SUBSIDENCE IN THREE DIMENSIONS: CENTER OF DILATION (42)

I Main Topics
- A Center of dilation in full-space
- B Center of dilation in a half-space
- C Gravitational and elastic analogs
- D Effects of a center of contraction

II Center of dilation in full-space
- A What is a center of dilation?
  - 1 A point at which the displacements radiate from
  - 2 An infinitely small spherical hole containing fluid at an infinite pressure (p dV = 1)
  - 3 A so-called nucleus of strain obtained by superposing, integrating or differentiating the effect of a force at a point
- B Radial displacements vary as $1/r^2$
- C Radial normal strains vary as $1/r^3$
- D Radial normal stresses vary as $1/r^3$

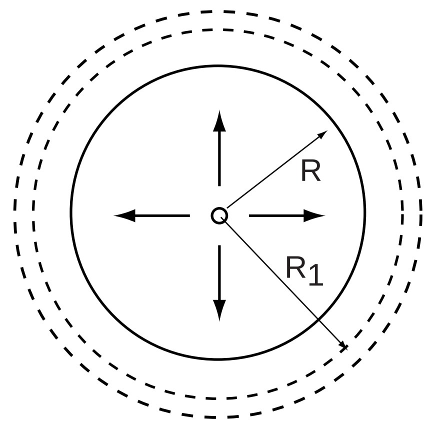
III Center of dilation in half-space
- A Construction of the solution by sources and images

Center of Dilation

Suppose a pressure energy souce (or a spherical volume) is inserted into a tiny spherical void in an infinte body, and this pressure (or inserted volume) does work on the body by displacing material radially away from the source. The material is homogeneous, isotropic, and continuous.

Area of a sphere = $4\pi R^2$

Volume of a sphere = $(4/3)\pi R^3$



Energy = work = Force•displacement
= pressure•area•radial displacement

$W = PAu_r$

$u_r = W/PA = W/(P4\pi R^2)$

So the radial displacements u_r

at the wall of the void go as $1/R^2$.

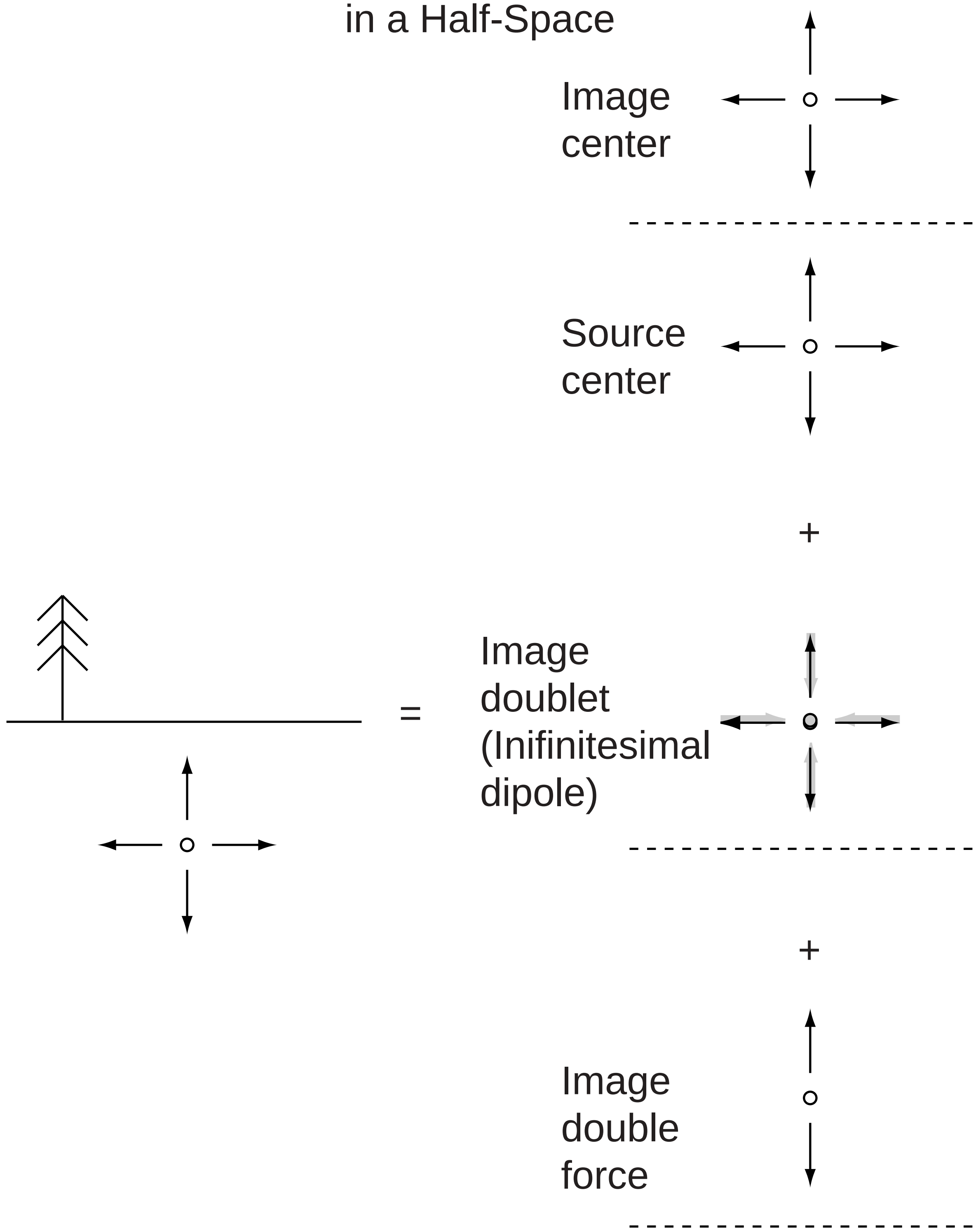
Now consider a concentric spherical surface outside the walls of the void - how do they displace? Consider the dashed circle in the diagram. Work is done on the inside of this surface by the expanding material inside it. Let the pressure exerted on the inside of this surface be $P^{(1)}$. Then

$W^{(1)} = P^{(1)}A^{(1)}u_r^{(1)}$

$u_r^{(1)} = W^{(1)}/P^{(1)}A^{(1)} = W^{(1)}/(P^{(1)}4\pi R^{(1)2})$

So at any arbitrary distance from the center of the void, the radial displacements $u_r$outside the void also go as $1/R^2$.

Construction of a Center of Dilation in a Half-Space



B Displacements for pressure energy E at a point source

	Full-space	Half-space
u_x	$E\frac{c_m}{4\pi}\frac{x}{R^3}$	$E\frac{c_m}{4\pi}x\left\{\frac{1}{R_1^3}+\frac{3-4\nu}{R_2^3}+\frac{6z(z+c)}{R_2^5}\right\}$
u_y	$E\frac{c_m}{4\pi}\frac{y}{R^3}$	$E\frac{c_m}{4\pi}y\left\{\frac{1}{R_1^3}+\frac{3-4\nu}{R_2^3}+\frac{6z(z+c)}{R_2^5}\right\}$
u_z	$E\frac{c_m}{4\pi}\frac{z}{R^3}$	$E\frac{c_m}{4\pi}\left\{\frac{z-c}{R_1^3}-\frac{(3-4\nu)(z+c)}{R_2^3}-\frac{6z(z+c)^2}{R_2^5}+\frac{2z}{R_2^3}\right\}$

c_m = poroelastic strain coefficient, dimension of pressure^{-1}.

ν = Poisson's ratio

c = depth of center of dilation

$R=\sqrt{x^2+y^2+z^2}$ distance from strain nucleus placed at the origin

$R_1=\sqrt{x^2+y^2+(z-c)^2}$ distance from a source strain nucleus at z = +c

$R_2=\sqrt{x^2+y^2+(z+c)^2}$ distance from an image strain nucleus at z = -c

Note that the half-space solutions yield the full space solutions if c is set to 0, R_1 is replaced by R, and the terms with R_2 are eliminated.

C Displacement field at the surface (z = 0) for a center of dilation

	Full-space	Half-space
u_x	$E\frac{c_m}{4\pi}\frac{x}{R^3}$	$E\frac{c_m}{4\pi}x\left\{\frac{1}{R_1^3}+\frac{3-4\nu}{R_1^3}\right\}=E\frac{c_m}{4\pi}x\left\{\frac{4(1-\nu)}{R_1^3}\right\}$
u_y	$E\frac{c_m}{4\pi}\frac{y}{R^3}$	$E\frac{c_m}{4\pi}y\left\{\frac{1}{R_1^3}+\frac{3-4\nu}{R_1^3}\right\}=E\frac{c_m}{4\pi}y\left\{\frac{4(1-\nu)}{R_1^3}\right\}$
u_z	$E\frac{c_m}{4\pi}\frac{z}{R^3}$	$E\frac{c_m}{4\pi}\left\{\frac{-c}{R_1^3}-\frac{(3-4\nu)(c)}{R_1^3}\right\}=E\frac{c_m}{4\pi}\left\{-\frac{4(1-\nu)(c)}{R_1^3}\right\}$

The half-space displacements at the surface of a half-space are proportional to the corresponding full-space displacements:

$$u_i^{half-space}=u_i^{full-space}[4(1-\nu)]\approx 3u_i^{full-space} \qquad (42.1)$$

IV Gravitational and elastic analogs (from Wang, H., 2000, Theory of linear poroelasticity: Princeton University Press, Princeton, New Jersey, 287 p.)

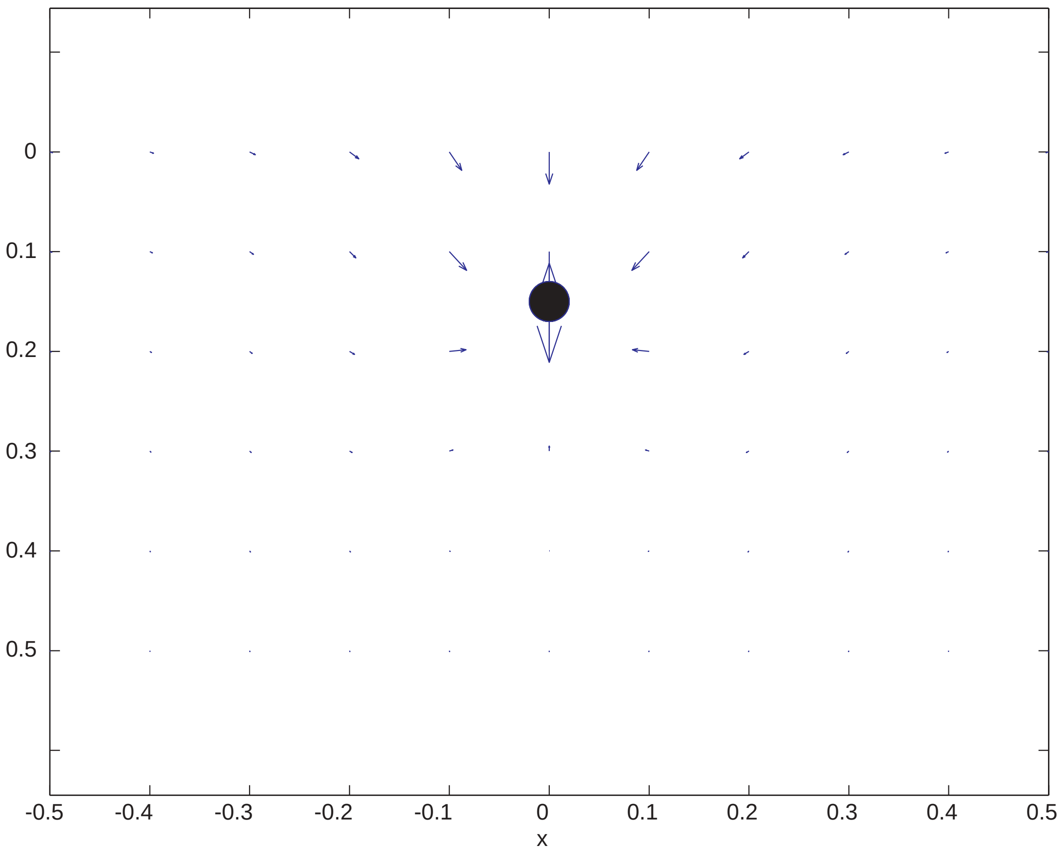
	Gravity	Elasticity Theory
Poisson's equation	$\nabla^2 U = -4\pi G\rho$ U = grav. potential ($U = mgz$) G = universal gravitational const. $-4\pi\rho$ = "density"	$\nabla^2 \Phi = c_m p = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \varepsilon$ Φ = displacement potential c_m = uniaxial poroelastic expansion coefficient p = pressure change
Gradients of potential	Gravitational force $g_i = \frac{\partial U}{\partial x_i}$	Displacement $u_i = \frac{\partial \Phi}{\partial x_i}$
Analogs	Acceleration g Potential U	Displacement u Potential Φ
Vertical component (z=0)	Gravitational force per unit mass $g^*_z(r,0) = G\frac{c}{\left(r^2+c^2\right)^{3/2}}$ positive g_z = down	Displacement per unit pressure energy amount $u^*_z{}^{HS}(r,0) = -\frac{c_m(1-\nu)}{\pi}\frac{c}{\left(r^2+c^2\right)^{3/2}}$ positive u_z = down
Horizontal (radial) component (z=0)	Gravitational force per unit mass $g^*_r(r,0) = -G\frac{c}{\left(r^2+c^2\right)^{3/2}}$ positive g_r = out	Displacement per unit pressure energy amount $u^*_r{}^{HS}(r,0) = \frac{c_m(1-\nu)}{\pi}\frac{r}{\left(r^2+c^2\right)^{3/2}}$ positive u_r = out

The factor of 4(1-ν) comes from the difference between the solutions of displacements in a half-space vs. a full-space

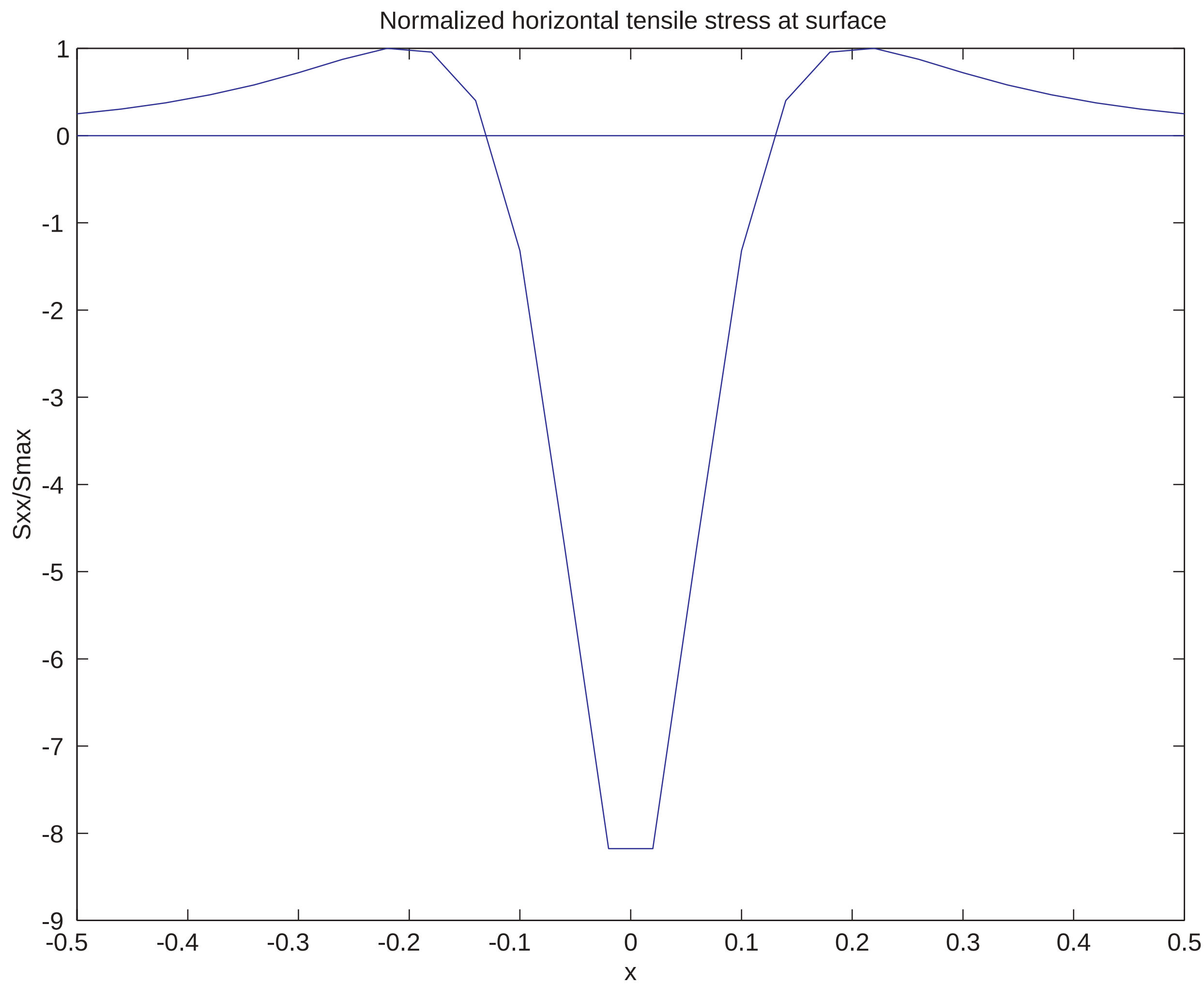
V Effects of a center of dilation or suction

	Center of dilation	Center of suction
Nearby radial stress	Compression increases	Compression decreases
Nearby hoop stress	Compression decreases	Compression increases
Horizontal tensile stress σ_{xx} at z=0	Decreases for $r > c/\sqrt{2}$ Increases for $r < c/\sqrt{2}$	Increases for $r > c/\sqrt{2}$ Decreases for $r < c/\sqrt{2}$

Displacement field associated with a source of suction



* For a center at a depth of 0.15.
* Note that positive z is down.
* The arrowhead that points up just above the center represents the upward displacement of a point beneath the center. Similarly, the arrowhead that points down just beneath the center represents the downward displacement of a point above the center.
* Note that the displacements at the surface are radial, but those between the surface and the center are not.



For a center at a depth of 0.15

$$u_r = K\frac{r}{\left(r^2+c^2\right)^{3/2}}$$

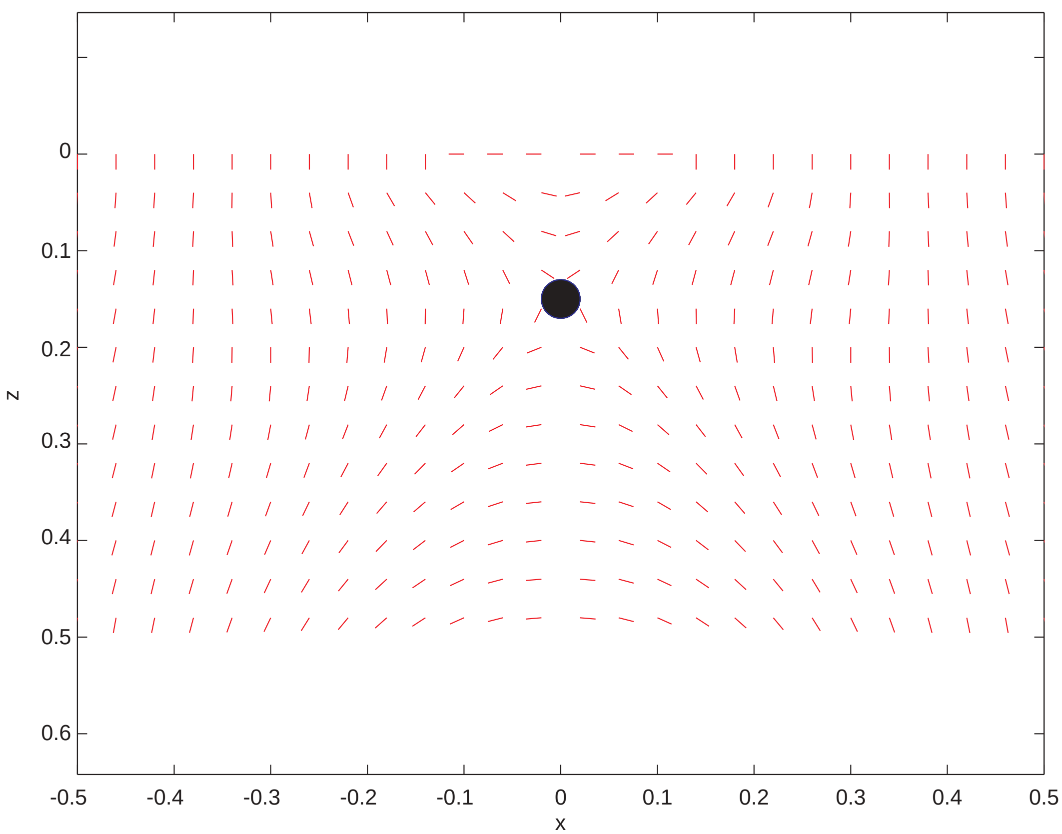
$$\frac{\partial u_r}{\partial r} = K\left[(r)(-3/2)\left(r^2+c^2\right)^{-5/2}(2r)+\left(r^2+c^2\right)^{-3/2}\right]$$

$$\frac{\partial u_r}{\partial r} = 0 \;\; if \;\; 0 = \left(-3r^2\right)+\left(r^2+c^2\right) = -2r^2+c^2$$

$$\frac{\partial u_r}{\partial r} > 0 \;\; if \;\; 2r^2 > c^2 \;\; or \;\; r > c/\sqrt{2}$$

$$\frac{\partial u_r}{\partial r} < 0 \;\; if \;\; 2r^2 < c^2 \;\; or \;\; r < c/\sqrt{2}$$

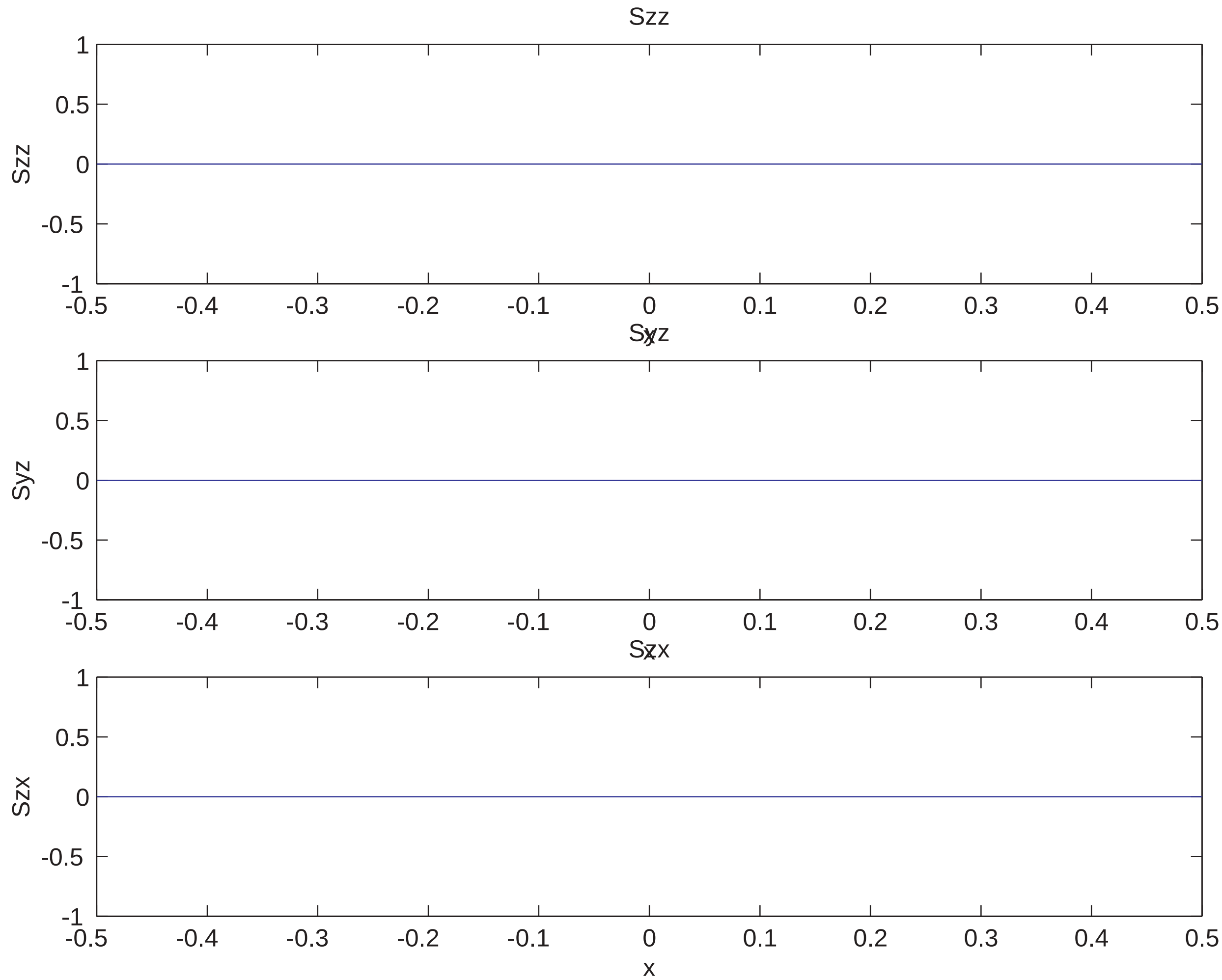
Trajectories of most compressive induced stress
due to a source of suction



* For a center at a depth of 0.15

* The induced compressive stresses are “roughly horizontal” within the hourglass shaped region above and below the center of suction, and they are “roughly vertical” elsewhere

Stresses acting on the surface z = 0



* This is a check on the tractions at the surface. They all equal zero as they should.