NUMERICAL SOLUTION OF THE 1-D DIFFUSION EQUATION (39)

I Main Topics
   A Motivation for using a numerical technique
   B Non-dimensionalizing the diffusion (heat flow) equation
   C Finite-difference solution to the 1-D heat equation (diffusion equation)

II Motivation for using a numerical technique
   A Insight into the second order PDE governing transient flow
   B Insight into effect of initial conditions and boundary conditions on the solution
   C To solve for a wide range of initial value/boundary value combinations and geometries
   D Finite-difference method a good learning tool
III Non-dimensionalizing the heat flow equation

A Start with the heat flow equation, where

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]

T = temperature (variable)
\( t = \) time (variable)
\( \alpha = \) thermal diffusivity (constant)
\( x = \) position (variable)

B "Nondimensionalizing" (or scaling) eliminates \( \alpha \)

C Select constant scaling terms

\( x_{\text{max}} \) (dimensions: length)
\( T_{\text{max}} \) (dimensions: °K)
\( \frac{x_{\text{max}}^2}{\alpha} \) (dimensions: time)

D Define dimensionless terms

\( x^* = \frac{x}{x_{\text{max}}} \)
\( T^* = \frac{T}{T_{\text{max}}} \)
\( t^* = t \left( \frac{x_{\text{max}}^2}{\alpha} \right) \)
III Non-dimensionalizing the heat flow equation

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]

C Select constant scaling terms

\( x_{\text{max}} \) (dimensions: length)
\( T_{\text{max}} \) (dimensions: K)
\[ \frac{x_{\text{max}}}{\alpha} \] (dimensions: time)

D Define dimensionless terms

\[ x^* = \frac{x}{x_{\text{max}}} \]
\[ T^* = \frac{T}{T_{\text{max}}} \]
\[ t^* = \frac{t}{\left(\frac{x_{\text{max}}^2}{\alpha}\right)} \]

E Recast dimensioned terms

\[ x = x^* x_{\text{max}} \]
\[ T = T^* T_{\text{max}} \]
\[ t = t^* \left(\frac{x_{\text{max}}^2}{\alpha}\right) \]

F Derivatives of \( x^* \), \( t^* \), and \( T \) (from previous slide)

\[ \frac{dx^*}{dx} = \frac{1}{x_{\text{max}}} \]
\[ \frac{dt^*}{dt} = \frac{\alpha}{x_{\text{max}}} \]
\[ \frac{dT^*}{dT} = \frac{T_{\text{max}}}{\alpha} \]

G Recast derivatives in heat flow equation using chain rule

\[ \frac{\partial T^*}{\partial t} = \frac{\partial T^*}{\partial t^*} \frac{dt^*}{dt} = \frac{\partial T^*}{\partial t^*} \left(\frac{x_{\text{max}}^2}{\alpha}\right) \frac{1}{\alpha} \]
\[ \frac{\partial T^*}{\partial x} = \frac{\partial T^*}{\partial x^*} \frac{dx^*}{dx} = \frac{\partial T^*}{\partial x^*} \left(\frac{x_{\text{max}}}{\alpha}\right) \]
\[ \frac{\partial^2 T^*}{\partial x^2} = \frac{\partial^2 T^*}{\partial x^2} \frac{dx^*}{dx} = \frac{\partial^2 T^*}{\partial x^2} \left(\frac{x_{\text{max}}}{\alpha}\right)^2 \]

\[ \frac{\partial^2 T^*}{\partial x^2} = \frac{\partial T^*}{\partial x^*} \frac{\partial T^*}{\partial x^*} \]
\[ \frac{\partial^3 T^*}{\partial x^3} = \frac{\partial^3 T^*}{\partial x^3} \frac{dx^*}{dx} = \frac{\partial^3 T^*}{\partial x^3} \left(\frac{x_{\text{max}}}{\alpha}\right)^3 \]

\[ \frac{\partial^4 T^*}{\partial x^4} = \frac{\partial^4 T^*}{\partial x^4} \frac{dx^*}{dx} = \frac{\partial^4 T^*}{\partial x^4} \left(\frac{x_{\text{max}}}{\alpha}\right)^4 \]
III Non-dimensionalizing the heat flow equation

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]

G Dimensioned derivatives in heat flow equation

\[ \frac{\partial T}{\partial t} = \frac{\partial T^*}{\partial t^*} \frac{T_{\text{max}}}{x_{\text{max}}} \]
\[ \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T^*}{\partial x^*^2} \frac{T_{\text{max}}}{x_{\text{max}}} \]

H Substitute for dimensioned derivatives in heat flow equation to obtain dimensionless form

\[ \frac{\partial T^*}{\partial t^*} \frac{\alpha}{x_{\text{max}}} = \frac{\partial^2 T^*}{\partial x^*^2} \frac{T_{\text{max}}}{x_{\text{max}}} \]

\[ \frac{\partial T^*}{\partial t^*} \frac{\alpha}{x_{\text{max}}} = \frac{\partial^2 T^*}{\partial x^*^2} \]

All terms here are dimensionless

III Finite-difference solution to the 1-D heat equation (diffusion equation)

A Set up a dimensionless grid

1. Let \( \Delta x^* = \Delta t^* = 1 \)
2. The row number \( I \) gives the time step
3. The column number \( j \) gives the position
III Finite-difference solution to the 1-D heat equation (diffusion equation)

B Explicit method

1 Approximate $\frac{\partial T^*}{\partial t^*}$

$$\frac{\partial T^*}{\partial t^*} \approx \frac{T^*_i(x, t^* + \Delta t^*) - T^*_i(x, t^*)}{\Delta t^*}$$

2 Approximate $\frac{\partial^2 T^*}{\partial x^* 2}$

$$\frac{\partial^2 T^*}{\partial x^* 2} \approx \frac{T^*_{i,j+1} - 2T^*_{i,j} + T^*_{i,j-1}}{\Delta x^*}$$

3 Set the derivatives equal

4 Solve for $T^*_{i+1,j}$

5 Solution prone to numerical error: approximations are at two different points
III Finite-difference solution to the 1-D heat equation (diffusion equation)

C Crank-Nicolson method

1 Evaluate $\delta^2 T/\delta x^2$ at $\bullet$ by averaging the second derivatives at $\Box$

2 Approximate $\partial T^*/\partial t^*$ at $\bullet$ by averaging values at $\Box$

3 Approximate $\partial^2 T^*/\partial x^*^2$ at $\bullet$ by averaging values at $\Box$
III Finite-difference solution to the 1-D heat equation (diffusion equation)

4 Now equate the dimensionless partial derivatives, setting $dx^* = dt^* = 1$

$$
\frac{\partial T^*}{\partial t^*} = \frac{\partial^2 T^*}{\partial x^*^2}
$$

5 Solve for $T^*_{i+1,j}^*$

$$
T^*_{i+1,j}^* \approx \frac{1}{2}(T^*_{i,j+1}^* - 2T^*_{i,j}^* + T^*_{i,j-1}^* + T^*_{i+1,j+1}^* - 2T^*_{i+1,j}^* + T^*_{i+1,j-1}^*)
$$

The value of $T$ at any node is equal to the average value of the two adjacent nodes at the same time step and the two nodes at the preceding time step.
III Finite-difference solution to the 1-D heat equation (diffusion equation)

D Recap
1 Convert dimensioned problem to dimensionless problem
2 Solve the dimensionless problem
3 Convert dimensionless temperature solution back to dimensioned solutions by multiplying by the scaling factor

\[ x^* = \frac{x}{x_{\text{max}}} \quad T^* = \frac{T}{T_{\text{max}}} \quad t^* = t \left( \frac{x_{\text{max}}^2}{\alpha} \right) \]

\[ T^*_{i,j+1} = \frac{T^*_{i+1,j} + T^*_{i-1,j} + T^*_{i,j+1} + T^*_{i,j-1}}{4} \]

\[ x = x^* x_{\text{max}} \quad T = T^* T_{\text{max}} \quad t = t^* \left( \frac{x_{\text{max}}^2}{\alpha} \right) \]

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III Finite-difference solution to the 1-D heat equation (diffusion equation)

E Closing comments
1 The finite-difference method is essentially an averaging procedure that describes how "information" propagates within a system
2 Solutions for 2-D Laplace equation and 1-D heat equation have revealing similarities and revealing differences

The value of \( T \) at the red node is approximately the average of the \( T \) values at the blue nodes