II Review of key points

A Heat flow analogous to flow of fluid during consolidation
\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \quad \frac{\partial u_{\text{excess}}}{\partial t} = C_v \frac{\partial^2 u_{\text{excess}}}{\partial z^2} \]

<table>
<thead>
<tr>
<th>Heat flow</th>
<th>Fluid flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>(u_{\text{excess}})</td>
</tr>
<tr>
<td>Temperature</td>
<td>Excess pore pressure</td>
</tr>
<tr>
<td>t</td>
<td>t</td>
</tr>
<tr>
<td>Time</td>
<td>Time</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>(C_v)</td>
</tr>
<tr>
<td>Thermal diffusivity</td>
<td>Coefficient of consolidation</td>
</tr>
<tr>
<td>z</td>
<td>z</td>
</tr>
<tr>
<td>position</td>
<td>position</td>
</tr>
</tbody>
</table>

Here \(T_v = t^* = t/(H^2/C_v)\)
II Review of key points

B Darcy’s law  
\[ q = \frac{Q}{A} = -k \frac{\partial H}{\partial x} \]
where
\( q \) = flux (m/sec)
\( Q \) = discharge rate (m³/sec)
\( A \) = area (m²)
\( K \) = hydraulic conductivity (m/sec)
\( \partial H/\partial x \) = head gradient (dimensionless)

C Darcy’s law (alternative form)  
\[ Q = -kA \frac{\Delta H}{\Delta L} \]
where
\( Q \) = discharge rate (m³/sec)
\( k \) = hydraulic conductivity (m/sec)
\( A \) = area (m²)
\( \Delta H \) = change in head (m)
\( \Delta L \) = length of flow path (m)

\( k \) is a function of the fluid and the porous medium

Relationship between hydraulic conductivity and intrinsic permeability

\[ k = K \frac{\rho g}{\mu} \]
\[ K = k \frac{\mu}{\rho g} \]

where
\( k \) = hydraulic conductivity (m/sec)
\( K \) = intrinsic permeability (m²)
\( \rho \) = density of the fluid (kg/m³)
\( g \) = acceleration due to gravity (m/sec²)
\( \mu \) = dynamic viscosity of fluid [kg/(m·sec)]
II Review of key points

B Controls on 1-D consolidation time \((t_c)\) of soils and sediments
1 The time for consolidation should scale with the volume of water squeezed out of the soil
2 The volume of expelled water is the product of three terms
   a Effective stress change \((\Delta \sigma')\)
   B Compressibility of the solid skeleton (i.e., \(m_v\), the coefficient of volume change)
   C Volume of the consolidating layer (or height \(H\) of the layer)

\[ t_c \propto \Delta \sigma' m_v H \]

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II Review of key points

B Controls on 1-D consolidation time \((t_c)\) of soils and sediments (cont.)
3 The time [for consolidation] should be inversely proportional to how fast the water can flow through the consolidating layer
4 From Darcy's law: the velocity of flow is scales with the product of the permeability \((k)\) and the hydraulic gradient
5 The hydraulic gradient matches the excess fluid pressure lost within the layer divided by the distance the pore fluid flows \((H)\)
6 The excess fluid pressure loss scales with \(\Delta \sigma'\)

\[ t_c \propto \frac{1}{k \Delta \sigma'/H} = \frac{H}{k \Delta \sigma'} \]
II Review of key points

B Controls on 1-D consolidation time ($t_c$) of soils and sediments (cont.)

7 $t_c$ increases with increasing compressibility (i.e., with increasing $m_v$)

8 $t_c$ increases rapidly with increasing volume (height) of the soil/sediment mass ($H$)

9 $t_c$ decreases with increasing permeability ($k$)

10 $t_c$ is independent of the magnitude of the effective stress change, assuming $m_v$ is a constant

$$t_c \propto \frac{\Delta \sigma' m_v H}{k \Delta \sigma'/H} = \frac{m_v H^2}{k}$$

II Review of key points

C Dimensional analysis illuminates consolidation time

1 1-D consolidation equation

$$\frac{\partial u}{\partial t} = C_v \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\text{pressure}}{\text{time}} = [c_v] \frac{\text{pressure}}{\text{length}^2} \Rightarrow [c_v] = \frac{\text{length}^2}{\text{time}}$$

where

$C_v = \text{coefficient of consolidation}$

$$= \frac{k}{\rho_{\text{water}} g m_v}$$

$m_v = \text{coefficient of volume change}$
II Review of key points

2 The dimensionless time used in dimensionless expressions for consolidation is found in the same way as in lecture 38

<table>
<thead>
<tr>
<th>Heat flow</th>
<th>Consolidation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$</td>
<td>$\frac{\partial u_e}{\partial t} = C_v \frac{\partial^2 u_e}{\partial x^2}$</td>
</tr>
<tr>
<td>$t_c \approx \frac{H^2}{\alpha}$</td>
<td>$t_c \approx \frac{H^2}{C_v}$</td>
</tr>
<tr>
<td>$t^* = \frac{t}{t_c} = \left( \frac{H^2}{\alpha} \right)$</td>
<td>$t^* = \frac{t}{t_c} = \left( \frac{H^2}{C_v} \right)$</td>
</tr>
</tbody>
</table>

II Review of key points

D The key length scale (L) is the flow path length

1 For double-sided drainage, L = half the layer thickness

2 For single-sided drainage, L = the whole layer thickness
III Calculating consolidation

\[ U(t) = \frac{\Delta H(t)}{\Delta H(t \rightarrow \infty)} = \frac{\Delta u_{\text{excess}}}{\Delta u_{\text{excess(max)}}} = 1 - \frac{u_{\text{excess}}}{u_0} \]

E Consolidation ratio for double-sided drainage of a uniform excess pressure pulse* [for analytical solution see Terzaghi (1943) or Lambe and Whitman (1969)]

1. For \( T_v = t/(C_vH^2) > 0.1 \)
   a. Excess pore pressure distribution \( (u) \) is approximately parabolic
   b. \( \bar{U} \) (ave. consolidation ratio)
      i. \( \bar{U} = 1 - \frac{\bar{u}}{u_0} \)
      ii. \( \bar{U} = 1 - \frac{\text{area beneath } u \text{-curve}}{\text{area beneath boxcar function}} \)
      iii. \( \bar{U} = 1 - \frac{2}{3} \frac{u_{\text{max}}}{u_0} \)

2. \( \sim 92\% \) consolidation at \( T_v=1 \)

II Review of key points

E Average consolidation ratio (\( \bar{U} \))

1. \( \bar{U}(t^* = 1) \approx 92\% \)
2. \( \bar{U}(t^* = 3) \approx 99\% \)
3. About 92\% of the ultimate primary consolidation occurs by the dimensionless time \( t^* = 1 \)
III Examples

A Consider layers of clay and sand, each 10’ thick, and suppose that the coefficient of volume change ("compressibility") of the sand is 1/5 that of the clay, and the permeability of the sand is 10,000 times that of the clay.

1 What is the ratio of the consolidation times of the sand and clay?

\[
\frac{t_{\text{clay}}}{t_{\text{sand}}} = \frac{H_{\text{clay}}}{H_{\text{sand}}} \frac{m_{\text{clay}}(\text{clay})/k_{\text{clay}}}{m_{\text{clay}}(\text{sand})/k_{\text{sand}}} = \frac{m_{\text{clay}}(\text{clay})/k_{\text{clay}}}{m_{\text{clay}}(\text{sand})/k_{\text{sand}}} = \frac{5/1}{1/10,000} = 50,000
\]

2 If a 10'-thick layer of clay reaches 90% consolidation in 10 years, how long would it take for a clay layer 40’ thick to reach that level of consolidation?

\[
\frac{t_{40'}_{\text{clay}}}{t_{10'}_{\text{clay}}} = \frac{m_{40'}_{\text{clay}}H_{40'}^2/k_{40'}_{\text{clay}}}{m_{10'}_{\text{clay}}H_{10'}^2/k_{10'}_{\text{clay}}} = \frac{m_{40'}_{\text{clay}}H_{40'}^2/k_{40'}_{\text{clay}}}{m_{10'}_{\text{clay}}H_{10'}^2/k_{10'}_{\text{clay}}} = \frac{40^2}{10^2} = 16 \quad 16 \times 10 \text{ years} = 160 \text{ years}
\]
III Examples

B  How long will a layer of clay take to reach 90% consolidation if the initial excess pore pressure distribution is constant across the layer and the layer is drained from its top and bottom? One dimensionless time unit \((t^*)\) is given by \(= 1\).

C  Calculating consolidation for double-sided drainage

4 Matlab script transient_heat4.m

So after one dimensionless time step [i.e. \(t^* = t/(H^2/C_v) = 1\)], about 92% (i.e., roughly 90%) of the ultimate consolidation will have occurred.
C Calculating consolidation for double-sided drainage

3 Matlab function consolidation_v2.m

function [U,v] = consolidation_v2(t,H,Cv)
% function [U,v] = consolidation_v2(t,H,Cv)
% Calculates consolidation factor for 1-D consolidation for a uniform excess pore pressure
% input arguments
% t = time
% H = coefficient of consolidation
% Cv = coefficient of consolidation
% output arguments
% U = consolidation ratio
% v = excess pore pressure
% initialize
% n = number of iterations
% k = position
% constant iniail temperature
% C = constant initial temperature
% initialize
% r = number of columns in the T matrix (number of points in one Qme)
% s = number of rows in the T matrix (number of points in space)
% initialize
% H = length of longest flow path
% a = number of rows in the T matrix (number of points in space)
% b = number of columns in the T matrix (number of points in time)
% numit = number of iterations at each time step
% TD = temperature at end of rod
% T1 = temperature at other end of rod
% initialize the temperature matrix
% T = zeros(a,b); % Fisrt index is position, second is time
% set the initial temperature distribution
% % Constant initial temperature
% T(:,1) = (linspace(T1,T1,a))'; % Constant initial temperature
% solve by finite difference method
% for k = 1:numit;
% for i = 2:a-1; % i is index for position
% T(i,j) = 0.25*(T(i+1,j-1)+T(i-1,j-1)+T(i+1,j-1)+T(i,j-1));
% end
% end
% end
% % Plot pore pressures at different times
% figure(1);
% plot(T(:,11:T1+1),T1:T1+1); % Plot columns of T versus index
title('Temperature at various times')
xlabel('Position')
ylabel('Temperature/T_max')
axis([1 a 0 1.05])

subplot(2,1,1)
plot(T,v)
% Plot columns of T versus index
title('Temperature at various times')
% ylabel('Temperature/T_max')
axis([1 a 0 1.05])

subplot(2,1,2)
t_star = (0:0.1):(a-1)/2);
plot(t_star,mean(T(:,1)),t_star,mean(T(:,2)),t_star,mean(T(:,3)));
% Plot consolidation ratio
xlabel('T_v')
ylabel('U = 1 - u/u_0')
title('Consolidation ratio for double-sided drainage')

end
end

C Calculating consolidation for double-sided drainage

4 Matlab script transient_heat4.m

% Solves for 1-D transient heat flow by finite differences
% H = length of longest flow path
a = 11; % number of rows in the T matrix (number of points in space)
b = 26; % number of columns in the T matrix (number of points in time)
umit = 21; % number of iterations at each time step
TD = 0; % temperature at one end of rod
T1 = 1; % temperature at other end of rod

% Initialize the "Temperature matrix"
T = zeros(a,b); % First index is position, second is time
% Set the initial temperature distribution
T(:,1) = (linspace(T1,T1,a))'; % Constant initial temperature
%
% Solve by finite difference method
% for k = 1:numit;
for i = 2:a-1; % i is index for position
T(i,j) = 0.25*(T(i+1,j-1)+T(i-1,j-1)+T(i+1,j-1)+T(i,j-1));
end
end

% Plot figures
figure(1)
cif
plot(T,
% Plot columns of T versus index
title('Temperature at various times')
xlabel('Position')
ylabel('Temperature/T_max')
axis([1 a 0 1.05])

subplot(2,1,1)
plot(T,v)
% Plot columns of T versus index
title('Temperature at various times')
% ylabel('Temperature/T_max')
axis([1 a 0 1.05])

subplot(2,1,2)
t_star = (0:0.1):(a-1)/2);
plot(t_star,mean(T(:,1)),t_star,mean(T(:,2)),t_star,mean(T(:,3)));
% Plot consolidation ratio
xlabel('T_v')
ylabel('U = 1 - u/u_0')
title('Consolidation ratio for double-sided drainage')
References