SUBSIDENCE MECHANICS: HEAT FLOW ANALOG (38)

I Main Topics
   A Motivation: Why investigate heat flow?
   B Development of 1-D heat flow equation as analog for consolidation
   C Finite-difference interpretation of heat flow equation
   D Dimensional analysis

II Motivation

A Heat flow equation has the same form as the consolidation equation but is easier to grasp
B Diffusion of heat analogous to diffusion of excess pore pressure
C Many analytic solutions for heat flow (e.g., Carslaw and Jaeger, 1984)
D Many analogous equations of great use
### E Flow analogs

<table>
<thead>
<tr>
<th>Flowing quantity</th>
<th>Incompressible Fluid</th>
<th>Heat</th>
<th>Chemical Species</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conserved quantity</td>
<td>Mass</td>
<td>Heat Energy</td>
<td>Molecules</td>
</tr>
<tr>
<td>1-D flux law</td>
<td>Darcy’s law $q = -k \frac{\partial H}{\partial x}$</td>
<td>Fourier’s law $q = -k \frac{\partial T}{\partial x}$</td>
<td>Fick’s law $J = -D \frac{\partial c}{\partial x}$</td>
</tr>
<tr>
<td>Flux term</td>
<td>$q =$ volume flux density $m^3/(m^2 \cdot sec)$</td>
<td>$q =$ heat flux density joules/(m$^2 \cdot sec)$</td>
<td>$J =$ diffusion flux moles/(m$^2 \cdot sec)$</td>
</tr>
<tr>
<td>Coefficient</td>
<td>$k =$ hydraulic conductivity $m/\text{sec}$</td>
<td>$k =$ thermal conductivity joules/(m$^2 \cdot K \cdot sec$)</td>
<td>$D =$ diffusivity $m^2/\text{sec}$</td>
</tr>
<tr>
<td>Potential term</td>
<td>$H =$ head (m)</td>
<td>$T =$ temperature ($^\circ \text{K}$)</td>
<td>$c =$ concentration (moles/m$^3$)</td>
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<tr>
<td>1-D diffusion law</td>
<td>$\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2}$</td>
<td>$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$</td>
<td>$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$</td>
</tr>
<tr>
<td>Coefficient</td>
<td>$\alpha =$ hydraulic diffusivity $m^2/\text{sec}$</td>
<td>$\alpha =$ thermal diffusivity $m^2/\text{sec}$</td>
<td>$D =$ diffusivity $m^2/\text{sec}$</td>
</tr>
<tr>
<td>Steady state flow (Term on left side of diffusion law = 0)</td>
<td>$\nabla^2 H = 0$</td>
<td>$\nabla^2 T = 0$</td>
<td>$\nabla^2 c = 0$</td>
</tr>
</tbody>
</table>

### III Development of 1-D heat flow equation as analog for consolidation

**A** Isotropic, uniform material

**B** Definition of terms

1. $U =$ heat energy (joules)
2. $x =$ position (meters)
3. $t =$ time (seconds)
4. $q =$ heat flux (joules/(meter$^2 \cdot$ sec))
   a. Rate of heat energy transfer per unit area per unit time
   b. Heat flux can vary with time and position, so $q = q(x,t)$
5. $T =$ temperature ($^\circ$)
   Temperature can vary with position and time, so $T = T(x,t)$
C Fourier's Law of Heat Conduction (1-D)

\[ q = -k \frac{\partial T}{\partial x} \]

1. Heat flow (q) scales with the temperature gradient (\( \partial T/\partial x \))
2. \( k \) = coefficient of thermal conductivity
   a. Dimensions: Joules sec\(^{-1}\) m\(^{-1}\) K\(^{-1}\)
   b. \( k \) assumed to be constant
3. Dimension check
   \[ \frac{\text{Joules}}{m^2 \text{ sec}} = \frac{\text{Joules} \cdot \circ K}{m \cdot \text{sec} \cdot \circ K \cdot m} \]

4. The minus sign
   a. For heat to flow from \( x_1 \) to \( x_2 \), where \( x_1 < x_2 \), \( T(x_1) > T(x_2) \).
   b. Positive heat flow corresponds to a drop in temperature, requiring \( k \) to be negative
5. Partial derivative used because \( T \) is a function of \( x \) and \( t \).
6. Finite difference approximation:
   \[ q = -k \frac{\Delta T}{\Delta x} \]
D Fluid flow analog (slow laminar flow)

\[ q = -k \frac{\partial H}{\partial x} \]

1. Volumetric flux (q) scales with the head gradient (\( \partial H/\partial x \))
2. \( k \) = hydraulic conductivity
   a. Dimensions: m/sec
   b. \( k \) assumed to be constant
   c. \( k \) depends on the intrinsic permeability of the material, the degree of saturation, and on the density and viscosity of the fluid
3. Dimension check

\[ \frac{m^3}{m^2 \text{ sec}} = \frac{m}{\text{ sec m}} \]

E Heat Flow Equation
(Conservation of energy)

- Change in heat energy = heat in – heat out

\[ \Delta U_{\text{heat}} = (\Delta T)(\text{mass})(\text{specific heat}) \]
\[ \Delta U_{\text{heat}} = (A)(\Delta x)[q(x = x_1) - q(x = x_2)] \]
\[ (\Delta T)(\text{mass})(\text{specific heat}) = (A)(\Delta x)[-\Delta q] \]
\[ (\Delta T)(\rho A \Delta x)(c) = (A)(\Delta x) \left[ -\frac{\partial q}{\partial x} \Delta x \right] \]
\[ \frac{(\Delta T)(\rho)(c)}{\Delta t} = \left[ -\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) \right] \]
\[ \frac{(\Delta T)(\rho)(c)}{\Delta t} \cdot k = \frac{\partial^2 T}{\partial x^2} \]
\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]
\[ \alpha = \text{thermal diffusivity} = \frac{k}{\rho c} \]
**F Heat Flow Equation**

- 1-D form: $K \frac{dT}{dt} = \frac{\partial^2 T}{\partial x^2}$\hspace{1cm}parabolic differential equation
- 2-D form: $K \frac{dT}{dt} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$
- 3-D form: $K \frac{dT}{dt} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

**G Laplace equation**

- Applies to steady state distribution of temperature
- Temperature does not change as a function of time
- $\frac{\partial T}{\partial t} = 0$

- 1-D: $\frac{\partial^2 T}{\partial x^2} = 0$
- 2-D: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$
- 3-D: $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$
- General: $\nabla^2 T = 0$

$Curvature(T_{v,w}) = \frac{d^2 T}{dx^2} \left[ \frac{d^2 T}{dx^2} \right]^{3/2} + \frac{1}{\sqrt{1 + \left( \frac{d^2 T}{dx^2} \right)^2}}$, so $d^2 T = 0$ means curvature($T_{v,w}$) = 0
H Relationship between 1-D temperature profile and heat change

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]

- To a good approximation, the rate of temperature change with time scales with the curvature of the temperature profile
- If the 1-D temperature profile isn't curved, then no change in heat energy occurs in the slab (i.e., steady state exists)

I Fluid Flow Equation

(Conservation of mass for incompressible fluid)

- Change in fluid mass = mass in – mass out

\[
\Delta \text{mass} = \frac{\Delta \text{volume}}{\Delta t} \cdot (\text{density})(\Delta H) \\
= (A)(\Delta t)(\text{density})[q(x=x_1) - q(x=x_2)] \\
= \frac{\Delta \text{volume}}{\Delta t} \cdot \frac{\Delta H}{\Delta t} = \frac{\Delta t}{\Delta H} \cdot \frac{\Delta H}{\Delta t} = \left( -\frac{\partial q}{\partial x} \right) \Delta x \\
S(\frac{\Delta H}{\Delta t}) = \left[ -\frac{\partial H}{\partial x} \right] \Delta x \\
S = \text{storativity} \\
\Delta H = k \frac{\partial^2 H}{\partial x^2} \\
\frac{\partial H}{\partial t} = \alpha \frac{\partial^2 H}{\partial x^2} \\
\alpha = \text{hydraulic diffusivity} = k/S \]

\[
Q = \text{volumetric flow rate} \\
\Delta H = \text{height} \\
H_1 > H_2 \\
\alpha = \frac{\partial H}{\partial t} = \alpha \frac{\partial^2 H}{\partial x^2} \\
S = \text{storativity} \\
\Delta H = k \frac{\partial^2 H}{\partial x^2} \\
\frac{\partial H}{\partial t} = \alpha \frac{\partial^2 H}{\partial x^2} \\
\alpha = \text{hydraulic diffusivity} = k/S \]
J Relationship between head profile and fluid volume change

\[ \frac{\partial H}{\partial t} = \alpha \frac{\partial^2 H}{\partial x^2} \]

- To a good approximation, the rate of fluid content change with time scales with the curvature of the head profile
- If the head profile isn’t curved, then no change in fluid content occurs in the slab

IV Finite difference interpretation of heat flow equation

- \( \nabla^2 T = 0 \)
- The value of \( T \) (here \( T = \) temperature) at a given point is the average of the values at the nearest neighboring points on a square grid (see notes on wave eqn)

\[ T_0 = \frac{T_1 + T_2 + T_3 + T_4}{4} \]
V Dimensional analysis

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]

A Question: How does the thickness \( H \) of a plate control the time the plate takes to cool?

B Consider the dimensions of the terms in the heat flow equation and the plate thickness \( H \)

\[ [t_c] = \text{cooling time} \]
\[ [T] = ^\circ \text{K} \]
\[ [\alpha] = (\text{length})^2 (\text{time})^{-1} \]
\[ [H] = [x] = \text{length} \]

C Now consider the cooling time at a point in the plate

D The equation for the cooling time \( t_c \) must be dimensionally consistent, and can only depend on the relevant factors. So

\[ t_c = CT^a \alpha^b H^c, \]

\[ [t_c] = \text{time} \quad [T] = ^\circ \text{K} \]
\[ [\alpha] = (\text{length})^2 (\text{time})^{-1} \quad [H] = \text{length} \]

where \( C \) is an unknown dimensionless constant

Hence

\[ [t_c] = [T]^a [\alpha]^b [H]^c \]

\[ (\text{time})^1 = (^\circ \text{K})^a \left( \frac{\text{length}^2}{\text{time}} \right)^b (\text{length})^c \]

\[ (\text{time})^1 = (^\circ \text{K})^a (\text{length})^{2bc} (\text{time})^{-b} \]

By inspection

\[ a = 0; b = -1; c = 2 \]

So

\[ t_c = C \alpha^{-1} H^2 \]
V Dimensional analysis

\[ \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \]

F Meaning of solution \( t_c = \frac{C}{\alpha^{\frac{1}{2}}} H^2 \)

1. The time for cooling the plate to some fraction of its initial temperature is proportional to \( 1/H^2 \)
2. Doubling the plate thickness quadruples the cooling time
3. If \( C \approx 1 \), then \( t_c = \frac{H^2}{\alpha} \)

This can be used for rough estimates of the cooling time

References