

ELASTICITY PROBLEMS IN POLAR COORDINATES (10)

I Main topics

- A Motivation
- B Cartesian Approach
- C Transformation of coordinates
- D Equilibrium equations in polar coordinates
- E Biharmonic equation in polar coordinates
- F Stresses in polar coordinates

II Motivation

- A Many key problems in geomechanics (e.g., stress around a borehole, stress around a tunnel, stress around a magma chamber) involve cylindrical geometries. Our reference frame should fit the features we examine.
- B Introduces the concept of stress concentration due to factors inside a body (as opposed to concentrated boundary loads).

III Approach

- A Transform elastic equations from xy form to polar form
- B Alternative: vector and tensor approaches (see C&P, Ch. 11)

IV Transformation of coordinates (See Fig. 34.1)

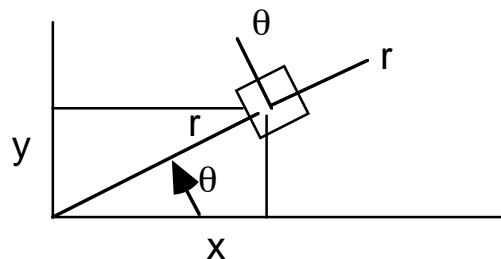
- A Rectangular to polar:

$$\theta = \tan^{-1}(y/x) \qquad r = (x^2 + y^2)^{1/2} \qquad (10.1)$$

- B Polar to rectangular:

$$x = r \cos\theta \qquad y = r \sin\theta \qquad (10.2)$$

- C Stress convention: Still use on-in convention ($\theta_{rr}, \theta_{r\theta}, \theta_{\theta r}, \theta_{\theta\theta}$)



V Equilibrium equations in polar coordinates

$$\sum F_r = 0 \quad (10.3)$$

We will assume body forces are negligible.

General form of the terms is $\sigma_{ij} = \{\sigma_{ij} \text{ ref} + (\sigma_{ij} \text{ gradient})(\text{distance})\}$.

$$\begin{aligned} & \left(\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r} \frac{dr}{2} \right) \left(r + \frac{dr}{2} \right) d\theta - \left(\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r} \frac{dr}{2} \right) \left(r - \frac{dr}{2} \right) d\theta + \\ & \cos \frac{d\theta}{2} \left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta} \frac{d\theta}{2} \right) dr - \cos \frac{d\theta}{2} \left(\sigma_{r\theta} - \frac{\partial \sigma_{r\theta}}{\partial \theta} \frac{d\theta}{2} \right) dr + \\ & - \sin \frac{d\theta}{2} \left(\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} \frac{d\theta}{2} \right) dr - \sin \frac{d\theta}{2} \left(\sigma_{\theta\theta} - \frac{\partial \sigma_{\theta\theta}}{\partial \theta} \frac{d\theta}{2} \right) dr = 0. \end{aligned} \quad (10.4)$$

This equation reduces to

$$\begin{aligned} & \left(2 \frac{\partial \sigma_{rr}}{\partial r} \frac{dr}{2} r d\theta \right) + \left(2 \sigma_{rr} \frac{dr}{2} d\theta \right) + \left(2 \frac{\partial \sigma_{rr}}{\partial \theta} \frac{dr}{2} \frac{dr}{2} d\theta \right) \\ & + \left(2 \frac{\partial \sigma_{r\theta}}{\partial \theta} \frac{d\theta}{2} dr \cos \frac{d\theta}{2} \right) - \left(2 \sigma_{\theta\theta} dr \sin \frac{d\theta}{2} \right) = 0 \end{aligned} \quad (10.5)$$

Dividing through by dr , and noting that for small angles $\sin(d\theta/2) \approx d\theta/2$ and $\cos(d\theta/2) \approx 1$, equation (10.5) reduces to

$$\left(\frac{\partial \sigma_{rr}}{\partial r} r d\theta \right) + \left(\sigma_{rr} d\theta \right) + \left(\frac{\partial \sigma_{rr}}{\partial r} \frac{dr^2}{2} d\theta \right) + \left(\frac{\partial \sigma_{r\theta}}{\partial \theta} d\theta \right) - \left(2 \sigma_{\theta\theta} \frac{d\theta}{2} \right) = 0. \quad (10.5)$$

The third term drops out because dr^2 is tiny relative to the other terms.

Dividing this (10.5) through by $r d\theta$ yields

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} + \frac{1}{r} \frac{\partial \sigma_{rr}}{\partial \theta} - \frac{\sigma_{\theta\theta}}{r} = 0. \quad (10.6)$$

This expression is commonly rearranged as (see eq. 5.41 in C&P):

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{rr}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0. \quad (10.7)$$

By summing the forces in the θ -direction one can obtain

$$\frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0. \quad (10.8)$$

Equations (10.7) and (10.8) are the equilibrium equations in polar coordinates.

VI Biharmonic equation in polar coordinates

Our starting point is the biharmonic equation

$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = 0, \quad (10.9)$$

and the stresses in terms of the stress function ϕ :

$$(a) \quad \sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad (b) \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad (c) \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \quad (10.10)$$

)

In order to transform these equations to polar form, we need to know how to express derivatives with respect to x and y in terms of r and θ . We get these relationships from the chain rule:

$$(a) \quad \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta} \frac{\partial \theta}{\partial x} \quad (b) \quad \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta} \frac{\partial \theta}{\partial y} \quad (10.11)$$

)

We therefore need to know how r and θ relate to x and y .

$$(a) \quad \theta = \tan^{-1}(y/x) \quad (b) \quad r = (x^2 + y^2)^{1/2} \quad (10.12)$$

)

$$(a) \quad x = r \cos \theta \quad (b) \quad y = r \sin \theta \quad (10.13)$$

)

Now to the derivatives. Starting with eq. (10.12b)

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} (2x) = \frac{x}{(x^2 + y^2)^{1/2}} = \frac{x}{r} = \cos \theta$$

(10.14)

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-1/2} (2y) = \frac{y}{(x^2 + y^2)^{1/2}} = \frac{y}{r} = \sin \theta$$

(10.15)

Now take derivatives of eq. (10.12a). Recalling that $d(\tan^{-1}u) = du/(1+u^2)$

$$\frac{\partial \theta}{\partial x} = \frac{\partial(\tan^{-1}(y/x))}{\partial x} = \frac{1}{\left(1 + \left(\frac{y}{x}\right)^2\right)^2} \frac{\partial(y/x)}{\partial x} = \frac{1}{\left(1 + \left(\frac{y}{x}\right)^2\right)^2} \frac{-y}{x^2} = \frac{-y}{r^2} = \frac{-\sin \theta}{r}$$

(10.16)

Similarly

$$\frac{\partial \theta}{\partial y} = \frac{\partial(\tan^{-1}(y/x))}{\partial y} = \frac{1}{\left(1 + \left(\frac{y}{x}\right)^2\right)^2} \frac{\partial(y/x)}{\partial y} = \frac{1}{\left(1 + \left(\frac{y}{x}\right)^2\right)^2} \frac{1}{x} \frac{x}{x} = \frac{x}{r^2} = \frac{\cos \theta}{r}$$

(10.17)

Now we can find the derivatives of ϕ in terms of r and θ by substituting eqs. (10.16) and (10.17) into chain rule equations (10.11):

$$(a) \quad \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \cos \theta - \frac{\partial \phi}{\partial \theta} \frac{\sin \theta}{r}, \quad (b) \quad \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r} \sin \theta + \frac{\partial \phi}{\partial \theta} \frac{\cos \theta}{r}.$$

(10.18)

Second derivatives are found by operating on first derivatives.

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \left(\frac{\partial \phi}{\partial x} \right)}{\partial x} = \frac{\partial \left(\frac{\partial \phi}{\partial r} \right)}{\partial r} \cos \theta - \frac{\partial \left(\frac{\partial \phi}{\partial x} \right)}{\partial \theta} \frac{\sin \theta}{r}. \quad (10.19)$$

)

Substituting eq. (10.18a) into eq. (10.19) yields

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial \left(\frac{\partial \phi}{\partial x} \right)}{\partial x} = \frac{\partial \left(\frac{\partial \phi}{\partial r} \cos \theta - \frac{\partial \phi}{\partial \theta} \frac{\sin \theta}{r} \right)}{\partial r} \cos \theta - \frac{\partial \left(\frac{\partial \phi}{\partial r} \cos \theta - \frac{\partial \phi}{\partial \theta} \frac{\sin \theta}{r} \right)}{\partial \theta} \frac{\sin \theta}{r}. \quad (10.20)$$

)

This can be expanded as

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} = & \frac{\partial^2 \phi}{\partial r^2} \cos^2 \theta - \frac{\partial^2 \phi}{\partial \theta \partial r} \frac{\sin \theta \cos \theta}{r} + \frac{\partial \phi}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2} \\ & - \frac{\partial^2 \phi}{\partial \theta \partial r} \frac{\sin \theta \cos \theta}{r} + \frac{\partial \phi}{\partial r} \frac{\sin^2 \theta}{r} + \frac{\partial^2 \phi}{\partial \theta^2} \frac{\sin^2 \theta}{r^2} + \frac{\partial \phi}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2} \end{aligned} \quad (10.21)$$

)

The expression for $\frac{\partial^2 \phi}{\partial y^2}$ can be found by the same procedure:

$$\begin{aligned} \frac{\partial^2 \phi}{\partial y^2} = & \frac{\partial^2 \phi}{\partial r^2} \sin^2 \theta + \frac{\partial^2 \phi}{\partial \theta \partial r} \frac{\sin \theta \cos \theta}{r} - \frac{\partial \phi}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2} \\ & + \frac{\partial^2 \phi}{\partial \theta \partial r} \frac{\sin \theta \cos \theta}{r} + \frac{\partial \phi}{\partial r} \frac{\cos^2 \theta}{r} + \frac{\partial^2 \phi}{\partial \theta^2} \frac{\cos^2 \theta}{r^2} - \frac{\partial \phi}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2} \end{aligned} \quad (10.22)$$

)

Equation (10.21) is the expression for σ_{yy} and eq. (10.22) is the expression for σ_{xx} . Notice the term-by-term "symmetry" between these two equations.

These equations can be added to give the expression for $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$.

Recalling that $\sin^2 \theta + \cos^2 \theta = 1$, we get

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = \nabla^2 \phi. \quad (10.23)$$

)

The biharmonic equation is obtained by allowing the harmonic equation (10.23) to operate on itself.

$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \phi = 0. \quad (10.24)$$

)

VII Stresses in polar coordinates

We are now left with the problem of how to determine the stresses in polar coordinates from the stress function ϕ . We know that the mean normal stress (and hence twice the mean stress) is an invariant term - it does not depend on the choice of the system of coordinates. As a result

$$2\sigma_{mean} = \sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\theta\theta} = \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = \nabla^2 \phi. \quad (10.25)$$

)

By comparing equations (10.23 and 10.25) we know that if one of the terms on the right side of eq. (10.24) equals $\sigma_{\theta\theta}$, then the other terms must sum to equal σ_{rr} . If we can show this for any particular position (r, θ) , then the result will hold for all positions. We therefore choose a simple case. Let the r -direction be along the x -axis and the θ -direction be parallel to the y -axis, so $\sigma_{rr} = \sigma_{xx}$ and $\sigma_{\theta\theta} = \sigma_{yy}$. The x -axis corresponds to $\theta = 0^\circ$ and the y -direction to $\theta = 90^\circ$. So

$$\frac{\partial^2 \phi}{\partial x^2} = \sigma_{yy} = \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \cos^2 \theta - \frac{\partial^2 \phi}{\partial \theta \partial r} \frac{\sin \theta \cos \theta}{r} + \frac{\partial \phi}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2} - \frac{\partial^2 \phi}{\partial \theta \partial r} \frac{\sin \theta \cos \theta}{r} + \frac{\partial \phi}{\partial r} \frac{\sin^2 \theta}{r} + \frac{\partial^2 \phi}{\partial \theta^2} \frac{\sin^2 \theta}{r^2} + \frac{\partial \phi}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2}. \quad (10.26)$$

)

For our case of $\theta = 0^\circ$, all terms with $\sin \theta$ are zero and this simplifies to

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \quad (10.27)$$

)

Similarly

$$\begin{aligned} \frac{\partial^2 \phi}{\partial y^2} = \sigma_{xx} = \sigma_{rr} = & \frac{\partial^2 \phi}{\partial r^2} \sin^2 \theta + \frac{\partial^2 \phi}{\partial \theta \partial r} \frac{\sin \theta \cos \theta}{r} - \frac{\partial \phi}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2} \\ & + \frac{\partial^2 \phi}{\partial \theta \partial r} \frac{\sin \theta \cos \theta}{r} + \frac{\partial \phi}{\partial r} \frac{\cos^2 \theta}{r} + \frac{\partial^2 \phi}{\partial \theta^2} \frac{\cos^2 \theta}{r^2} - \frac{\partial \phi}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2} \end{aligned} \quad (10.28)$$

)

Again $\theta = 0^\circ$, so all terms with $\sin \theta$ are zero, and this simplifies to

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}. \quad (10.29)$$

)

Note that $\sigma_{\theta\theta} + \sigma_{rr}$ does indeed equal $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$.The shear stress $\tau_{r\theta}$ can be determined from σ_{xy} and is given by

$$(a) \quad \sigma_{r\theta} = \frac{-\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) \quad \text{or} \quad \sigma_{r\theta} = \frac{-1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial \phi}{\partial \theta}. \quad (10.30)$$

)

Stresses in Polar Coordinates

$$r = (x^2 + y^2)^{1/2}$$

$$\theta = \tan^{-1}(y/x)$$

