# SUBSIDENCE MECHANICS REVIEW (41)

I Main Topic: Review of key points in consolidation theory and examples
(From Lambe, T.W., and Whitman, R.V.,1969, Soil mechanics, Wiley, New York, 553p.)

A Main Point 1: three parameters control consolidation time

"The time [for consolidation] should be directly proportional to the volume of water which must be squeezed out of the soil. This volume of water must in turn be related to the product of the [effective] stress change, the compressibility [coefficient of volume change] of the mineral skeleton, and the volume of the soil."

$t \propto \Delta\sigma' m_v H$ (this is the time for a given height change) (41.1)

t= time

$\Delta\sigma'$ = change in effective stress

$m_v$ = coefficient of volume change = $[\Delta V/V_0]/\Delta\sigma'$

H = height of soil column

"The time [for consolidation] should be inversely proportional to how fast the water can flow through the soil. From fluid mechanics we know that the velocity of flow is related to the product of the permeability and the hydraulic gradient, and that the gradient is proportional to the [excess] fluid pressure lost within the soil divided by the distance through which the pore fluid must flow."

The excess fluid pressure loss $\propto \Delta\sigma'$, and the relevant distance is H, so

$$t \propto \frac{1}{k(\Delta\sigma' / H)} \quad (41.2)$$

k= hydraulic conductivity of the porous, permeable solid

From (41.1) and (41.2)

$$t \propto \frac{\Delta\sigma' m_v H}{k(\Delta\sigma' / H)} \quad (41.3)$$

hence

$$t \propto \frac{m_v H^2}{k} \quad (41.4)$$

From (41.4) the consolidation time t:

1 Increases with increasing compressibility (i.e., with increasing $m_v$)
2 Increases rapidly with increasing volume (height) of soil mass
2 Decreases with increasing permeability
4 Is independent of the magnitude of the effective stress change(!)

B Main Point 2: dimensional analysis illuminates consolidation time
The one-dimensional consolidation equation for saturated clayey layers is:

$$\frac{\partial u_e}{\partial t} = c_v \frac{\partial^2 u_e}{\partial x^2} \qquad \frac{pressure}{time} = c_v \frac{pressure}{length^2} \Rightarrow c_v = \frac{length^2}{time} \qquad (41.5)$$

where

$c_v$ = coefficient of consolidation = $\frac{k}{\rho_{water} g m_v}$

$m_v$ = coefficient of volume change

The dimensionless time used in the dimensionless form of (41.5) is found in the same way as in lecture 39. The underlined terms below are dimensioned variables; they are multiplied by dimensioned "scaling constants" to yield dimensionless (starred) variables:

Let $x^* = \underline{x}/x_{max}$, $t^* = c_v \underline{t}/x_{max}^2$, and $u_e^* = \underline{u_e} / u_{e\,max}$

or $\underline{x} = x^* x_{max}$, $\underline{t} = t^* x_{max}^2 / c_v$, and $\underline{u_e} = u_e^* u_{e\,max}$.

Now $H = x_{max}$, so the second relationship in the line above becomes

$$t = t^* H^2 / c_v = t^* H^2 \frac{\rho_{water} g m_v}{k} \qquad (41.6)$$

Compare (41.6) with (41.4) - they have the same form.

C Main Point 3: the key length scale is the flow path length

1 For double drainage, H = half the layer thickness
2 For single drainage, H = the whole layer thickness

3 Average consolidation ratio $U = \frac{consolidation\ at\ time\ t}{ultimate\ consolidation}$

a $U(t^*=1) \approx 92\%$ (using the dimensionless time defined above)
b $U(t^*=3) \approx 99\%$
c About 92% of the ultimate primary consolidation occurs in the time given by $x_{max}^2 / c_v$

D Examples

Consider layers of clay and sand, each 10’ thick, and suppose that the coefficient of volume change (“compressibility”) of the sand is 1/5 that of the clay, and the permeability of the sand is 10,000 times that of the clay.

<u>What is the ratio of the consolidation times of the sand and clay</u>?

Using (41.4)

$$\frac{t_{clay}}{t_{sand}} = \frac{H_{clay}{}^2 m_v(clay)/k_{clay}}{H_{sand}{}^2 m_v(sand)/k_{sand}} = \frac{m_v(clay)/k_{clay}}{m_v(sand)/k_{sand}} = \frac{5/1}{1/10{,}000} = 50{,}000$$

<u>If a 10’-thick layer of clay reaches 90% consolidation in 10 years, how long would it take for a clay layer 40’ thick to reach that level of consolidation</u>?

Using (41.4)

$$\frac{t_{40'}}{t_{10'}} = \frac{m_{v40'} H_{40'}{}^2/k_{40'}}{m_{v10'} H_{10'}{}^2/k_{10'}} = \frac{H_{40'}{}^2}{H_{10'}{}^2} = \frac{40^2}{10^2} = 16$$

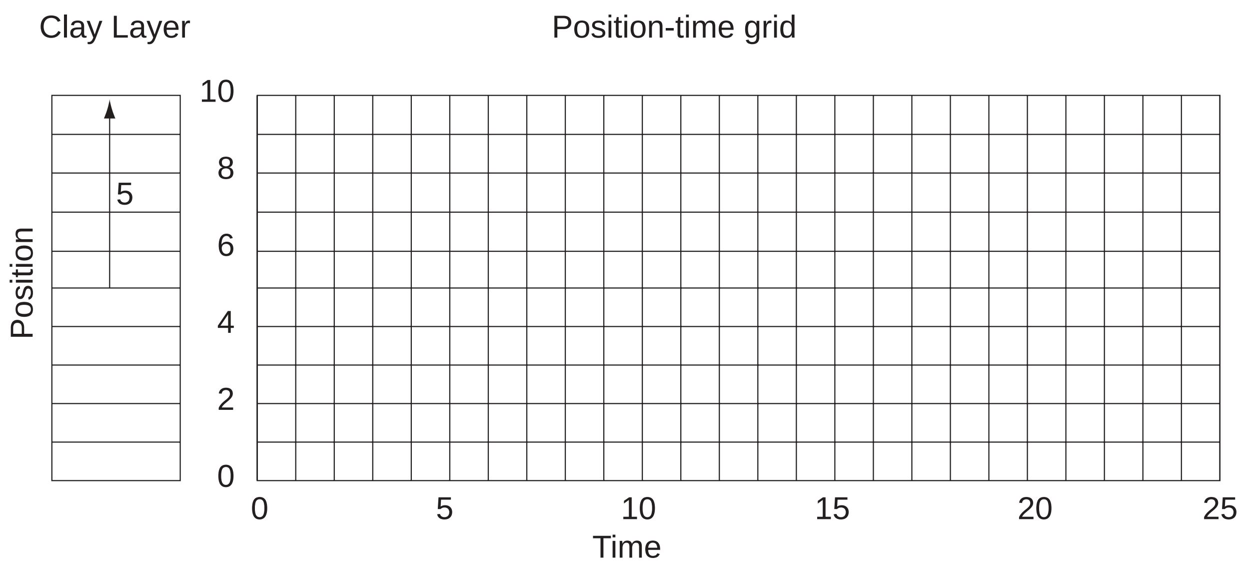
16 x 10 years = 160 years

Quadrupling the thickness increases the time for consolidation by a factor of 16.

<u>How long will a layer of clay take to reach 90% consolidation if the initial excess pore pressure distribution is constant across the layer and the layer is drained from its top and bottom?</u> One dimensionless time unit (t*) is given by $t^* = c_v t/x_{max}{}^2 = 1$.

We need to define our space-time grid to use the numerical procedure of lecture 39.

1 We start by dividing our layer up into slices of equal thickness. Let’s pick 10 slices. These 10 slices are defined by 11 points.
2 We need to pick a length scale ($x_{max}$). Let’s pick a length scale that corresponds to the longest distance fluid has to flow. For our example here, that is half the thickness of the layer. So $x_{max}$ is 5 increments.
3 How much time does 1 dimensionless time interval correspond to?
$t = t^* x_{max}{}^2/c_v = 5^2/c_v = 25/c_v$. These 25 time intervals are defined by 26 points.


Clay Layer
Position-time grid
Position
5
10
8
6
4
2
0
0
5
10
15
20
25
Time

```
% Matlab script transient_heat3.m
% Solves for 1-D transient heat flow by finite differences
% H = length of longest flow path
a = 11;         % Number of rows in the T matrix (# of points in space)
b = 26;         % Number of columns in the T matrix (# of points in time)
numit = 21;     % Number of iterations at each time step
T0 = 0;         % Temperature (excess pore pressure) at one end of rod
T1 = 1;         % Temperature (excess pore pressure) at other end of rod

% Initialize the "Temperature matrix"
T=zeros(a,b);               % First index is position, second is time step

% Set the intial temperature (or excess pore pressure distribution)
T(:,1) = (linspace(T1,T1,a))'           % Constant initial excess pore pressure

% Solve by finite difference method
for j=2:b;                              % j is the index number for the time step
        for k = 1:numit;
            for i = 2:a-1;              % i is the index number for the position
                T(i,j) = 0.25*( T(i+1,j-1)+T(i-1,j-1)+T(i+1,j)+T(i-1,j) );
            end
        end
end

% Plot the figures
figure(1)
clf
subplot(2,1,1)
plot(T);        % This plots the columns of T versus their index
title('Excess pore pressure distribution at various times')
xlabel('Position')
ylabel('Excess pore pressure/u0')
axis([1 a 0 1.05])

subplot(2,1,2)
plot([0:b-1]/(((a-1)/2)^2),mean(T),'-.b')
%plot([0:b-1]/(a-1)^2,mean(T),'-.b')
hold on
plot([0:b-1]/(((a-1)/2)^2),1-mean(T),'r')
%plot([0:b-1]/(a-1)^2,1-mean(T),'r')
title('Mean excess pore pressure (blue dash) and U (red solid) at various times')
xlabel('Dimensionless time')
ylabel('Mean excess pore pressure/u0 (or consolidation ratio)')
```

T =

Columns 1 through 7

| | | | | | | |
|---|---|---|---|---|---|---|
| 1.0000 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1.0000 | 0.7320 | 0.4226 | 0.3468 | 0.2930 | 0.2583 | 0.2307 |
| 1.0000 | 0.9282 | 0.7621 | 0.6249 | 0.5471 | 0.4860 | 0.4369 |
| 1.0000 | 0.9807 | 0.9132 | 0.8170 | 0.7317 | 0.6609 | 0.5978 |
| 1.0000 | 0.9945 | 0.9681 | 0.9130 | 0.8417 | 0.7688 | 0.6995 |
| 1.0000 | 0.9972 | 0.9813 | 0.9406 | 0.8773 | 0.8052 | 0.7341 |
| 1.0000 | 0.9945 | 0.9681 | 0.9130 | 0.8417 | 0.7688 | 0.6995 |
| 1.0000 | 0.9807 | 0.9132 | 0.8170 | 0.7317 | 0.6609 | 0.5978 |
| 1.0000 | 0.9282 | 0.7621 | 0.6249 | 0.5471 | 0.4860 | 0.4369 |
| 1.0000 | 0.7320 | 0.4226 | 0.3468 | 0.2930 | 0.2583 | 0.2307 |
| 1.0000 | 0 | 0 | 0 | 0 | 0 | 0 |

Columns 8 through 14

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.2078 | 0.1879 | 0.1701 | 0.1541 | 0.1397 | 0.1266 | 0.1148 |
| 0.3945 | 0.3570 | 0.3234 | 0.2931 | 0.2657 | 0.2409 | 0.2184 |
| 0.5415 | 0.4908 | 0.4449 | 0.4033 | 0.3657 | 0.3315 | 0.3006 |
| 0.6352 | 0.5764 | 0.5227 | 0.4740 | 0.4298 | 0.3897 | 0.3534 |
| 0.6674 | 0.6058 | 0.5495 | 0.4984 | 0.4519 | 0.4098 | 0.3715 |
| 0.6352 | 0.5764 | 0.5227 | 0.4740 | 0.4298 | 0.3897 | 0.3534 |
| 0.5415 | 0.4908 | 0.4449 | 0.4033 | 0.3657 | 0.3315 | 0.3006 |
| 0.3945 | 0.3570 | 0.3234 | 0.2931 | 0.2657 | 0.2409 | 0.2184 |
| 0.2078 | 0.1879 | 0.1701 | 0.1541 | 0.1397 | 0.1266 | 0.1148 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

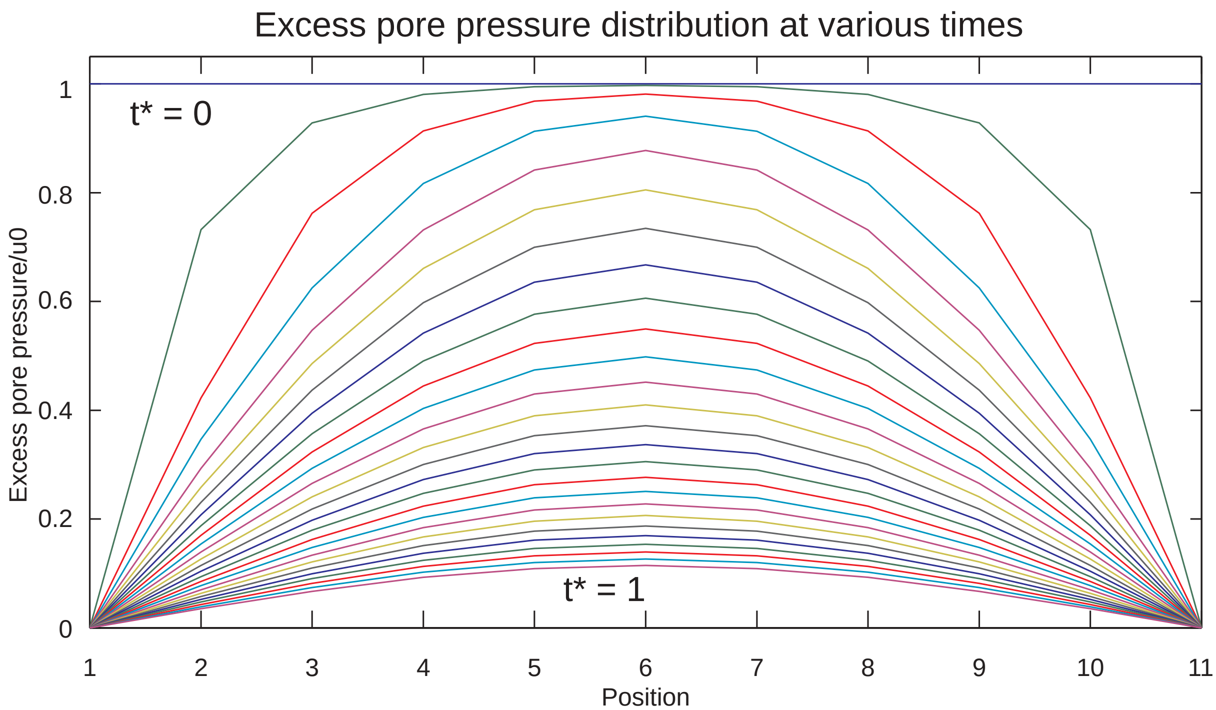
Columns 15 through 21

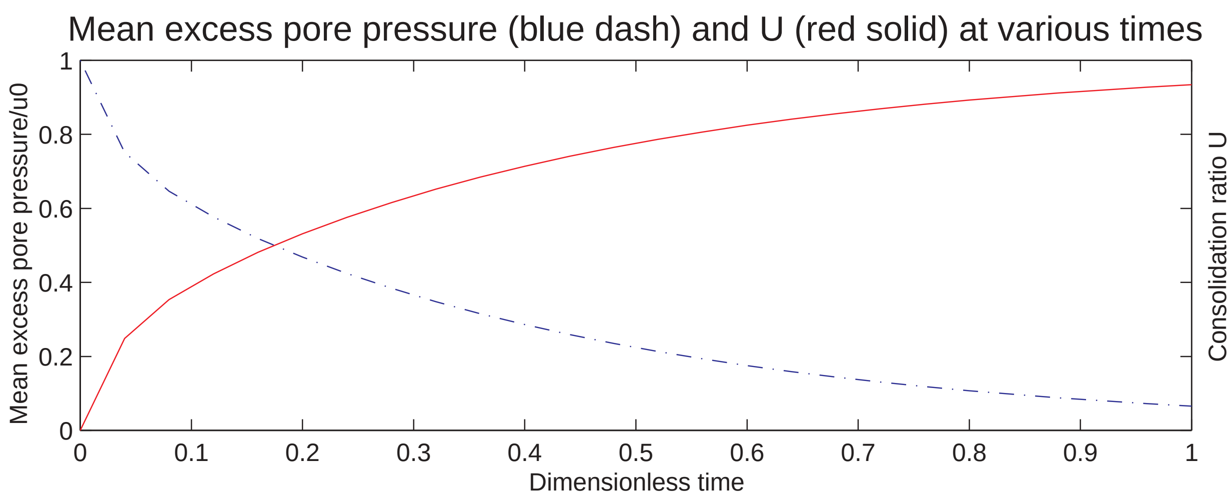
| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1041 | 0.0944 | 0.0856 | 0.0776 | 0.0704 | 0.0638 | 0.0578 |
| 0.1980 | 0.1795 | 0.1628 | 0.1476 | 0.1338 | 0.1213 | 0.1100 |
| 0.2725 | 0.2471 | 0.2240 | 0.2031 | 0.1842 | 0.1670 | 0.1514 |
| 0.3204 | 0.2905 | 0.2634 | 0.2388 | 0.2165 | 0.1963 | 0.1780 |
| 0.3369 | 0.3054 | 0.2769 | 0.2511 | 0.2277 | 0.2064 | 0.1872 |
| 0.3204 | 0.2905 | 0.2634 | 0.2388 | 0.2165 | 0.1963 | 0.1780 |
| 0.2725 | 0.2471 | 0.2240 | 0.2031 | 0.1842 | 0.1670 | 0.1514 |
| 0.1980 | 0.1795 | 0.1628 | 0.1476 | 0.1338 | 0.1213 | 0.1100 |
| 0.1041 | 0.0944 | 0.0856 | 0.0776 | 0.0704 | 0.0638 | 0.0578 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Columns 22 through 26

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0.0524 | 0.0475 | 0.0431 | 0.0391 | 0.0354 |
| 0.0997 | 0.0904 | 0.0820 | 0.0743 | 0.0674 |
| 0.1373 | 0.1245 | 0.1129 | 0.1023 | 0.0928 |
| 0.1614 | 0.1463 | 0.1327 | 0.1203 | 0.1091 |
| 0.1697 | 0.1539 | 0.1395 | 0.1265 | 0.1147 |
| 0.1614 | 0.1463 | 0.1327 | 0.1203 | 0.1091 |
| 0.1373 | 0.1245 | 0.1129 | 0.1023 | 0.0928 |
| 0.0997 | 0.0904 | 0.0820 | 0.0743 | 0.0674 |
| 0.0524 | 0.0475 | 0.0431 | 0.0391 | 0.0354 |
| 0 | 0 | 0 | 0 | 0 |

```
»mean(T)

ans =

  Columns 1 through 7

    1.0000    0.7516    0.6467    0.5767    0.5186    0.4685    0.4240

  Columns 8 through 14

    0.3841    0.3482    0.3156    0.2861    0.2594    0.2352    0.2133

  Columns 15 through 21

    0.1934    0.1753    0.1590    0.1441    0.1307    0.1185    0.1074

  Columns 22 through 26

    0.0974    0.0883    0.0801    0.0726    0.0658

»1-mean(T)

ans =

  Columns 1 through 7

         0    0.2484    0.3533    0.4233    0.4814    0.5315    0.5760

  Columns 8 through 14

    0.6159    0.6518    0.6844    0.7139    0.7406    0.7648    0.7867

  Columns 15 through 21

    0.8066    0.8247    0.8410    0.8559    0.8693    0.8815    0.8926

  Columns 22 through 26

    0.9026    0.9117    0.9199    0.9274    0.9342
```

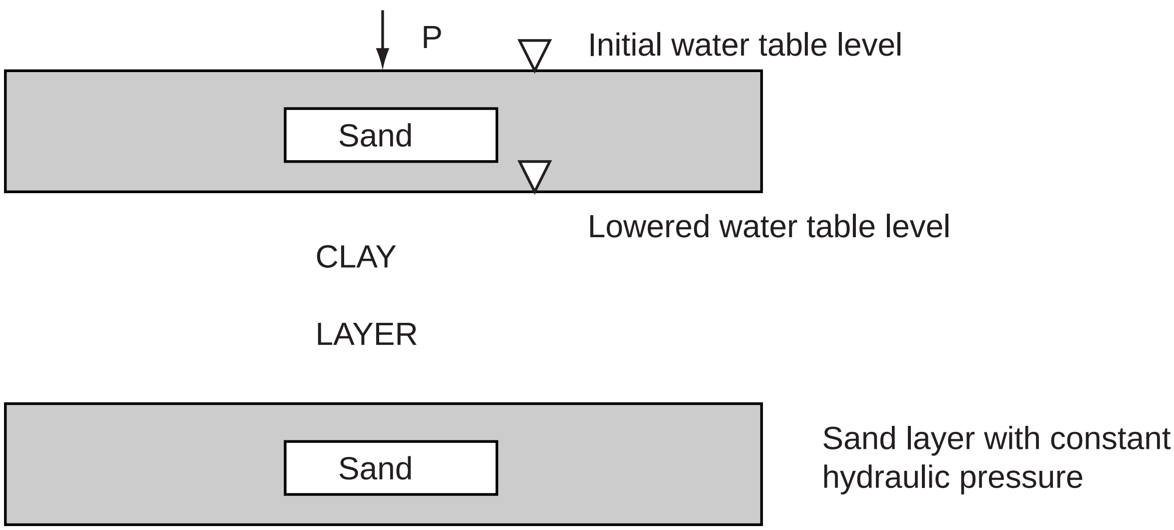



So after one dimensionless time step, about 92% (i.e., roughly 90%) of all the consolidation that will occur will have occurred.

An analogous procedure could be used to solve for the problem where the drainage is single sided and the initial excess pore pressure distribution is not constant but varies linearly.

Water Table Level and Subsurface Layers




Saturated clay layer (of thickness $H_0$) with double drainage

Pore Pressure Levels and Excess Pore Pressure