SUBSIDENCE MECHANICS: HEAT FLOW ANALOG (38)

I Main Topics
- A Motivation: Why investigate the flow of heat?
- B Development of 1-D heat flow equation as analog for consolidation
- C Solution of heat flow equation using finite-difference approximation

II Motivation: Why investigate the flow of heat?
- A The form of the heat flow equation (i.e., diffusion equation) is identical to the 1-D consolidation equation but is easier to understand and derive from a mechanical standpoint
- B Diffusion of heat is analogous to diffusion of excess pore pressure (i.e., pore pressure in excess of hydrostatic pressure)
- C Hundreds of analytic solutions for heat flow problems exist; (see Carslaw, H.S., and Jaeger,J.C., 1984, Conduction of heat in solids, Clarendon Press, Oxford, 510 p.)
- D The diffusion equation for the diffusion of solutes in water also has the same form as the heat equation, so solutions to the heat equation have many varied and important practical uses.

Conduction Analogies

	Fluid	Heat	Electrical	Chemical
Potential	Total head h (cm)	Temperature T (°)	Voltage V (volts)	Concentration c (moles/cm^3)
Storage	Fluid volume W (cm^3/cm^3)	Thermal energy U (cal/cm^3)	Charge Q (Coulombs)	moles
Conductivity	Hydraulic k (cm/sec)	Thermal k (cal/°/cm/sec)	Electrical (volts/sec)	Diffusion coeff. (moles/sec)
Flow (discharge)	Q (cm^3/sec)	Q (cal/sec)	Current I (amps)	J (moles/ sec)
Flux (Spec. dis.)	q (cm/sec)	q (cal/[sec•cm^2])	amps/cm^2	moles sec^{-1}cm^2
Gradient	$i=-\frac{\partial h}{\partial x}$ (cm/cm)	$i=-\frac{\partial T}{\partial x}$ (°/cm)	$i=-\frac{\partial V}{\partial x}$ (volts/cm)	$i=-\frac{\partial c}{\partial x}$ (moles/ cm^4)
Conduction	Darcy's Law $Q=-k\frac{\partial h}{\partial x}A$	Fourier's Law $Q=-k\frac{\partial T}{\partial x}A$	Ohm's Law $I=-\sigma\frac{\partial V}{\partial x}A=\frac{V}{R}$	Fick's Law $J=-D\frac{\partial c}{\partial x}A$
Capacitance	Coefficient of volume change	Volumetric heat C (cal/°/ cm^3)	Capacitance C (farads)	- - - - - - - - - -
Steady state flow	$\nabla^2 h=0$	$\nabla^2 T=0$	$\nabla^2 V=0$	$\nabla^2 c=0$
1-D diffusion	$\frac{\partial h}{\partial t}=c_v\frac{\partial^2 h}{\partial x^2}$	$\frac{\partial T}{\partial t}=\frac{k}{C}\frac{\partial^2 T}{\partial x^2}$	$\frac{\partial V}{\partial t}=\frac{\sigma}{C}\frac{\partial^2 V}{\partial x^2}$	$\frac{\partial c}{\partial t}=k\frac{\partial^2 c}{\partial x^2}$

II Development of 1-D heat flow equation as analog for consolidation

A Isotropic, uniform material

B Definition of terms

1 Position (x) Dimensions of meters

2 Time (t) Dimensions of seconds

3 Heat flux (q)

a Rate of heat energy transfer across a given unit area per unit time. The flux can vary with time and position, so q = q(x,t).

b Dimensions: Joules/($meter^2$ · sec)

c 1 heat flow unit (hfu) = 10^{-6} cal cm^{-2} sec^{-1} = 0.04184 W m^{-2}

4 Temperature (T) Dimensions of degrees (°K)

Temperature can vary with time and position, so T = T(x,t).

C Fourier's Law of Heat Conduction (1-D)

$$q = -k\frac{\partial T}{\partial x} \tag{38.1}$$

where $\frac{\partial T}{\partial x}$ = temperature gradient

1 Heat flow is proportional to the temperature gradient

2 k = coefficient of thermal conductivity

Dimensions: Joules sec^{-1} m^{-1} $°K^{-1}$

k assumed to be constant [i.e., k ≠ k(x); k ≠ k(t);k ≠ k(T)]

3 Dimension check: $q = -k\frac{\partial T}{\partial x}$

$$\frac{Joules}{meter^2 \sec} = \left(\frac{Joules}{\sec\ m\ °K}\right)\left(\frac{°K}{m}\right)$$

4 Partial derivative used because T is a function of x and t.

5 The minus sign: for heat to flow from x_1 to x_2 , where $x_1 < x_2$, $T(x_1) > T(x_2)$. So positive heat flow corresponds to a drop in temperature.

6 Finite difference approximation: $q = -k\frac{\Delta T}{\Delta x}$

6 Analog: Darcy's Law

$$q = -k\frac{\partial H}{\partial x} \tag{38.2}$$

q = volumetric flow rate per unit area (*specific* discharge)

k = hydraulic conductivity

$\frac{\partial H}{\partial x}$ = head gradient

C The heat equation

1 Conservation of energy (see handout)

2 1-D form: $C\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ a parabolic differential equation

3 2-D form: $C\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$

4 3-D form: $C\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

5 Laplace Equation (2-D or 3-D)

Applies to steady state distribution of temperature

a Temperature does not change as a function of time

b $\frac{\partial T}{\partial t} = 0$

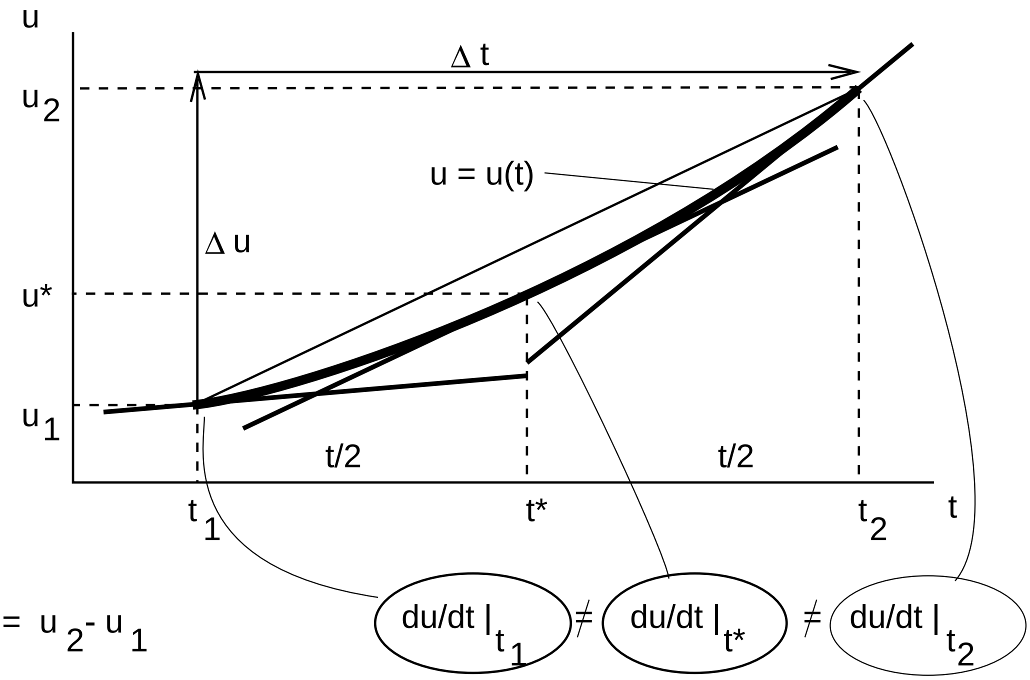
c $0 = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$ or $0 = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$ or $0 = \nabla^2 T$

d In finite-difference form, the Laplace equation means that the value of the function T (here T=temperature) at a given point is the average of the values at the nearest neighboring points, where the points are on a square grid (see notes on wave eqn).

6 Initial conditions and boundary conditions

The equations above indicate how temperature will change as a function of time and position within a body. If we know the temperature conditions at the boundaries of a body at specific points in time and space, then we can solve these equations to find the temperature distribution within the body

DERIVATIVES AND PARTIAL DERIVATIVES



$\Delta u = u_2 - u_1$

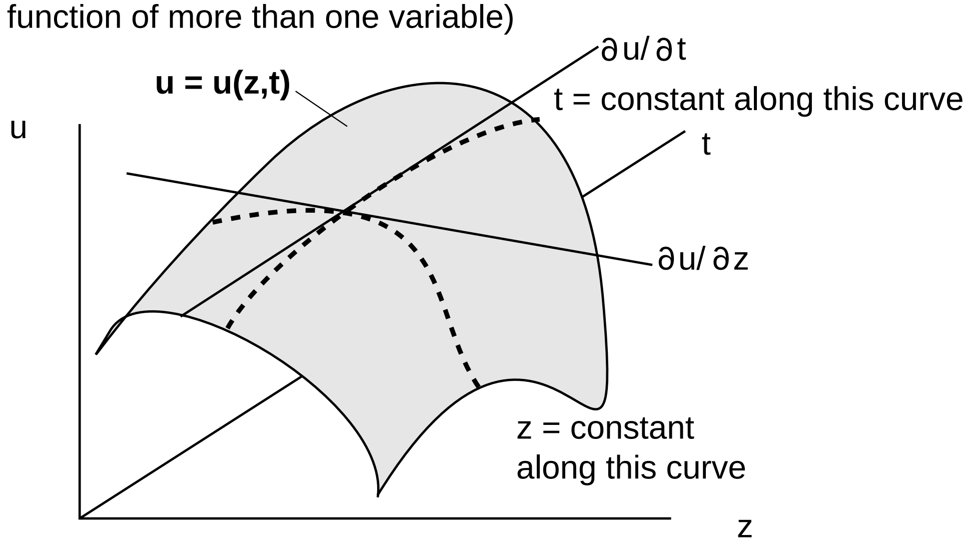
$\Delta t = t_2 - t_1$

$\Delta t/2 = t^* - t_1 = t_2 - t^*$

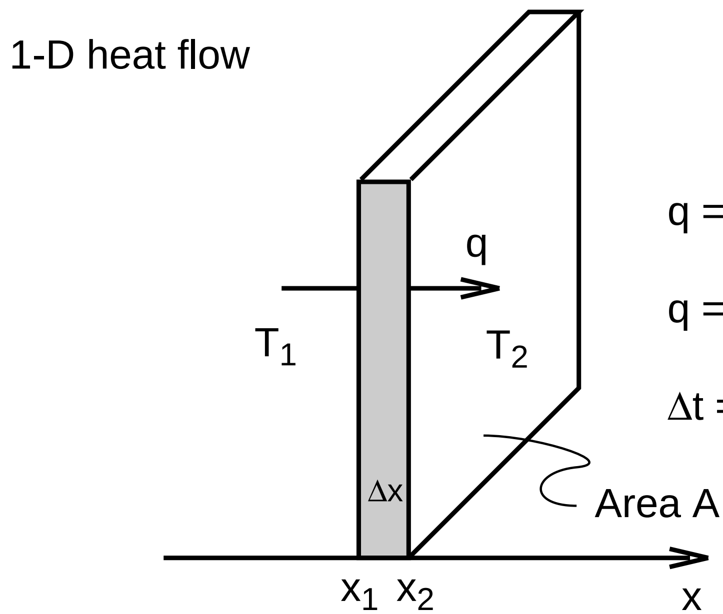
However, $du/dt|_{t^*} \approx \dfrac{u_2 - u_1}{t_2 - t_1}$, and

$$du/dt|_{t^*} \approx \frac{du/dt|_{t_1} + du/dt|_{t_2}}{2}$$

THE PARTIAL DERIVATIVE
(u is a function of more than one variable)



Heat Flow



T = Temperature
$T_1 > T_2$

$q = -k\ (\partial T/\partial x)$

q = heat flux

Δt = increment of time

Change in heat energy = heat in - heat out
(in the slab)

$(\Delta T)(\text{mass})(\text{spec. heat}) = A\Delta t\ [q(x1) - q(x2)] = A\Delta t\ [q(x1) - \{q(x1)+\Delta q\}]$

$(\Delta T)(\rho\ V)(c) = -A\Delta t\ [\Delta q]$

$(\Delta T)(\rho\ A\Delta x)(c) = -A\Delta t\ [(\partial q\ /\partial x)\ \Delta x\ ]$, but $q = -k\ \ \partial T\ /\partial x$, so

$(\Delta T)(\rho\ A\Delta x)(c) = A\Delta t\ k[\ \underline{(\ \partial^2 T\ /\partial x^2)}\ \Delta x\ ]$

If the temperature profile isn't curved (i.e., if the underlined term is zero), then there can be no change in heat energy in the slab.

$(\Delta T/\Delta t)(\rho\ c/k) = (\ \partial^2 T\ /\ \partial x^2)$ $K = \rho\ c/k$ = diffusivity

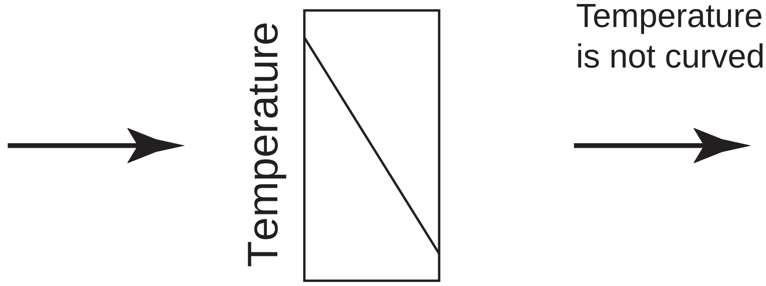
$$\boxed{(\partial T/\partial t)(\rho\ c/k) = (\ \partial^2 T\ /\ \partial x^2)}$$ ⟵ 1-D Heat Flow Eqn

To a good approximation, the rate of temperature change with time is proportional to the curvature of the temperature profile.

EFFECT OF TEMPERATURE GRADIENT CHANGES ON HEAT FLOW

Heat in = Heat out

Temperature profile
is not curved

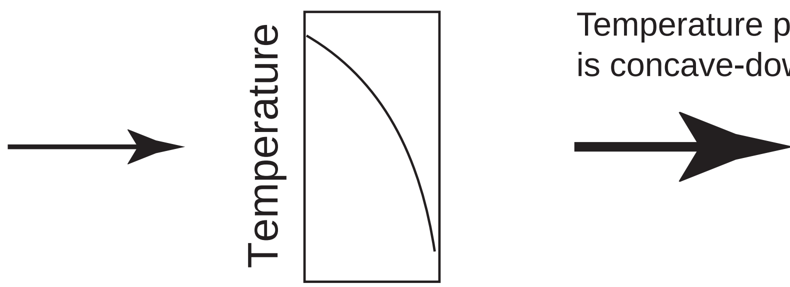


-Temperature gradient going in = -Temperature gradient going out

No heat is stored in the box

Heat in < Heat out

Temperature profile
is concave-down

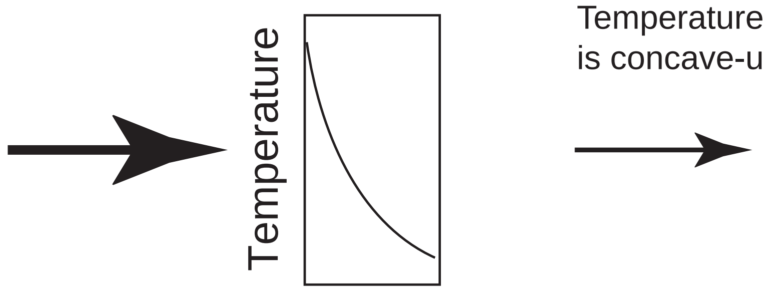


-Temperature gradient going in < -Temperature gradient going out

Heat is lost from the box

Heat in > Heat out

Temperature profile
is concave-up

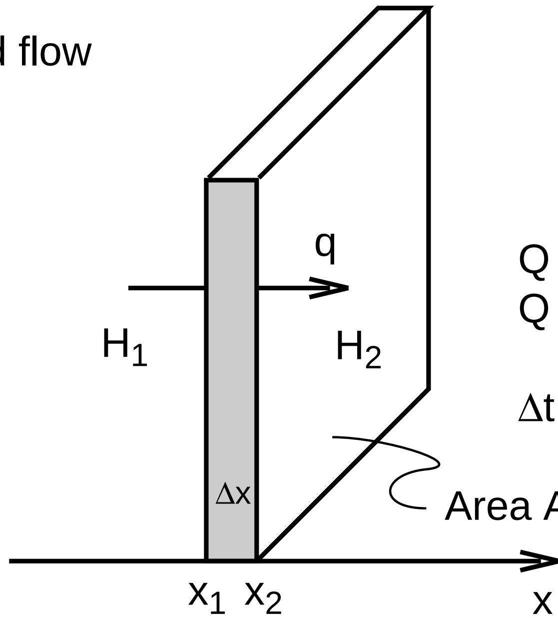


-Temperature gradient going in > -Temperature gradient going out

Heat is gained in the box

Fluid Flow

1-D fluid flow



H = head
$H_1 > H_2$

Q = volumetric flow rate
$Q = -k\ (\partial H/\partial x)\ A$

Δt = time increment

Change in fluid volume = volume in - volume out
(in the slab)

$$(\Delta V_W) = \Delta t\ [Q(x1)-Q(x2)] = \Delta t\ [Q(x1)-\{Q(x1)+\Delta Q\}]$$

$$(\Delta V_W) = \Delta t\ [-\Delta Q]$$

$$(\Delta V_W) = -\Delta t\ [(\partial Q/\partial x)\ \Delta x\ ],\quad \text{but } Q = -kA\ \partial H/\partial x, \text{ so}$$

$$(\Delta V_W) = kA\ \Delta t\ [(\underline{\partial^2 H/\partial x^2})\ \Delta x\ ]$$

If the head profile isn't curved (i.e., if the underlined term is zero), then there can be no change in water volume in the slab.

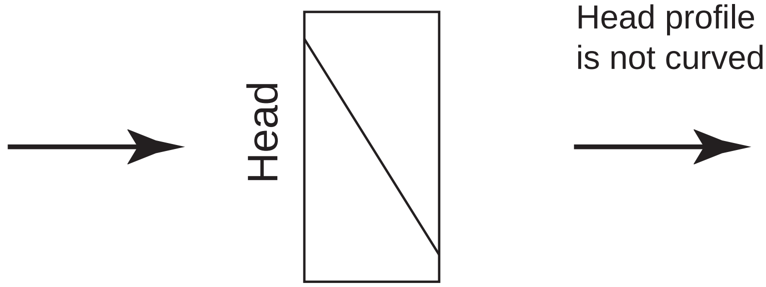
$$(\Delta V_W/\Delta V_{box})/\Delta t) = k(\partial^2 H/\partial x^2)$$

To a good approximation, the rate of fluid volume change with time is proportional to the curvature of the head profile.

EFFECT OF HEAD GRADIENT CHANGES ON FLUID FLOW

Fluid in = Fluid out

Head profile
is not curved



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Head profile
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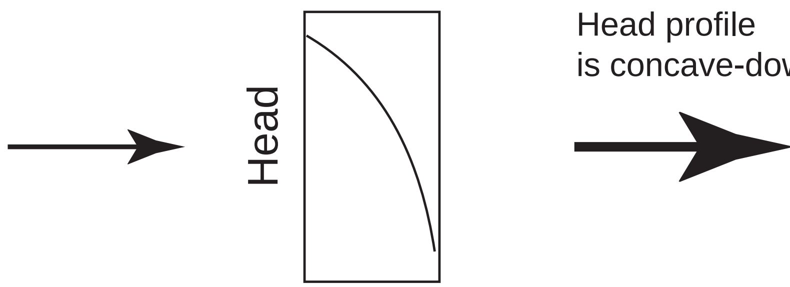

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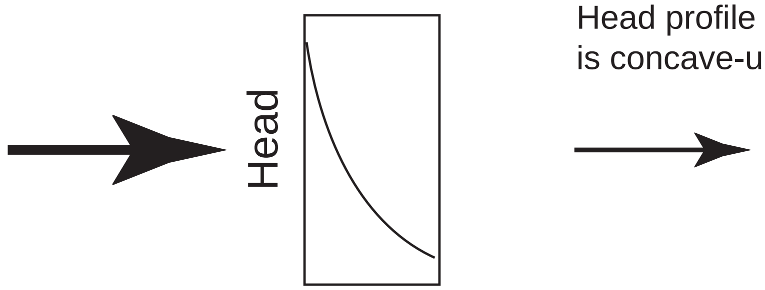

|Head gradient going in > -Head gradient going out

Fluid is gained in the box