\varphi}_y = -\bar{U}(y) \), the linearization of (2.1) can be
written as
\[
\frac{\partial}{\partial t} \xi + \bar{U} \frac{\partial}{\partial x} \xi + \frac{\partial}{\partial x} \left( \beta \bar{U}_y \right) \frac{\partial \varphi'}{\partial x} - \lambda^{-2} \frac{\partial \varphi'}{\partial t} = -\bar{U} \frac{\partial h_B}{\partial x}, \tag{3.1}
\]
where \( \xi' \) and \( \varphi' \) are perturbation quantities. Attention in this paper is restricted to single-wave topography in the
zonal direction,
\[
h_B(x, y) = H(y) \left\{ h_k e^{ikx} + h_k^* e^{-ikx} \right\}, \tag{3.2a}
\]
where an asterisk denotes complex conjugation. The shape \( H(y) \) of the topography in the meridional direction is arbitrary, subject to the constraint that it have nonzero projection onto the meridional profile of the basic flow,
\[
\int_0^\pi \bar{U}(y) H(y) dy \neq 0. \tag{3.2b}
\]

Assuming a solution with streamfunction pattern similar to (3.2a),
\[
\varphi(x, y, t) = \phi(y) \{ A(t) e^{ikx} + A^*(t) e^{-ikx} \}, \tag{3.3}
\]
resonance will occur if \( \bar{U} \) and \( \phi \) satisfy
\[
\bar{U} \left( \frac{d^2}{dy^2} - k^2 \right) \phi + (\beta - \bar{U}_y) \phi = 0. \tag{3.4}
\]
This is a generalized resonance condition for nonuniform basic flow.

For convenience of notation, let
\[
\bar{U} = U g(y), \tag{3.5}
\]
where \( g(y) \) is a specified function and \( U > 0 \) is the characteristic speed of the basic flow. With boundary condition (2.2), Eq. (3.4) then becomes a typical eigenvalue problem:
\[
L_0 \phi = \left\{ g \left( \frac{d^2}{dy^2} - k^2 \right) - g_{yy} \right\} \phi = -\rho \phi; \tag{3.6a}
\]
\[
\phi(0) = \phi(\pi) = 0. \tag{3.6b}
\]
The eigenvalue \( \rho \) and corresponding eigenfunction \( \phi(y) \) determine the amplitude \( U \) of the resonant basic flow
and the meridional structure of the resonant response, respectively. Notice that \( \rho = \beta / U \) is an inverse Rossby
number.

For climatological midlatitude flows \( g(y) \) does not vanish over \([0, \pi]\); hence, we do not consider here the
singular case of critical lines (see, however, Tung and Lindzen 1979). For nonvanishing westerly flow, \( g(y) \geq g_0 > 0 \) over \([0, \pi]\), (3.4–3.6) is a regular Sturm–Liouville problem (e.g., Courant and Hilbert 1953;
Ghil 1976) with a countable infinity of real eigenvalues \( \{ \rho_l \} \) and eigenfunctions \( \{ \phi_l \} \). The latter form a complete orthonormal basis for the expansion of squareintegrale functions over \((0, \pi)\).

The simplest example is
\[
g(y) = 1,
\]
which represents a uniform basic flow. In this case,
\[
\rho_l = k^2 + l^2, \tag{3.7a}
\]
\[
\phi_l = \sin ly, \quad l = 1, 2, \cdots \infty. \tag{3.7b}
\]

The corresponding resonant zonal flow speeds \( U_l \) of the basic flow, \( \bar{U}(y) = U_l g(y) \), are
\[
U_l = \beta/(k^2 + l^2). \tag{3.7c}
\]

The latter is a well-known condition for the resonance of stationary Rossby waves (Egger 1978; Tung and Lindzen 1979). For fixed \( k \), different meridional wavenumbers \( l \) give very different resonant flow speeds \( U_l \); as \( l \) increases, \( U_l \) decreases rapidly.

In general, \( g \approx 1 \), but one still has a countable eigenset \( \{ \rho_l, \phi_l(y) \} \), with
\[
\rho_1 < \rho_2 < \rho_3 \cdots;
\]
although, \( \rho_l \) and \( \phi_l \) must be calculated numerically. For convenience, hereafter, we refer to \( l = 1 \) as the CDV-type monopole resonance and to \( l = 2 \) as the LG-type dipole resonance respectively, and justify this nomenclature in connection with Eq. (3.8), Table 1, and Figs. 1 and 2 below.

From (3.7c) it follows that, for ultra-long planetary waves, \( k \ll l \), the required jet speed for the first resonance
will be 3–4 times larger than that for the second one. This is also true for general nonuniform basic flows. To give an example, we take
\[
g(y) = v_0 + \alpha_1 \sin y + \alpha_2 \sin 2y \tag{3.8}
\]
and consider the \( \beta \)-channel centered at 45°N. The corresponding zonal wavenumber is
\[
k = \tilde{k}/2.83, \quad \tilde{k} = 0, 1, 2 \cdots,
\]
for a channel width of 5000 km.

The eigenvalue problem (3.4–3.6) is solved by spectral expansion of \( \phi \) in terms of \( \{ \sin ly \} \). Table 1 lists
the first few eigenvalues \( \{ \rho_l \} \) for different zonal wavenumbers \( k \) and different meridional truncations, \( 0 \leq j \leq N \), in the case of \( v_0 = 0, \alpha_1 = 1, \) and \( \alpha_2 = 0 \). As we see from the table, a meridional truncation of \( N = 32 \)
is sufficient in this case for the first few eigenvalues to converge. At this truncation, the ratio of the first two eigenvalues $\rho_2/\rho_1$ is larger when the zonal wavenumber is smaller. For wavenumber 2, it is about four, while for $k = 4$ it is less than two. This ratio also depends on channel width.

In the long-wave limit $k \to 0$, the first resonance eventually disappears, $U_1 = \beta/\rho_1 \to +\infty$. Tung and Rosenthal (1985) pointed out that CDV-type monopole resonance requires a zonal forcing too strong to be realistic. Clearly the second, dipole resonance is more realistic, in the sense that a much weaker zonal forcing, $U_2 = \beta/\rho_2$, is required: for a channel width of 5000 km and $k = 2$, the required zonal flow speed is 100 m s$^{-1}$ for the monopole and 23 m s$^{-1}$ for the dipole resonance.

The meridional structure $\phi(y)$ of the perturbation streamfunction (3.3) is shown in Fig. 1, for two basic profiles $g(y)$ [panels (a) and (b)]. The first eigenfunction has no sign change throughout the channel (solid), the second one has one such change (dashed), and the $l$th eigenfunction $\phi_l(y)$ has $l - 1$ zeros (not shown for $l \geq 3$). It is also found that $\alpha_1$ affects the eigenvalue only very little, but it does affect the eigenfunction substantially. When $\alpha_2 > 0$, the basic jet is located towards the south side of the channel—cf. Eq. (3.8)—while the resonant structure exhibits a larger amplitude towards the north side (Figs. 1a,b). For $\alpha_2 < 0$, the situation is reversed (not shown).

The total streamfunction fields, $\Psi(y) + A\Psi'(x, y)$, with $A = 0.5$, are shown in Figs. 2a,b. The first resonance exhibits a simple ridge and trough pattern, as noted by CDV for $\alpha_2 = 0$. The second resonance can exhibit a dipole pattern upstream of the topography when the wave amplitude is strong enough. This difference in the linearly resonant response has, as we shall see in section 5, an important consequence for the nonlinear interaction between the resonant wave and the topography.

In Figs. 2c,d, we show the zonal-mean mountain torque distribution corresponding to these two flow patterns. The torque is computed for the profile $H(y) = \sin \frac{3}{2} \left( y - \frac{\pi}{6} \right)$, $\pi/6 \leq y \leq 5\pi/6$, (3.9) and $H(y) = 0$ outside this interval. This profile satisfies condition (3.2b) and the positive topography associated with (3.2a, 3.9) and $k = 2$ is shaded in Figs. 2a,b. Although their signs and amplitudes depend on the phase difference between flow and mountain, the structures in Figs. 2c,d are quite different from each other. For the first resonance, the form drag tends to accelerate the basic zonal flow, while the second resonance affects mostly the meridional shear. These differences will be discussed further, as they pertain to oscillatory solutions, in section 7.

### 4. Weakly nonlinear analysis

In Eq. (2.1), $\Psi^*$ maintains $\Psi(y)$ against dissipation, so we expect the characteristic speed $U$ of the fully nonlinear mean flow to be comparable to and less than $U^*$. $U \approx U^*$, since part of the kinetic energy of $\Psi^*$ is used to maintain the finite-amplitude waves as well (see LG). Furthermore, since the eigenfunctions $\{\phi_l(y)\}$ form a complete basis for the expansion of solutions in the meridional direction, we expect a significant spectral peak of $\psi(x, y)$ at $l$ if $U \approx U_l$. In other words, if the basic flow is nearly resonant with a mode of the topography in the linearized problem, a wavy response dominated by the $\phi_l(y)$ structure will arise from Eq. (2.1). This suggests a way of exploring the nonlinear behavior of the system near the linear resonance.

Pedlosky (1981) discussed this problem in the simple case of $g(y) = 1$ and confirmed the low-order truncation results obtained by CDV on topographic or form-drag instability, and on the multiple equilibria this instability gives rise to. In the following, we extend the discussion of form drag instability to the general case (3.5) and focus our attention on Hopf bifurcation rather than saddle-node bifurcation. While the latter

---

**Table 1.** Eigenvalues $\rho_l$ of linear resonance problem for sinusoidal profile of imposed zonal flow; $k$ is the zonal wavenumber of the topography and $N$ the meridional truncation used in computing the eigenvalues $\rho_l$, $l = 1, 2, \cdots, N$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>0.44</td>
<td>2.36</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.42</td>
<td>2.05</td>
<td>5.49</td>
<td>10.42</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.41</td>
<td>1.87</td>
<td>4.19</td>
<td>7.58</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.41</td>
<td>1.79</td>
<td>3.81</td>
<td>6.61</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>0.41</td>
<td>1.77</td>
<td>3.68</td>
<td>6.27</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>1.50</td>
<td>2.40</td>
<td>4.37</td>
<td>6.95</td>
</tr>
<tr>
<td>$k = 0$</td>
<td>32</td>
<td>0.0</td>
<td>1.58</td>
<td>3.45</td>
<td>6.05</td>
</tr>
</tbody>
</table>
\[
\gamma = \epsilon^2 R, \quad h_B(x, y) = \epsilon^3 h(x, y), \quad (4.3c,d)
\]
\[
\tau = \epsilon^2 \tau, \quad \frac{\partial}{\partial t} = \epsilon^2 \frac{\partial}{\partial \tau}; \quad (4.3e,f)
\]

notice that \( \eta = O(\epsilon^2) \) henceforth. Substituting (4.3) into (2.1) and rearranging the equation into terms of the same order \( \epsilon^n \), a simplified, closed system for the first two orders obtains by following the procedure outlined in appendix B.

(a) First-order problem

For the \( O(\epsilon) \) problem, we simply obtain the linear form (3.1) of the original Eq. (2.1), which we rewrite with the intervening notation as
\[
\mathcal{L} \psi^{(1)} = \frac{\partial}{\partial x} \left\{ U_i g \xi^{(1)} + (\beta - U_i g^*) \psi^{(1)} \right\} = 0; \quad (4.4)
\]
primes denote \( y \)-derivatives from now on. The solution of this problem can be written as
\[
\psi^{(1)} = \phi_t(y) \left\{ A(\tau) e^{ikx} + A^*(\tau) e^{-ikx} \right\}. \quad (4.5)
\]
This solution describes a nearly stationary Rossby wave with a slowly varying amplitude \( A(\tau) \). The time evolution of this amplitude is determined by considering higher-order problems, which represent the nonlinear aspects of the resonance due to topographic forcing.

(b) Second-order problem

The \( O(\epsilon^2) \) term in the expansion (4.3) is given by
\[
\mathcal{L} \psi^{(2)} = -J(\psi^{(1)}, \xi^{(1)}). \quad (4.6a)
\]
For the uniform basic flow discussed by Pedlosky (1981), \( \phi_t = 0 \), and the right-hand side (RHS) vanishes in (4.6a). For nonuniform basic flows, \( \phi_t(y) \neq 0 \) and the RHS does not vanish. In general,
\[
J(\psi^{(1)}, \xi^{(1)}) = \left\{ ikA^2(\phi_t \phi_t^* - \phi_t^* \phi_t) e^{i2kx} \right\} + \{ \}^*, \quad (4.6b)
\]
where \( \{ \}^* \) is the complex conjugate of the term in the first set of curly brackets, and formal closure of the weakly nonlinear expansion (4.3) has to be sought at a higher order than in Pedlosky (1981). Still, the zonal term \( k = 0 \) that arises formally in (4.6b) is identically equal to zero, \( J(\psi, \xi^*) = 0 \). We assume here that the two linear resonances for wavenumbers \( k \) and \( 2k \) are sufficiently well separated, which is basically valid as seen from Table 1, so that (4.6b) does not cause a higher-order resonance.

Then (4.6a,b) has the special solution:
\[
\psi^{(2)} = F(y) \left\{ [A^2(\tau) e^{i2kx}] + \{ \}^* \right\}; \quad (4.7)
\]
here \( F(y) = F_t(y) \) is the particular solution of
\[
\mathcal{L}_2 F(y) = -\frac{1}{2} (\phi_t \phi_t^* - \phi_t^* \phi_t) \quad (4.8a)
\]
with
\[ L_2 = U_l \left[ \left( g \left( \frac{\partial^2}{\partial y^2} - 4k^2 \right) - g^* \right) + \beta \right]. \] (4.8b)

Allowing for a zonal-flow correction \( \psi^{(2)}(y, \tau) \) to be determined later, the complete second-order solution is
\[ \psi^{(2)} = \psi^{(2)}_0(y, \tau) + \psi^{(2)}_1(x, y, \tau). \] (4.9)

Figures 3a,b show the solution \( F'(y) \) of (4.8) corresponding to the two panels of Fig. 1. The second-order wave correction \( F'(y) \) is very small for the first resonance, \( l = 1 \), and is rather large for the second resonance, \( l = 2 \). This implies that nonlinear wave-wave interaction is much stronger near the dipole resonance than near the monopole resonance. This may explain why LG found a more complicated bifurcation structure in the vicinity of their second resonance than near the first, CDV-type resonance in their model.

c. Third-order problem

Closure of the system governing the first-order solution is achieved by considering the \( O(\epsilon^3) \) problem. This yields
\[ L \psi^{(3)} = -\left( \frac{\partial}{\partial \tau} \hat{\xi}^{(1)} - \chi \frac{\partial}{\partial \tau} \hat{\xi}^{(1)} \right) \]
\[ - R \hat{\xi}^{(1)} - U_l \frac{\partial \hat{h}}{\partial y} - J(\psi^{(2)}_0 + \psi^{(2)}_1, \hat{\xi}^{(1)}) \]
\[ = J_3, \] (4.10)

while the zonal average for the \( O(\epsilon^4) \) problem gives
\[ \frac{\partial}{\partial \tau} \left( \hat{\xi}^{(2)}_0 - \chi \hat{\xi}^{(2)}_0 + R(\hat{\xi}^{(2)}_0 - \Psi^{(2)}) \right) \]
\[ = -\frac{\partial}{\partial y} \left( \frac{\partial \hat{\xi}^{(1)}}{\partial x} h - \frac{\partial \hat{\xi}^{(1)}}{\partial y} \right) - \frac{\partial \hat{\xi}^{(3)}}{\partial x} + \frac{\partial \hat{\xi}^{(3)}}{\partial y}, \] (4.11)

where \( \Psi^{(2)} \) is defined in Eq. (4.3b). For solvability of (4.10–11), two constraints are derived following appendix B and form a closure for the first- and second-order solutions (4.4–6):

\[ \frac{\partial}{\partial \tau} A + R' A + i \omega |A|^2 A + i \hat{\xi}(\tau) A - ib \hat{h} = 0, \] (4.12a)

\[ \frac{\partial}{\partial \tau} \left( \frac{\partial^2}{\partial y^2} - \chi \frac{\partial^2}{\partial y^2} \right) \psi^{(2)}_0 + R \frac{\partial^2}{\partial y^2} \left( \psi^{(2)}_0 - \Psi^{(2)} \right) \]
\[ = R' Y'(y) |A|^2 + Y_2(y) \text{Im}(\hat{h} A^*); \] (4.12b)

here,
\[ R' = -R \langle \phi (\phi \phi_1 - k^2 \phi_1) \rangle / D_l, \] (4.13a)
\[ D_l = -\langle \phi (\phi \phi_1 - k^2 \phi_1) \rangle + \chi \langle \phi^2 \rangle, \] (4.13b)
\[ \omega = k \langle G(y) \phi(y) \rangle / D_l, \] (4.13c)
\[ G(y) = 2 \left[ F(\phi^2 - k^2 \phi_1) - \phi (F^2 - 4k^2 F) \right] + [- \phi (F'' - 4k^2 F') + F' (\phi - k^2 \phi)] \] (4.13d)
\[ \hat{Z}(\tau) = k \langle \psi^{(2)}_1 (\phi \phi_1 - k^2 \phi_1) / D_l \rangle, \] (4.13e)
\[ b = k U_l \langle \phi \hat{H} \beta_1 \rangle / D_l, \] (4.13f,g)
\[ Y_1(y) = \chi \frac{\partial}{\partial y} \left( \frac{1}{U_l} \psi^{(2)}_0 \phi \phi_1 - \langle \phi_1^2 \phi \phi_1 \rangle \right) \] (4.13h)
\[ Y_2(y) = 2k \langle g H(y) \phi \phi_1 \rangle \frac{\partial}{\partial y} \]
\[ \times \left[ \frac{1}{U_l} (\phi \phi_1 - k^2 \phi_1) - \chi \frac{\partial}{\partial y} \right] / D_l, \] (4.13i)

and \( \langle \cdot \rangle \) denotes averaging in the meridional direction.

Equations (4.12a,b) now form a complete system for \( A(\tau) \) and \( \psi^{(2)}_0(y, \tau) \), describing the nonlinearly resonant dynamics near the \( l \)th linear resonance. Equation (4.12a), in particular, is a Landau–Ginzburg equation (Haken 1985) for the slow evolution of the wave amplitude \( A(\tau) \). The cubic term represents wave-wave interaction effects, while \( \hat{Z}(\tau) \) denotes the wave-zonal flow interaction and \( \hat{h} \) is the topographic forcing.

It is easy to show that if \( g = 1 \), then
\[ \omega = 0, \quad Y_1(y) = 0, \]
and the results become identical to those of Pedlosky (1981) when \( \lambda^{-2} = 0 \). In the following two sections, we study mechanisms generating low-frequency oscillations through Hopf bifurcations due to form-drag instability, when \( g(y) \approx 1 \).

### 5. Hopf bifurcation and wave–wave interaction

In this section, we carry out the weakly nonlinear analysis for \( \lambda^{-2} = 0 \). Then Eqs. (4.12a,b) can be further simplified by separating out the \( \tau \)-dependence of \( \psi_0^{(2)} \) in (4.9).

\[
\psi_0^{(2)}(y, \tau) = Z(\tau) \tilde{\phi}(y) + \Psi^{(2)}(y). \tag{5.1}
\]

This yields a set of ordinary differential equations (ODEs) for the wave amplitude \( A \) and its second-order correction \( Z \),

\[
\dot{A} + RA + i\omega |A|^2 A + i(aZ + \delta) A - ibA' = 0, \tag{5.2a}
\]

\[
\dot{Z} + RZ = \text{Im}(\tilde{h}A^*), \tag{5.2b}
\]

where \( (\cdot)' = \frac{d}{dt}, Z(\tau) \) is defined implicitly by (5.1), and

\[
a = k\langle \phi_\tau(\phi_\tau'' - k\phi_\tau) \rangle/D_1, \tag{5.3a}
\]

\[
\delta = k\langle \phi_\tau(\Psi^{(2)}''(\phi_\tau'' - k\phi_\tau) - \phi_\tau\Psi^{(2)}''(\phi_\tau'' - k\phi_\tau)) \rangle/D_1. \tag{5.3b}
\]

The formal angular frequency of \( A \) is \( aZ + \delta + \omega |A|^2 \), and thus wave–wave interaction tends to reduce the frequency, for small amplitudes, when \( \omega \) and \( aZ + \delta \) have different signs.

When \( R = 0 \) and \( \omega = 0 \), it has been shown that Eqs. (5.2a,b) have two different types of periodic solution (Plumb 1981a,b; Jin and Zhu 1986). If \( R \neq 0 \) and \( \omega = 0 \), Eqs. (5.2a,b) have only equilibria as their long-term solutions—there is no Hopf bifurcation and no periodic solution possible. Therefore, the form-drag instability of forced-dissipative flow discussed by Pedlosky and CDV in the presence of one meridional mode only, does not generate periodic solutions.

However, when \( \omega \neq 0 \)—i.e., when the effect of wave–wave interaction is taken into account—the frequency correction due to this effect can change the stability of the equilibria. To show this point analytically, we consider the case of weak friction,

\[
R = \varepsilon \ll 1. \tag{5.4}
\]

Without loss of generality, one may assume

\[
h = h_0 = \text{real}. \tag{5.5}
\]

Then (5.2) has the equilibrium solution

\[
\bar{A} = \bar{A}_r + i\bar{A}_i, \quad \bar{A}_r = \frac{bh_0}{\omega_1}, \tag{5.6a,b}
\]

\[
B = \bar{A}_i/(a\bar{Z} + \delta + \omega |\bar{A}_r|^2), \tag{5.6c}
\]

where \( \bar{Z} \) satisfies the nonlinear algebraic equation

\[
\bar{Z} = \left[ \frac{(\omega + a) \bar{Z} + \delta}{(\omega + a)^2 + R^2} \right] + R^2 = 0. \tag{5.6d}
\]

Linearizing Eq. (5.2) about this equilibrium yields

\[
\frac{d}{d\tau} A'_r + \varepsilon_1 A'_r - \omega_1 A'_r - \omega A_0 BZ' = 0, \tag{5.7a}
\]

\[
\frac{d}{d\tau} A'_i + \varepsilon_2 A'_i + \omega_2 A'_i + aA_0 Z' = 0, \tag{5.7b}
\]

\[
\frac{d}{d\tau} Z' + \varepsilon Z' = -\bar{h}_A A'_i; \tag{5.7c}
\]

here \( A'_r \) and \( Z' \) are perturbations, \( A'_i = A'_i + iA'_i \), and

\[
\varepsilon_1 = \varepsilon(1 - 2\omega B\bar{A}_0^2), \quad \omega_1 = \omega\bar{A}_0^2 + aZ + \delta, \tag{5.7d,e}
\]

\[
\varepsilon_2 = \varepsilon(1 + 2\omega B\bar{A}_0^2), \quad \omega_2 = 3\omega\bar{A}_0^2 + aZ + \delta. \tag{5.7f,g}
\]

Neglecting terms of order \( \varepsilon^2 \) and higher, the eigenvalue equation for (5.7) is

\[
(\sigma + \varepsilon^3 + (\omega_1 - \omega_0 - a\bar{A}_0 h_r)(\sigma + \varepsilon) - \omega a\bar{A}_0 h_r = 0. \tag{5.8}
\]

Asymptotically for \( \varepsilon \to 0 \), (5.8) has three roots:

\[
\sigma_{1,2} = \pm i\sqrt{\tilde{\Delta}} - \varepsilon \left(1 + \frac{a\bar{A}_0 h_r}{2\Delta} \right), \tag{5.9a,b}
\]

\[
\sigma_3 = \varepsilon \left(-1 + \frac{a\bar{A}_0 h_r}{2\Delta} \right), \tag{5.9c}
\]

where

\[
\tilde{\Delta} = \omega_1 \omega_2 - a\bar{A}_0 h_r. \tag{5.9d}
\]

The case \( \tilde{\Delta} < 0 \) is the one of purely exponential, form-drag instability, as found by CDV and by Pedlosky (1981). When \( \varepsilon \neq 0, \tilde{\Delta} > 0 \), the real part of \( \sigma_1 = \sigma_2^* \) is always negative, provided \( \omega = 0 \), since

\[
(a\bar{Z} + \delta)^2 - a\bar{A}_0 h_r > 0
\]

implies

\[
1 + \frac{a\bar{A}_0 h_r}{2(a\bar{Z} + \delta)^2 - a\bar{A}_0 h_r} > 0.
\]

Only \( \sigma_3 \) can be either negative or positive, depending on the steady solution (5.6). This result shows clearly that form drag by itself does not induce oscillation in this case.

On the other hand, when \( \tilde{\Delta} > 0 \) and \( \omega \neq 0 \)—that is, the effect of wave–wave interaction is taken into account—the real part of \( \sigma_1 = \sigma_2^* \) becomes

\[
\sigma_{1,2} = -\left(1 + \frac{a\bar{A}_0 h_r}{2(\omega_1 \omega_2 - a\bar{A}_0 h_r)} \varepsilon \right). \tag{5.10}
\]
When \( ab \) is positive, then \( \sigma_{1,2}^{(r)} > 0 \) requires
\[
\bar{\omega}_r < 0 \quad \text{(or } \omega_1 < 0), \quad \omega_2 > 0, \quad (5.11a,b)
\]
and
\[
\frac{abh_r^2}{2} < \omega_1^2 \omega_2 < abh_r^2. \quad (5.12a,b)
\]
Therefore, a Hopf bifurcation to finite-amplitude periodic solutions is possible when
\[
\omega > 0. \quad (5.13)
\]
When \( ab < 0 \) conditions (5.11) and (5.12) are reversed, and Hopf bifurcation is possible only when \( \omega < 0 \).
The sign of \( \omega \) depends [cf. Eqs. (4.13b-d)] on the resonant wavenumber \( l \) and the basic flow structure \( g(y) \), which determine \( \phi_i(y) \). This Hopf bifurcation is subresonant or superresonant according to whether \( \omega_1 < 0 \) or \( \omega_1 > 0 \). The two cases are discussed in greater detail in appendix A.

To clarify the bifurcation criteria (5.11-13), we rewrite the system (5.2a,b) as follows:
\[
\dot{X}_1 + RX_1 - (\delta + X_3)X_2 - \bar{\omega}(X_1^2 + X_2^2)X_2 = 0,
\]
\[
(5.14a)
\]
\[
\dot{X}_2 + RX_2 + (\delta + X_3)X_1 + \bar{\omega}(X_1^2 + X_2^2)X_1 = 1,
\]
\[
(5.14b)
\]
\[
\dot{X}_3 + RX_3 = -cX_2;
\]
\[
(5.14c)
\]
\[
\text{Fig. 5. Projections onto the } (X_1, X_2) \text{-plane of trajectories of the O } (\varepsilon) \text{ system (5.14) for (a) } \delta = 1.0, \text{ (b) } \delta = 1.75, \text{ (c) } \delta = 2.05 \text{ and (d) } \delta = 2.15. \text{ Values of } R, \bar{\omega}, \text{ and } c \text{ in this and the subsequent figures are the same as in Fig. 4.}
\]

Here
\[
X_1 + iX_2 = A/bh_r, \quad X_3 = aZ, \quad (5.15a,b)
\]
\[
c = abh_r^2, \quad \bar{\omega} = \omega(bh_r)^2. \quad (5.15c,d)
\]

Following the analysis in Eqs. (5.6-5.13), it can be shown that Hopf bifurcation exists in system (5.14) when
\[
\frac{1}{3} < \bar{\omega}c < 1. \quad (5.16)
\]

Figures 4a,b show the asymptotic approach to an oscillating solution of system (5.14). For parameter values chosen as in Fig. 4, the Hopf bifurcation takes place around \( \delta = 0 \). From Fig. 4a it is clear that the real part of the wave component is predominantly negative, so that the ridge of the wave is located predominantly upstream of the topographic maximum.

The successive increase of complexity in the solutions as \( \delta \) increases is shown in Figs. 5-8. Figures 5 and 7 display phase-plane projections of trajectories for values of \( \delta \) between 1.0 and 2.15, and between 2.2 and 2.75, respectively. In Figs. 6 and 8 appear the corresponding time plots \( X_1(t) \) of one coordinate of the solution. It is clear from these figures that, as \( \delta \) increases, the wave amplitude becomes larger and that a period-doubling cascade leads to chaotic solutions (Feigenbaum 1979; Legras and Ghil 1983; LG; Lin et al. 1989). Periodic windows with periods equal approximately to an odd multiple of the basic period also appear in the chaotic parameter domain. Thus, the period in Fig. 7a

Fig. 4. A solution of the first-order system (5.14) for \( R = 0.1, \bar{\omega} = 0.5 \) and \( c = 1.0 \). Time evolution of (a) \( X_1 \), and (b) \( X_2 \). The evolution of \( X_2(t) \) (not shown) is very similar to that of \( X_1(t) \) [panel (a)], because of the symmetry of Eqs. (5.14a,b) with respect to \( X_1 \) and \( X_2 \).
is three times that in Fig. 5a, approximately (see Fig. 7 of Lin et al. 1989, and discussion there), and it gives rise to a new cascade of period doubling. It is not our main interest here to study the details of the Feigenbaum scenario for transition to turbulence in such a prototype model, since they can give us only limited insight on low-frequency oscillations in the real atmosphere.

It is worth pointing out, however, that the model is quite robust with respect to both the dominant frequency of the oscillatory solutions and the strong preference for a given geographic position of the wave. In other words, the frequency $\Delta^{1/2}$ that is determined by (5.9d) at the first Hopf bifurcation point remains the dominant frequency for all parameter values examined, even though the solutions do become quite complicated and eventually aperiodic as $\delta$ increases (compare also with Fig. 8 in Legras and Ghil 1983). Likewise, the ridge of the wave oscillates back and forth in the zonal direction, but stays almost exclusively upstream of the topography.

In system (5.2a,b) or (5.14a–c), the conventional Rossby wave frequency is given by $\delta + X_3$, being due to advection by the zonal flow. However, the modified frequency $\Delta^{1/2}$ of the oscillations related to the bifurcation above, as given by Eqs. (5.7e,g; 5.9d), is strongly dependent on the wave amplitude $A_\tau$. Hence, a nearly resonant, strong-wave solution can give a very low frequency $\Delta^{1/2}$ that is quite different from the advective frequency. Taking the supercriticality parameter $\eta$ about 0.1–0.2, which corresponds to a flow speed range of 18–28 m s$^{-1}$ for the dipole resonance, and our dimensional time unit of $\tau$ in Eqs. (4.2, 4.3) as 5–10 days, the period of oscillation in the system is about 30–60 days, which is in the intraseasonal band. This low-frequency Hopf bifurcation is a novel result of our analysis, being distinct from the one due to wave–zonal flow interaction. The latter was present already in CDV and Yoden (1985), and is discussed in the next section.

It should be pointed out that if we neglect form drag—i.e., let $h_r = 0$ in Eq. (5.7c)—the Hopf bifurcation will disappear. In this sense, form drag does play
a key role in producing low-frequency oscillations in this weakly nonlinear, barotropic β-channel model. Due, however, to their omission of wave–wave interaction, CDV (1979) and Pedlosky (1981) could not find this kind of oscillatory solution in their models.

6. Hopf bifurcations due to wave–zonal flow interaction

In this section, we discuss the complete system (4.12a,b) with \( \lambda^{-2} \neq 0 \). This system is more complicated since the zonal-flow part is no longer separable with respect to \( \tau \) and \( y \). Thus, we assume, instead of (5.1),

\[
\psi_0^{(2)}(y, \tau) = \Psi^{(2)}(y) + \sum_{i=1}^{\infty} Z_i(\tau) \cos ly. \tag{6.1}
\]

Then Eqs. (4.12a,b) can be written as

\[
\dot{A} + R' A + i\omega |A|^2 A + i\ddot{Z}(\tau) A - ib\dot{h} = 0, \tag{6.2a}
\]

\[
\dot{Z}_l + R_l Z_l = RK_l |A|^2 + C_l \text{Im}(A^* \ddot{h}), \quad l = 1, \ldots, \infty. \tag{6.2b}
\]

Fig. 7. Projection onto \((X_1, X_2)\)-plane of trajectories of the system (5.14) for (a) \( \delta = 2.2 \), (b) \( \delta = 2.3 \), (c) \( \delta = 2.35 \) and (d) \( \delta = 2.75 \).

Fig. 8. Evolution of \( X_1(t) \) corresponding to Figs. 7a–d: (a) \( \delta = 2.2 \), (b) \( \delta = 2.3 \), (c) \( \delta = 2.35 \) and (d) \( \delta = 2.75 \).
Here, $R$ is given by (4.3c), $R'$ by (4.13a) and

$$ R_l = R_l^2 / (l^2 + \lambda^{-2}), \quad \text{(6.2c)} $$

$$ K_l = -\langle Y_1(y) \cos y \rangle / (l^2 + \lambda^{-2}), \quad \text{(6.2d)} $$

$$ C_l = -\langle Y_2(y) \cos y \rangle / (l^2 + \lambda^{-2}), \quad \text{(6.2e)} $$

$$ \hat{Z} = \delta + \sum_{l=1}^{\infty} a_l Z_l, \quad \text{(6.2f)} $$

$$ a_l = k \langle \phi_l[(\cos y)\phi_l^* - k^2 \phi_l] - \phi_l(\cos y)\phi_l^* \rangle / D_l. \quad \text{(6.2g)} $$

$Y_1(y)$ and $Y_2(y)$ are given by Eqs. (4.13h,i).

Truncating the meridional expansion at $l = N$, let (6.2f) be replaced by

$$ \hat{Z}(\tau) = \delta + Z^{(N)}(\tau) = \delta + \sum_{l=1}^{N} a_l Z_l. \quad \text{(6.3)} $$

An equilibrium solution similar to (5.6) is given by

$$ Z = \delta + Z^{(N)}, \quad \text{(6.4a)} $$

$$ \bar{A} = \bar{A}_r + i e B, \quad \text{(6.4b)} $$

$$ B = \bar{A}_r / (\delta + Z^{(N)} + \omega \bar{A}_r^2), \quad \text{(6.4c)} $$

and an equation very similar to (5.6d) obtains for determining $Z^{(N)}$.

The linear perturbation equations for the stability of this equilibrium are

$$ \frac{d}{d\tau} A_l' + e A_l' - \omega_l A_l' - e B \sum_{l=1}^{N} a_l Z_l = 0, \quad \text{(6.5a)} $$

$$ \frac{d}{d\tau} A_l' + e A_l' - \omega_l A_l' - \bar{A}_r \sum_{l=1}^{N} a_l Z_l = 0, \quad \text{(6.5b)} $$

$$ \frac{d}{d\tau} Z_l + e_l Z_l = \epsilon R \Delta A, \quad l = 1, \cdots N; \quad \text{(6.5c)} $$

here,

$$ e' = R'(1 - 2 \omega B \bar{A}_r^2), \quad \text{(6.6a)} $$

$$ e'' = R'(1 + 2 \omega B \bar{A}_r^2), \quad \text{(6.6b)} $$

$$ \bar{e} = R', \quad \text{(6.6c)} $$

$$ e_l = R_l, \quad \text{(6.6d)} $$

$$ \omega_l = \omega A_r^2 + \delta + Z^{(N)}, \quad \text{(6.6e)} $$

$$ \omega_l = 3 \omega A_r^2 + \delta + Z^{(N)}. \quad \text{(6.6f)} $$

A general eigenanalysis of system (6.5) is quite difficult. However, an asymptotic analysis of Hopf bifurcation is still possible in the weak friction limit, $R' = \epsilon \ll 1$.

Neglecting some higher-order terms, an asymptotically valid eigenvalue equation is

$$ (\sigma + \bar{e})^{N-1} \left[ (\sigma + \bar{e}) \left( \sigma + e' \right) \right. $$

$$ + \omega_1 \omega_2 (\sigma + \epsilon) - \epsilon \omega_2 \bar{B} \sum_{l=1}^{N} a_l c_l \frac{\sigma + \bar{e}}{\sigma + \epsilon_l} $$

$$ - A_r \bar{h}(\sigma + \epsilon_1) \sum_{l=1}^{N} a_l c_l \frac{\sigma + \bar{e}}{\sigma + \epsilon_l} $$

$$ - 2 A_r^2 \omega_1 \epsilon_1 \sum_{l=1}^{N} K_l \delta_l \frac{\sigma + \bar{e}}{\sigma + \epsilon_l} \right] = 0. \quad \text{(6.7)} $$

Setting all $\bar{e}$, $e'$, $\epsilon_1$, $\epsilon_l$ to zero, one obtains

$$ \sigma_j = 0, \quad i = 1, \cdots, N, \quad \text{(6.8a)} $$

$$ \sigma^2 + \delta = 0, \quad \delta = \omega_1 \omega_2 - A_r \bar{h} \sum_{l=1}^{N} a_l c_l $$

$$ = \omega_1 \omega_2 - \bar{a} \bar{A}_r \bar{h}. \quad \text{(6.8b,c)} $$

These results are similar to the first-order roots in system (5.8), with $\bar{a}$ defined in the obvious way by (6.8c). To obtain the roots for small $\epsilon$, we assume

$$ \sigma_j = \sigma_j^{(0)} + \epsilon \sigma_j^{(1)} + \cdots, \quad i = 1, \cdots, N + 2. \quad \text{(6.9a)} $$

Because we are interested in Hopf bifurcation, only the $\epsilon$-correction to the pair of purely imaginary eigenvalues $\sigma_{1,2} = \pm i \sqrt{\delta}$ is considered here. When $\epsilon \rightarrow 0$ and $\delta \neq 0$, for $j = 1, 2$ and $1 \leq l \leq N$,

$$ \frac{\epsilon_j + \bar{e}}{\epsilon_j + \epsilon_l} \sim 1, \quad \bar{e} = \sum_{l=1}^{N} \epsilon_l / N. \quad \text{(6.9b)} $$

Then,

$$ \sigma_{1,2} = \pm \sqrt{\delta} - \epsilon \quad \text{(6.9c)} $$

$$ \times \left[ \frac{\bar{a} \bar{A}_r \bar{h} \left( \frac{1 + \epsilon_j - \epsilon}{\epsilon} \right) - \epsilon / \epsilon_2 \omega_1 \bar{K} A_r^2}{1 + \frac{\epsilon_2 \omega_2}{2 \Delta}} \right]. \quad \text{(6.9d)} $$

where $\bar{K} = \sum_{l=1}^{N} K_l \delta_l$. When $\bar{K} = 0$ and $\bar{e} = \epsilon$, (6.9) becomes the same as Eqs. (5.9a) and (5.10). However, when $\lambda^{-2} \neq 0$ and $g \neq 1$, then $\bar{K} \neq 0$, so even for $\omega = 0$ it is still possible to have

$$ Re \sigma_{1,2} > 0. \quad \text{(6.10a)} $$

if

$$ \omega_1 \bar{K} > 0. \quad \text{(6.10b)} $$

This combination of inequalities, distinct from (5.16), outlines another mechanism for Hopf bifurcation. Because $\bar{K}$ is related to meridional momentum transport by the forced wave [cf. Eqs. (6.2d,g)] and this momentum transport typically results in the zonal jet shifting its location, we refer to the related Hopf bifurcation as jet-shifting bifurcation.
Since $\omega_1 \bar{K} > 0$ is a precondition for the bifurcation, the latter is subresonant when $\bar{K} < 0$ and superresonant when $\bar{K} > 0$. The frequency associated with this bifurcation is low compared to synoptic frequencies for reasons similar to those discussed at the end of section 5. Hence, this interaction mechanism between a wave and the zonal flow can yield low-frequency oscillations when more than one meridional mode is considered in a CDV-type model. CDV noted the presence of a periodic solution in their model when using two, rather than one, meridional mode. Yoden (1985) investigated numerically Hopf bifurcation in the CDV model with two meridional modes, and LG allowed nine meridional modes, as well as wave–wave interaction. The analytic study here emphasizes that both form drag and wave–mean flow interactions are responsible for the existence of this kind of Hopf bifurcation.

We shall not investigate here the detailed parameter dependence of solutions to system (6.2a,b), but show an example of periodic solution. To do so, we use the following reduced system that can be shown to possess the same bifurcation (by the Center Manifold Theorem, cf. Carr 1981; or Ghil and Childress 1987, section 12.2):

$$\dot{X}_1 + RX_1 - (\delta + X_3)X_2 - \bar{\omega}(X_1^2 + X_2^2)X_2 = 0,$$

(6.11a)

$$\dot{X}_2 + RX_2 - (\delta + X_3)X_1 - \bar{\omega}(X_1^2 + X_2^2)X_1 = 1,$$

(6.11b)

$$\dot{X}_3 + RX_3 = -cX_2 - R(KX_1^2 + X_2^2).$$

(6.11c)

This system differs from (5.14) by the presence of the last term in (6.11c), which incorporates the effect of zonal flow–wave interaction due to wave–momentum transport. As seen from Fig. 9, even if $\omega = 0$, Hopf bifurcation still occurs in (6.11), which confirms our analysis of Eqs. (6.9, 6.10).

The nondimensional period of the solution is very close to the basic period in section 5. The combination of wave–wave interaction and wave–zonal flow interaction will alter the trajectories in phase space, and higher-order resonances may arise. But the basic features of the oscillations discussed in section 5 and here are robust over a wide range of parameter values. In particular, we notice from Fig. 9 that the ridge of the wave is still upstream of the topographic ridge throughout the entire cycle.

7. Discussion and conclusions

Low-frequency oscillations with periods of 30–60 days over the NH middle latitudes have been identified in 700 mb geopotential height data (Ghil and Mo 1990) and in GCM simulations (Marcus et al. 1990), as well as in simple models (LG; Keppenne 1989; Tribbia and Ghil 1990). Ghil (1987), based on results from Legras and Ghil (1985: abbreviated here as LG) and personal communications with M. Kimoto and K. Mo (both in 1986), hypothesized that topographically induced instability of nonzonal midlatitude flow is responsible for extratropical low-frequency oscillations. We have shown here that form-drag–related Hopf bifurcation can be due to either jet-shifting wave–zonal flow interaction or to wave–wave interactions. Both types of bifurcation give rise to a low frequency that is determined by the amplitude of the topographically forced wave and by the zonal flow structure.

Namias (1950) in his study of the 4–6 week index cycle during NH winter showed that there is a very significant shifting of the subtropical jet maximum (his Fig. 5) associated with the cycle’s change in value of the index. The meridional structure and oscillatory nature of LG’s second resonance in the NH extratropics resembles qualitatively these observed features. As pointed out in section 3 here, and emphasized already by LG, this resonance requires lower, more realistic values of mean zonal wind than their first, CDV-like resonance. Hence, the former is more likely to occur in the atmosphere than the latter.

In the weakly nonlinear, barotropic framework of this paper, a zonally asymmetric vorticity forcing $\nabla^2 \psi(x, y)$ of thermal origin, say, can replace topography and still yield a system of PDEs similar to (4.12a,b) by dropping the form-drag term in Eqs. (2.1, 4.12a,b), and substituting for $ibh$ in (4.12a) the amplitude of the vorticity forcing. Then the Hopf bifurcation discussed in section 5 disappears, although nonlinear resonance through wave–wave interaction is still
possible. The Hopf bifurcation due to the wave–zonal flow interaction, however, is present [cf. (6.10, 6.11)].

The only way to distinguish between the two types of oscillations with jet shifting, of topographic and of thermal origin, would be in the framework of a baroclinic model.

The low-frequency Hopf bifurcations analyzed here are truly nonlinear in the sense that both zonal and wave components of the circulation must be involved simultaneously. Simmons et al. (1983) found linearly unstable low-frequency modes in a barotropic model without topography, but with a nonzonal climatological basic flow: their underlying idea was that such a flow would be maintained by topography and vorticity forcing in a nonlinear barotropic model. We have shown here, however, that form drag and eddy-momentum transport can modify the zonal component of the flow in an oscillatory, rather than steady state fashion. Any oscillation in atmospheric angular momentum (AAM) is a wavenumber zero phenomenon by necessity. The 30–60 day oscillation in AAM, on the one hand, and in eddy geopotential fields, on the other, are thus only two different aspects of the complete intraseasonal oscillatory phenomenon in the NH extratropics. Dickey et al. (1990) have analyzed AAM data in separate latitude bands and have shown, in fact, that a 40-day oscillation in the NH extratropics is strongest in the 26°N–44°N belt, where root-mean-square topography is largest.

The low-frequency oscillations revealed by the present analysis exhibit quite different flow patterns and physical processes interacting with the topography for distinct resonances. If the ridge of the flow's wave component is not in phase with the wavy shaped mountain sitting in the flow field as in Fig. 2, then a nonzero zonally averaged mountain torque results, accelerating or decelerating the zonal flow. For the flow pattern associated with the first, CDV-type resonance (Fig. 2a), this mountain torque does not change sign across the channel (Fig. 2c), and the amplitude of the zonal flow is effectively modified. For the second, LG-type resonance (Fig. 2b), the associated mountain torque can change sign across the channel (Fig. 2d), tendency to shift meridionally the zonal jet maximum, rather than to modify its amplitude. This results in an oscillatory component X(t) in the periodic solution as shown in Fig. 4b [see Eqs. (5.1, 5.15c)].

Figure 10 shows streamfunction plots including the two-mode basic flow, as used in Fig. 1b, and the first-order wave solution of Fig. 5a, corresponding to the second resonance in Eqs. (5.2). The four states (a, b, c, d) correspond to the uppermost, rightmost, lowermost and leftmost points on the trajectory in Fig. 5a. The cycle includes pronounced phases of zonal flow (Fig. 10b, near the origin in Fig. 5a) and of blocking (Fig. 10d, farthest from the basic flow in Fig. 5a). Blocking features are stronger and more pronounced when they occur upstream of the mountain (Fig. 10d) than when they occur downstream (Fig. 10a). The back-and-forth movement of the wave field relative to the mountain results in the acceleration and deceleration of the zonal-flow component, forming part of the oscillatory phenomenon.

As pointed out in section 4, the second-order wave field correction is substantial for the LG-type, dipole resonance. Adding this correction to the plots of Figs. 10a–d results in Figs. 11a–d. The overall features of the oscillation are the same, including the transition between zonal flow (Fig. 11b) and blocking (Fig. 11d), and the shifting of the wave across the topography. The interference between wavenumbers and makes the field slightly more complicated and realistic. The upstream blocking (Fig. 11d) is stronger, more localized and shifted northward, while the downstream blocking (Fig. 11a) is about the same as in Fig. 10a; i.e., even weaker with respect to Fig. 11d.

The wave–wave interaction process tends to give rise to more localized, spatially aperiodic patterns. It can be shown that, when the channel width becomes narrow or, in other words, the zonal wavenumber approaches 0, the nonlinear wave equation (4.12a) will be replaced.
by a KdV-type equation. This kind of localization due to nonlinearity may balance the linear dispersion of Rossby waves (Patioine and Warn 1982; Malguzi and Malanote–Rizzoli 1984). It is plausible to conjecture, therefore, that the low-frequency Hopf bifurcation due to wave–wave interaction might be as important as the jet-shifting bifurcation, or even more so.

The oscillations discussed qualitatively here exhibit certain features that do not agree entirely with observational and GCM results (Ghil and Mo 1990; Marcus et al. 1990). The latter are characterized largely, but not exclusively, by a geographically fixed, standing wave pattern dominated by zonal wavenumber two. The oscillatory pattern in our simple, weakly nonlinear barotropic model moves back and forth across the mountain in better agreement with the results of the fully nonlinear, high-resolution barotropic model of Tribbia and Ghil (1990). This discrepancy between the barotropic results and the observed or GCM-simulated atmosphere may be due to the lack of capability of barotropic models to include transients resulting from baroclinic instability and asymmetric thermal forcing.

Lau (1988) recently showed that there is significant low-frequency variability in the second-moment statistics of the transients over the storm-track regions. Both topography and thermal forcing are important for the stationary planetary waves that determine the position and intensity of the storm tracks (Held 1983), while the synoptic waves localized along the latter could interact through wave–wave interaction with the forced planetary waves. Low-frequency, oscillatory interactions between ultralong topographic waves and shorter baroclinic waves have also been observed in rotating-annulus experiments with simple bottom topography (Bernardet et al. 1990). We expect to pursue both nonlinear localization and baroclinic aspects in future work on low-frequency atmospheric variability.

In summary we conclude that

- the dipole resonance associated with a second meridional mode appears to be more important than the monopole, CDV-type resonance for realistic midlatitude flows;
- form-drag-related Hopf bifurcations are possible through both wave–zonal flow and wave–wave interactions.

These Hopf bifurcations lead to periodic or nearly periodic oscillations with a robust low frequency, which is determined by the topographic forcing and zonal flow structure. This frequency belongs to the intraseasonal band for a wide range of relevant parameters. The three-way interaction of zonal flow with a complex meridional structure, of finite-amplitude waves on such a zonal flow, and of midlatitude topography seems to provide a plausible mechanism for an inherently extratropical low-frequency oscillation. A more complete understanding of extratropical variability in the intraseasonal band requires a more complete, baroclinic and thermally active, description of the dynamics as well as a study of interactions with tropical variability in the same band.

Acknowledgments. We owe a debt of gratitude to M. Kimoto and K. Mo, who helped set us on the right track. The LXT-ISO club provided stimulation and encouragement. The manuscript was improved by comments from P. Bernardet, R. Pierrehumbert, and three anonymous referees. B. Gola, C. Monroe, and C. Wong turned many illegible drafts and even less legible sets of corrections into a neat typescript. NSF and NASA supported this work through Grants ATM86-15424, ATM90-13217 and NAG-5713.

APPENDIX A

Hopf Bifurcations, Resonance and Criticality

We briefly describe here two basic concepts used throughout this paper. To illustrate the difference between sub- and super-resonance of a Hopf bifurcation, let us consider the simple linear equation

\[ U_0 \frac{\partial}{\partial x} \zeta' + \beta \frac{\partial \psi'}{\partial x} = -U_0 \frac{\partial h}{\partial x}, \quad (A.1) \]

where \( U_0 \) is a uniform basic flow. For a wave-like topographic forcing and the associated streamfunction solution

\[ \begin{pmatrix} \psi' \\ h \end{pmatrix} = \begin{pmatrix} A \\ h_k \end{pmatrix} e^{ikx} \sin ly; \quad (A.2a,b) \]

we have

\[ A = h_0 / (U_0 - U_s), \quad U_s = \beta / (k^2 + l^2), \quad (A.3a,b) \]

where \( U_s \) is the resonant flow speed. The solution is subresonant or super-resonant, cf. Figs. A1a,b, respectively, when \( U_0 < U_s \) or \( U_0 > U_s \). Passage through resonance corresponds to a change in the phase of the wave with respect to the topography by \( \pi \) (e.g., Ghil and Childress 1987, section 6.3). The presence of dissipation and nonlinearity will shift the flow patterns in the two panels somewhat.

Fig. A1a,b. Schematic flow patterns at (a) subresonance, and (b) superresonance. Positive topography is shaded.
The purely exponential topographic instability found by CDV can be explained dynamically in a simple way (for an alternative simple explanation, see Revell and Hoskins 1984). The super-resonant topographically forced flow pattern shown in Fig. A1b is clearly unstable. A westerly perturbation on the flow pattern in this panel will produce, due to the anomalous advection, a wave perturbation with high pressure east of the mountain and a low east of the valley. This perturbation results in an eastern mountain torque acting on the atmosphere, so as to accelerate the westerly perturbation further. A reverse, easterly perturbation on the super-resonant flow will lead, likewise, to easterly acceleration. Purely exponential topographic instability is the result. It is easy to show in the same idealized way that the subresonant flow is stable.

In section 5, it is shown, however, that phase shifts by friction and by wave–wave interaction can produce a subresonant oscillatory instability near both the CDV and LG resonances. In section 6 we show, moreover, that friction and wave–zonal flow interaction can produce a phase shift that leads to a super-resonant Hopf bifurcation near the monopole resonance, and a subresonant one near the dipole resonance; the latter requires less forcing and hence is more realistic than the former.

Subcritical and supercritical Hopf bifurcation are entirely different in nature from the two types of exponentially growing, stationary resonance described above. For clarity, a quick description is given here; details can be found in Ghil and Childress (1987, section 12.2) and elsewhere (see references there).

For a dynamical system with a control parameter \( \mu \), Hopf bifurcation takes place when a steady solution of the system becomes unstable as a pair of complex eigenvalues \( \sigma = \sigma_r \pm i\sigma_i \) of the linearization about that solution has \( \sigma_r(\mu) > 0 \) and \( \sigma_i = \pm \omega \) for \( \mu > \mu_c \), and while all the other eigenvalues have negative real part. The \( \mu_c \) characterizes the so-called bifurcation point, where \( \sigma_r(\mu_c) = 0 \), while \( \sigma_r(\mu) < 0 \) for \( \mu < \mu_c \). Super-critical Hopf bifurcation is the case shown in Fig. A2a in which a stable, periodic solution appears just when \( \mu \geq \mu_c \). In the subcritical case, a periodic solution appears even for \( \mu < \mu_c \), but is unstable near the bifurcation point (Fig. A2b). In the present paper \( \eta = \mu - \mu_c = O(\epsilon^2) \) and all bifurcations are supercritical.

**APPENDIX B**

**General Formalism for Weakly Nonlinear Analysis**

The general procedure for the derivation in section 4 can be outlined as follows (e.g. Nayfeh, 1973). Substituting (4.3) into (2.1) and rearranging yields

\[ \mathcal{F}(0) + \epsilon \mathcal{F}(1) + \epsilon^2 \mathcal{F}(2) + \cdots = 0. \]  

(B.1)

Hence, \( \epsilon \) being arbitrary,

\[ \mathcal{F}(i) = 0, \quad i = 0, 1, 2, \ldots \]  

(B.2)

In this system \( \mathcal{F}(0) = 0 \), and \( \mathcal{F}(1) \) gives a homogeneous equation

\[ \mathcal{L} \psi^{(1)} = 0, \]  

(B.3)

where \( \mathcal{L} \) is the linear operator defined by Eq. (4.4). Equation (B.3) has a unique nontrivial solution with undetermined amplitude. The higher-order balances can be written as

\[ \mathcal{L} \psi^{(n)} = G_n(\psi^{(n-1)}, \psi^{(n-2)}, \ldots, \psi^{(1)}), \]  

\[ n = 2, 3, \ldots \]  

(B.4)

In general, the “forcing” term \( G_n \) does not vanish identically, due to nonlinearity and external forcing. In order to make the expansion (4.3) valid; i.e.,

\[ O(\psi^{(n)})/O(\psi^{(n-1)}) \leq M_n < \infty; \]  

(B.5)

it is required that the apparent “forcing” term be perpendicular to the first-order solution,

\[ \langle \langle \psi^{(1)} G_n \rangle \rangle = 0; \]  

(B.6)

here \( \langle \langle \cdots \rangle \rangle = \int_0^{2\pi} \int_0^\infty (\ldots) dx dy / 4\pi^2 \) is the average over the whole flow domain and \( \psi^{(1)*} \) is the complex conjugate of \( \psi^{(1)} \). Equation (B.6) is the so-called solvability condition. These conditions for \( n = 2, 3, \ldots \), \( n_0 \) provide the equations necessary to determine the solution we are interested in up to order \( n_0 - 1 \).

**REFERENCES**
