II 1-D consolidation equation

A Analogy between consolidation and flow of heat (from Terzaghi, 1943, p. 272): “The loss of water (consolidation) corresponds to the loss of heat (cooling) and the absorption of water (swelling) to an increase of the heat content of a solid body. The existence of the thermodynamic analogue is useful in two different ways. First of all it eliminates in some cases the necessity of solving the differential equation 99(6) because a great variety of solutions has already been obtained in connection with thermodynamic problems. Second, in contrast to the phenomena of consolidation and swelling, the processes of cooling and heating are familiar to everybody from daily experience. Therefore the knowledge of the existence of the analogue facilitates the visualization of the mechanics of consolidation and swelling.”
II 1-D consolidation equation

B Variation of pore pressure

1 Initial conditions

2 Immediately after surcharge

3 After a ‘short’ time

4 After a ‘long’ time

Excess pore pressure pulse dissipates quickly in sand; its pore water rapidly flows sideways.
Excess pressure remains in clay.

C Start with Darcy’s Law

\[ Q = -kiA \]
\[ Q = \text{discharge (L}^3/\text{t)} \]
\[ k = \text{conductivity (L/t)} \]
\[ i = \text{head gradient (L/L)} \]
\[ A = \text{area (L}^2) \]

or

\[ q = \frac{Q}{A} = -ki \]
\[ q = \text{flux (L/t)} \]

D Express the head gradient, in terms of the pore pressure and elevation

\[ i = \frac{\partial H}{\partial z} = \frac{\partial [z + (u/\rho g)]}{\partial z} \]
\[ i = \frac{\partial z + \left( \left[ u_{\text{hydrotatic}} + u_{\text{excess}} \right]/\rho g \right)}{\partial z} \]
II 1-D consolidation equation

E The total pore pressure \( u \) is the sum of the excess pore pressure \( u_{\text{excess}} \) and hydrostatic pressure \( u_{\text{hydrostatic}} \).

\[
\frac{\partial H}{\partial z} = \frac{\partial}{\partial z} \left[ z + \left( u_{\text{excess}} + u_{\text{hydrostatic}} \right) \right] = \frac{\partial}{\partial z} \left[ z + \left( \frac{\rho g (Z - z)}{\rho g} \right) \right] = \frac{\partial}{\partial z} \left[ z + \frac{1}{\rho g} \partial u_{\text{excess}} \right] = 0
\]

\( Z \) (water column height) is a constant.

F The rate of change in the elevation head \( z \) and the hydrostatic pressure head cancel each other out exactly, so head change scales with the excess pressure change.

\[
\frac{\partial H}{\partial z} = \frac{\partial}{\partial z} \left( u_{\text{excess}} \right) = \frac{1}{\rho g} \frac{\partial u_{\text{excess}}}{\partial z}
\]

\[
i_2 - i_1 = \frac{1}{\rho g} \frac{\partial^2 u_{\text{excess}}}{\partial z^2}
\]

G The head gradient \( i_1 \) at elevation \( z \) at the base of a slice of clay of thickness \( dz \) is:

\[
i_1 = \frac{1}{\rho g} \frac{\partial u_{\text{excess}}}{\partial z}
\]

H The head gradient \( i_2 \) at the top of the clay slice (elevation \( z + dz \)) is:

\[
i_2 = \frac{1}{\rho g} \left( u_{\text{excess}} + \frac{\partial u_{\text{excess}}}{\partial z} dz \right)
\]

\[
i_2 = \frac{1}{\rho g} \left( \frac{\partial u_{\text{excess}}}{\partial z} \left( u_{\text{excess}} + \frac{\partial u_{\text{excess}}}{\partial z} dz \right) \right)
\]

\[
i_2 = 1 + \frac{\partial^2 u_{\text{excess}}}{\partial z^2} dz
\]

Water column of height \( Z \)

Marined clay type of thickness \( h \) with double drainage
II 1-D consolidation equation

I The change in water flux out of the top of the clay slice reflects the water loss from the clay slice (per unit area)

\[ \Delta q = q_2 - q_1 = -k(i_2 - i_1) \]

\[ i_2 - i_1 = \frac{1}{\rho g} \left( \frac{\partial^2 u_{\text{excess}}}{\partial z^2} \right) dz \]

\[ \Delta q = -k \frac{\partial^2 u_{\text{excess}}}{\partial z^2} dz \]

II 1-D consolidation equation

M The change in the flux times the horizontal area (A) gives the water volume loss in the clay with respect to time:

\[ A \Delta q = \Delta Q = \frac{\Delta \text{volume}_{\text{water}}}{\Delta t} \]

K The right hand side of flux equation (upper) resembles the right side of the heat equation.

L What about the left side?
II 1-D consolidation equation

\[ A \Delta q = \Delta Q = \frac{\Delta \text{volume}_{\text{water}}}{\Delta t} \]

N The water volume loss equals the void volume loss in the clay.

O Dividing the volume loss per time by the unit area A gives rate of change of the height (thickness) of the clay slice with respect to time

\[ \Delta q = \frac{\partial (\Delta H)}{\partial t} \]

Here, \( H \) is the height of the clay layer, not water head.

P The height change, in turn, is the product of the vertical strain (\( \varepsilon_z \)) and the original layer height \( H_0 = dz \), so...

\[ \Delta q = \frac{\partial (\Delta H) H_0}{\partial t} = \frac{\partial (\Delta H) dz}{\partial t} = \left( \frac{\Delta H}{H_0} \right) \frac{\partial (\Delta H)}{\partial t} dz = \frac{\partial \varepsilon_z}{\partial t} dz \]

II 1-D consolidation equation

\[ \Delta q = -k \frac{\partial^2 u_{\text{excess}}}{\partial z^2} dz \]

Flux equations

Q From Lec. 37, the coefficient of compressibility (\( m_v \)) relates the volumetric (and vertical) strain to the difference between the excess pore pressure and initial pore pressure (\( u_0 \))

\[ \varepsilon_z = \frac{\Delta H}{H_0} = -m_v (u_{\text{excess}} - u_0) \]

R The derivative of the vertical strain with respect to time is

\[ \frac{\partial \varepsilon_z}{\partial t} = \frac{\partial}{\partial t} \left[ -m_v (u_{\text{excess}} - u_0) \right] \]

\[ \frac{\partial \varepsilon_z}{\partial t} = -\frac{\partial m_v}{\partial t} + \frac{\partial u_{\text{excess}}}{\partial t} \]

\[ \frac{\partial \varepsilon_z}{\partial t} = -\frac{\partial m_v}{\partial t} \]

S Substituting the last expression into the second flux equation yields

\[ \Delta q = -m_v \frac{\partial u_{\text{excess}}}{\partial t} dz \]
II 1-D consolidation equation

\[ \Delta q = \frac{-k}{\rho g} \frac{\partial^2 u_{\text{excess}}}{\partial z^2} \, dz \]

Flux equations

T Equate these expressions for flux

\[ \frac{-m_{v}}{\partial t} \frac{dz}{\partial t} = \frac{-k}{\rho g} \frac{\partial^2 u_{\text{excess}}}{\partial z^2} \, dz \]

\[ \frac{\partial u_{\text{excess}}}{\partial t} = \frac{k}{\rho g m_{v}} \frac{\partial^2 u_{\text{excess}}}{\partial z^2} \]

\[ \frac{\partial u_{\text{excess}}}{\partial t} = C_v \frac{\partial^2 u_{\text{excess}}}{\partial z^2} \]

\( C_v = \text{coefficient of consolidation} \)

II 1-D consolidation equation

U The 1-D equations for consolidation and heat flow have exactly the same form ...

\[ \frac{\partial u_{\text{excess}}}{\partial t} = C_v \frac{\partial^2 u_{\text{excess}}}{\partial z^2} \]

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2} \]

so the 1-D diffusion of excess pore pressure in a soil is analogous to the 1-D diffusion of heat by conduction

V Doubling the layer thickness quadruples the time to achieve the same amount of consolidation

W Key difference: the hydraulic conductivity of earth materials and \( C_v \) generally decrease as the void ratio decreases; this would increase the consolidation time relative to theoretical prediction
III Calculating consolidation

A Recall the consolidation ratio

\[ U(t) = \frac{\Delta H(t)}{\Delta H(t \to \infty)} = \frac{\Delta u_{\text{excess}}}{\Delta u_{\text{excess(max)}}} = 1 - \frac{u_{\text{excess}}}{u_0} \]

B Caveats regarding the equation for the consolidation ratio U

1. Unrealistically large values of settlement predicted for large values of Δu
   
   Example: if |(Δu)(m_v)| > 1, then |Δh_{calculated}| > H_0. A layer cannot achieve a negative thickness.

2. Equation most accurate for |Δu| ≪ m_v

2. Past a certain pore pressure change, the consolidation is not recoverable or linearly related to pore pressure change
III Calculating consolidation

\[ U(t) = \frac{\Delta H(t)}{\Delta H(t \to \infty)} = \frac{\Delta u_{\text{max}}}{\Delta u_{\text{max}}} = 1 - \frac{u_{\text{max}}}{u_0} \]

C Consolidation ratio for double-sided drainage of a uniform excess pressure pulse* [for analytical solution see Terzaghi (1943) or Lambe and Whitman (1969)]

1 For \( T_v = t/(C_vH^2) > 0.1 \)
   a Excess pore pressure distribution (\( u \)) is approximately parabolic
   b \( U \) (ave. consolidation ratio)
      i \[ U = 1 - \frac{u}{u_0} \]
      ii \[ \bar{U} = \frac{1}{\Delta u_{\text{max}}} \text{area beneath } u \text{ curve} \]
      iii \[ \bar{U} = 1 - \frac{2}{3} \frac{u_{\text{peak}}}{u_0} \]

2 \~92\% consolidation at \( T_v = 1 \)

---

C Calculating consolidation for double-sided drainage

3 Matlab function

```matlab
function [U, u] = consolidation_v2(t, Cv, H)
% Calculates time factor for 1-D consolidation for a uniform excess pressure pulse (i.e., a boxcar function)
% Output arguments
% U = consolidation ratio
% u = excess pore pressure
% Input arguments
% t = time
% Cv = coefficient of consolidation
% H = layer thickness
% Example
% t = 0:0.01:1; Cv = 1; H = 1;
% [U, u] = consolidation_v2(t, Cv, H);
% % Calculate consolidation ratio
% U(t) = 1 - sum(u(:,t))/(length(z)-1);
% % Plot pore pressures at different times
% figure(1)
% plot(z-1/2, u(:,11), z-1/2, u(:,21), z-1/2, u(:,31), z-1/2, u(:,41), z-1/2, u(:,51), ...
% z-1/2, u(:,61), z-1/2, u(:,71), z-1/2, u(:,81), z-1/2, u(:,91), z-1/2, u(:,101));
% xlabel('z/H')
% ylabel('u/\mu_0')
% title('Pore pressure decay for double-sided drainage')
% legend('T_v = 0.1', 'T_v = 0.2', 'T_v = 0.3', 'T_v = 0.4', 'T_v = 0.5',
% 'T_v = 0.6', 'T_v = 0.7', 'T_v = 0.8', 'T_v = 0.9', 'T_v = 1.0');
% % Plot consolidation ratio
% figure(2)
% plot(t, U)
% xlabel('T_v = t/(C_vH^2)')
% ylabel('U = 1 - u/\mu_0')
% title('Average consolidation ratio for double-sided drainage')
% end
```

---

% Plot pore pressures at different times
figure(1)
% t = 0:0.01:1;
% plot(z-1/2, u(:,11), z-1/2, u(:,21), z-1/2, u(:,31), z-1/2, u(:,41), z-1/2, u(:,51), ...
% z-1/2, u(:,61), z-1/2, u(:,71), z-1/2, u(:,81), z-1/2, u(:,91), z-1/2, u(:,101));
% xlabel('z/H')
% ylabel('u/\mu_0')
% title('Pore pressure decay for double-sided drainage')
% legend('T_v = 0.1', 'T_v = 0.2', 'T_v = 0.3', 'T_v = 0.4', 'T_v = 0.5',
% 'T_v = 0.6', 'T_v = 0.7', 'T_v = 0.8', 'T_v = 0.9', 'T_v = 1.0');
% % Plot consolidation ratio
% figure(2)
% plot(t, U)
% xlabel('T_v = t/(C_vH^2)')
% ylabel('U = 1 - u/\mu_0')
% title('Average consolidation ratio for double-sided drainage')
% end
References

• http://www.geoengineer.org/component/k2/item/448-karl-terzaghis-legacy-in-geotechnical-engineering
• http://en.wikipedia.org/wiki/Karl_von_Terzaghi
• Online version of Terzaghi (1943)
  • https://labmekanikatanah.files.wordpress.com/2013/04/karl_terzaghi_theoretical_soil_mechanicsbookfi-org.pdf