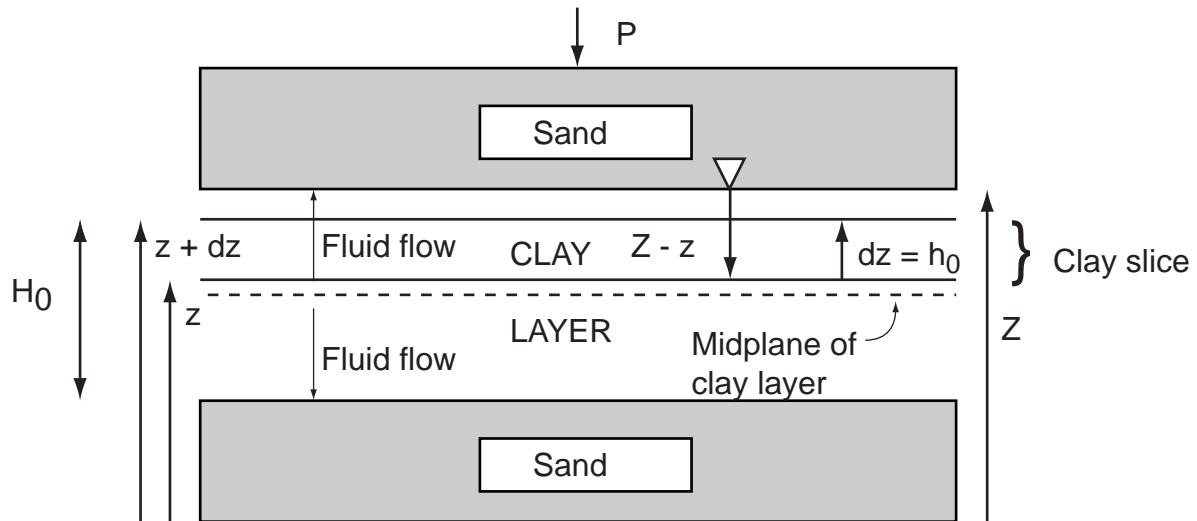


TERZAGHI'S 1-D CONSOLIDATION EQUATION (40)

I Main Topics

- A The one-dimensional consolidation equation analog to heat flow
- C Calculating consolidation for double-sided drainage

II The one-dimensional consolidation equation analog to heat flow



Saturated clay layer (of thickness H_0) with double drainage

At time $t=0$, a pressure P is applied to the top of our sand-clay-sand sandwich. The increase in load will be initially born by the water; the water pressure in all the layers goes up. This excess pore pressure (water pressure above the hydrostatic [equilibrium] level) will dissipate quickly in the sand layers because the water flows rapidly through the high-permeability sand (it flows sideways in a "violation" of our one-dimensional assumptions). The excess pore pressure will dissipate slowly in the clay. The water in the clay has to flow vertically because of the long, slow horizontal flow path in the clay. Experience shows that the sand will consolidate little. Individual sand grains are stiff, and their volumes change little with the applied loads. Unless the sand grains change their packing, the collective volume of the sand won't change significantly either. The clay, however, will consolidate significantly.

We start by looking at how the water flows, using Darcy's Law.

$$Q = -kiA \quad (40.1)$$

Q=discharge; k=conductivity; i=head gradient; A=Area

Dimensions

Q: L³/t k: L/t i: L/L (dimensionless) A: L²

$$q = Q/A = \text{flux} = -ki \quad (40.2)$$

q = unit discharge (discharge/unit area). Dimensions of velocity.

Now we investigate the head gradient, which drives fluid flow:

$$i = \frac{\partial H}{\partial z} = \frac{\partial(z + [u/\rho g])}{\partial z} = \frac{\partial(z + [(u_{hydrostatic} + u_{excess})/\rho g])}{\partial z} \quad (40.3)$$

The excess pore pressure (u_{excess}) in (40.3) is the difference between the actual pore pressure (u) and hydrostatic pressure ($u_{hydrostatic}$). The rate of change in the elevation head (z) and the hydrostatic pressure head cancel each other out exactly:

$$\frac{\partial(z + [u_{hydrostatic}/\rho g])}{\partial z} = \frac{\partial(z + [\rho g(Z - z)/\rho g])}{\partial z} = \frac{\partial(z + (Z - z))}{\partial z} = \frac{\partial Z}{\partial z} = 0$$

Without this cancellation, then water at the bottom of a still swimming pool might flow to the top of the pool! So equation (40.3) simplifies:

$$i = \frac{\partial H}{\partial z} = \frac{\partial(z + [(u_{hydrostatic} + u_{excess})/\rho g])}{\partial z} = \frac{1}{\rho g} \frac{\partial u_e}{\partial z} \quad (40.4)$$

The head gradient at the base of the slice (i.e., at elevation "z") is:

$$i_1 = \frac{1}{\rho g} \frac{\partial u_e}{\partial z} \quad (40.5)$$

The head gradient at the top of the slice (i.e., at elevation "z+dz") is:

$$i_2 = \frac{1}{\rho g} \frac{\partial}{\partial z} (u_e + \Delta u_e) = \frac{1}{\rho g} \frac{\partial}{\partial z} \left(u_e + \frac{\partial u_e}{\partial z} dz \right) = \frac{1}{\rho g} \left(\frac{\partial u_e}{\partial z} + \frac{\partial^2 u_e}{\partial z^2} dz \right) \quad (40.6)$$

The change in unit discharge [i.e., the net flow of water out of the clay slice (per unit area)] reflects the water loss in the slice (per unit area):

$$\Delta q = q_2 - q_1 = -k(i_2 - i_1) = \frac{-k}{\rho g} \frac{\partial^2 u_e}{\partial z^2} dz \quad (40.7)$$

Two comments. First, we have held k , the hydraulic conductivity, constant - is that OK? Second, the right hand side of (40.7) looks somewhat like one side of the heat equation. What about the left side? The change in unit discharge of the slice times the area (A) of the slice gives the water volume loss with respect to time:

$$\Delta q A = \frac{\partial V_{water}}{\partial t} \quad \left(\frac{L}{T} L^2 = \frac{L^3}{T} \right) \quad (40.8)$$

The water volume loss is also the void volume loss in the clay. If both sides of (40.8) are divided by the area A , then the right side is the change in void volume/unit area with respect to time, or in other words, the height change of the slice with respect to time. The height change, in turn, is the product of the vertical strain (ϵ_z) and the original slice height $h_0 = dz$, so

$$\Delta q = \frac{\partial(V_{water}/A)}{\partial t} = \frac{\partial(\Delta h)}{\partial t} = \frac{\partial(\Delta h/h_0) dz}{\partial t} = \frac{(\partial \epsilon_z) dz}{\partial t} \quad (40.9)$$

Let's substitute this into the left side of equation (40.7):

$$\frac{(\partial \epsilon_z) dz}{\partial t} = \frac{-k}{\rho g} \frac{\partial^2 u_e}{\partial z^2} dz \quad (40.10a)$$

$$\frac{\partial \epsilon_z}{\partial t} = \frac{-k}{\rho g} \frac{\partial^2 u_e}{\partial z^2} \quad (40.10b)$$

This looks even more like the heat equation, but note that $\epsilon_z \neq u_e$.

Equation (40.10b) expresses how the vertical strain changes with time relative to the second partial derivative of the excess pore pressure with respect to position. We seek to find how the vertical strain changes as a function of the effective stress. The coefficient of compressibility (m_v), also known as the coefficient of volume change, is defined as the change in volumetric strain divided by the change in effective stress. For our 1-D case:

$$m_v = \frac{\Delta V/V_0}{\Delta \sigma'} = \frac{\Delta h/h_0}{\Delta \sigma'} \quad (40.11)$$

In our case here, the change in effective stress is exactly opposite to the change in the excess pore pressure (i.e., the increase in load picked up by the soil skeleton equals the decrease in the excess water pressure). So:

$$m_v = \frac{\Delta h/h_0}{\Delta \sigma'} = \frac{-\Delta h/h_0}{\Delta u_e} = \frac{-\varepsilon_z}{u_e - u_0} \quad (40.12)$$

Solving for the strain, which appears on the left side of (40.10b), gives

$$\varepsilon_z = -m_v(u_e - u_0) \quad (40.13)$$

Inserting (40.13) into (40.10b) yields

$$\frac{\partial(m_v(u_e - u_0))}{\partial t} = \frac{k}{\rho g} \frac{\partial^2 u_e}{\partial z^2} \quad (40.14)$$

Now differentiate the left side of (40.14), noting that u_0 is a constant:

$$m_v \frac{\partial(u_e - u_0)}{\partial t} = m_v \left(\frac{\partial u_e}{\partial t} - \frac{\partial u_0}{\partial t} \right) = m_v \frac{\partial u_e}{\partial t} = \frac{k}{\rho g} \frac{\partial^2 u_e}{\partial z^2} \quad (40.15)$$

This can be simplified by grouping all the constants:

$$\frac{\partial u_e}{\partial t} = \frac{k}{m_v \rho g} \frac{\partial^2 u_e}{\partial z^2} = C_v \frac{\partial^2 u_e}{\partial z^2} \quad (40.16)$$

The term C_v is called the coefficient of consolidation. This equation is exactly analogous to the heat equation:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \quad (40.17)$$

So the diffusion of heat in an insulated bar is analogous to the diffusion of excess pore pressure in a soil. A key difference between the two phenomena is that the hydraulic conductivity of earth materials generally decreases as the porosity or void ratio decreases; this effect would increase the consolidation time.

III Calculating consolidation for double-sided drainage

At this point we return to some practical questions: How much will a clay layer will consolidate? What is the ultimate consolidation? How does the consolidation relate to the pore pressure? We start with our definition for strain, and focus on the thickness of the clay layer (H).

$$\varepsilon_{vertical} = \frac{\Delta H}{H_0} = \frac{\Delta V_{voids}}{V_{total}} \quad (40.18)$$

The change in void volume can be expressed in terms of the void ratio e :

$$\frac{\Delta V_{voids}}{V_{total}} = \frac{\frac{V_{voids}(final)}{V_{solids}} - \frac{V_{voids}(initial)}{V_{solids}}}{\frac{V_{total}}{V_{solids}}} = \frac{\Delta e}{\frac{V_{solids} + V_{voids}(initial)}{V_{solids}}} = \frac{\Delta e}{1 + e_0} \quad (40.19)$$

So putting (40.18) and (40.19) together:

$$\Delta H = H_0 \frac{\Delta e}{1 + e_0} \quad (40.20)$$

This provides a way to limit the ultimate consolidation. We know that Δe cannot exceed the initial void ratio, so

$$\Delta H_{max} = H_0 \frac{e_0}{1 + e_0} \quad (40.21)$$

If we know the ultimate change in effective stress, then the coefficient of volume change can be used to get the thickness change. In our 1-D situation, volume changes occur as a result of height changes, so

$$m_v = \frac{\Delta H / H_0}{\Delta \sigma'} \quad (40.22)$$

Solving for H_0 in the above expression yields

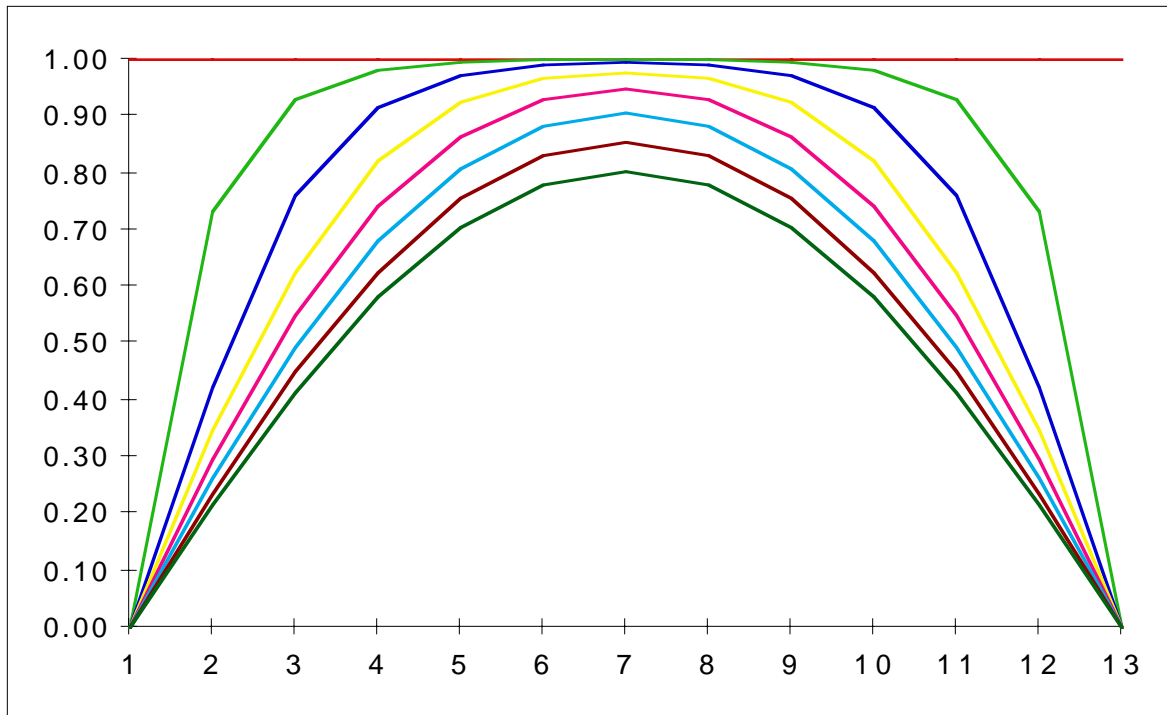
$$H_0 m_v \Delta \sigma' = \Delta H \quad (40.23)$$

This not only can be used to give the ultimate consolidation, but because we can calculate the change in effective stress in a layer from the change in excess pore pressure, we can calculate how the consolidation varies as a function of time. The excess pore pressure will vary through a layer, so the average excess pore pressure (with the overbar) in the layer is what is used:

$$\frac{\Delta H(t)}{\Delta H(t=\infty)} = \frac{H_0 m_v \Delta \bar{\sigma}'}{H_0 m_v \Delta \sigma'_{\infty}} = \frac{\Delta \bar{\sigma}'}{\Delta \sigma'_{\infty}} = \frac{u_{excess,t=0} - \bar{u}_{excess}(t)}{u_{excess,t=0}} = \frac{\Delta \bar{u}_{excess}(t)}{u_{excess,t=0}} = U \quad (40.24)$$

This ratio is known as the consolidation ratio U . One note of caution in the use of equations (40.23) and (40.24): they yield unrealistically large values of settlement for large values of $\Delta \sigma'$. For example, if $(\Delta \sigma')(m_v) > 1$, then $\Delta H > H_0$. The layer obviously cannot achieve a negative thickness, so m_v should only be considered a constant for a finite range of $\Delta \sigma'$.

At a given instant in time, this consolidation ratio can be visualized in the following plot, where the horizontal axis is position across a layer, and the vertical axis is the excess pore pressure.



These curves show how the excess pore pressure decays with time. The ratio of the area above a curve to the area of the entire box is the consolidation ratio U at a given time. The lower the curve, the longer the time for the excess pore pressure to diffuse.