

## STRESS AT A POINT

## I Main Topics

- A Stress vector (traction) on a plane
- B Stress at a point
- C Principal stresses
- D Initial 2-D stress transformation equations

II Stress vector (traction) on a particular plane:  $\vec{\tau}$ 

- A  $\vec{\tau} = \lim_{A \rightarrow 0} \vec{F}/A$ . Dimensions of force per unit area
- B Traction vectors can be added vectorially.
- C A traction vector can be resolved into normal and shear components.
  - 1 A normal traction  $\vec{\tau}_n$  acts perpendicular to a plane
  - 2 A shear traction  $\vec{\tau}_s$  acts parallel to a plane
- D The magnitudes of tractions depend on the orientation of the plane

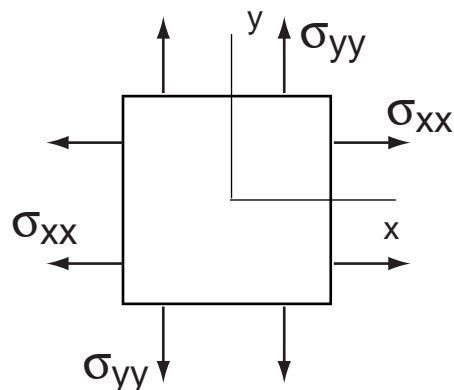
## III Stress at a point

- A Stresses refer to **balanced** internal "forces (per unit area)". They differ from force vectors, which, if unbalanced, cause accelerations

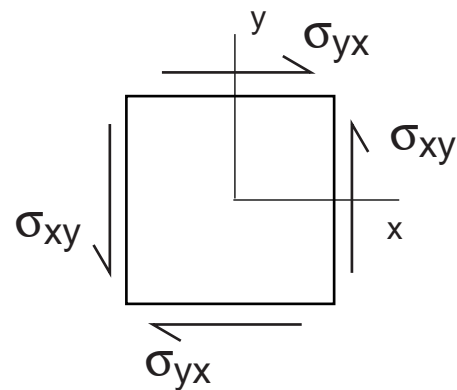
- B **"On -in convention": The stress component  $\sigma_{ij}$  acts on the plane normal to the i-direction and acts in the j-direction**

- 1 Normal stresses:  $i=j$

- 2 Shear stresses:  $i \neq j$



Normal stresses



Shear stresses

$$C \quad \sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad \text{or} \quad \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

D In 3-D, nine components are needed to define the state of stress at a point. These nine components are the traction components that act on three perpendicular planes (planes with normals in the x-, y-, and z-directions)

E The state of stress can (and usually does) vary from point to point.

F For rotational equilibrium,  $\sigma_{xy} = \sigma_{yx}$ ,  $\sigma_{xz} = \sigma_{zx}$ ,  $\sigma_{yz} = \sigma_{zy}$

#### IV **Principal Stresses** (these have magnitudes and orientations)

A Principal stresses act on planes which feel no shear stress

B The principal stresses are normal stresses.

C Principal stresses act on perpendicular planes

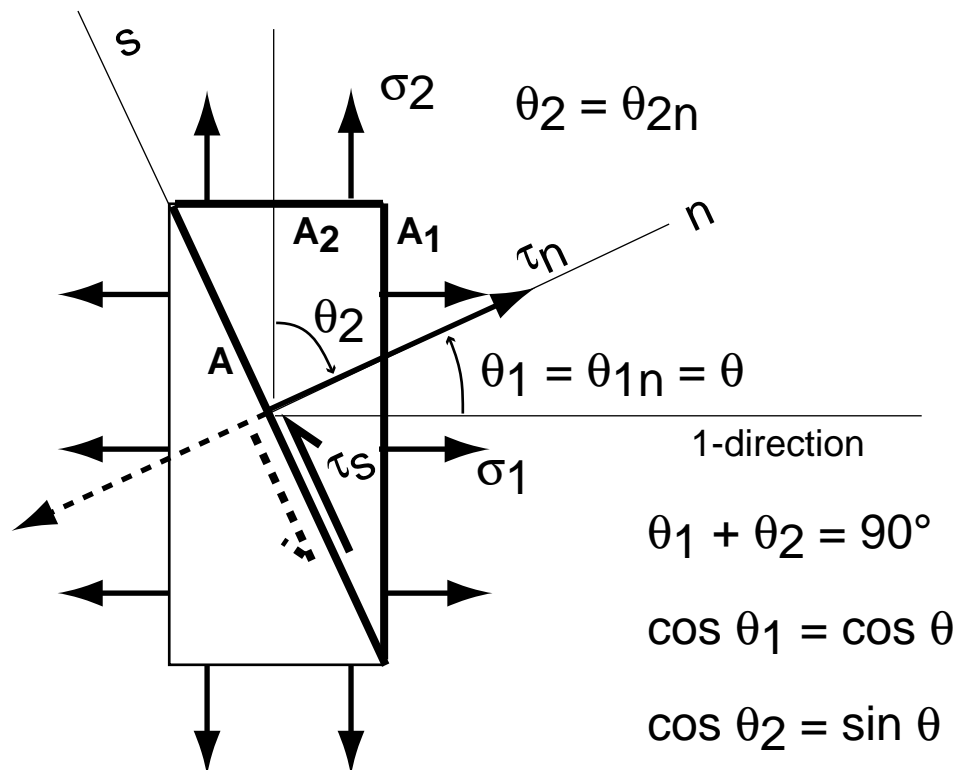
D The maximum, intermediate, and minimum principal stresses are usually designated  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$ , respectively. Note that the principal stresses have a single subscript.

E The principal stresses represent the stress state most simply.

$$F \quad \sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \quad \text{3-D} \quad \text{or} \quad \sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \quad \text{2-D}$$

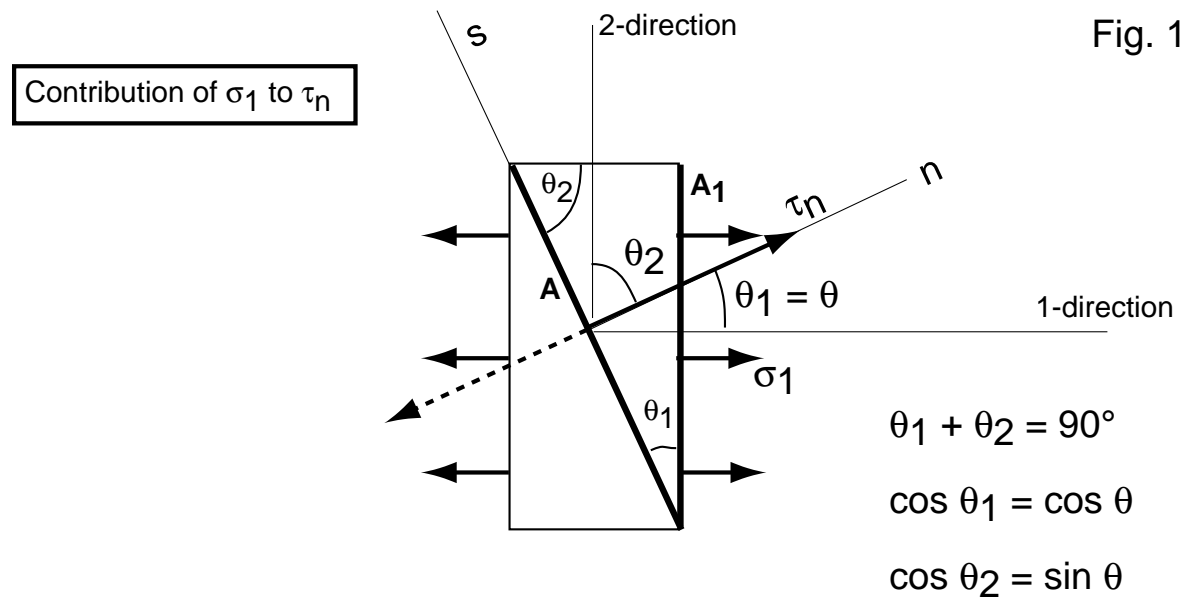
## V Initial 2-D stress transformation equations

A Consider three planes A, A1, and A2 that form the sides of a triangular prism; these have normals in the n-, 1-, and 2-directions, respectively. Plane A1 is acted on by known normal stress  $\sigma_1$ . Plane A2 is acted on by known normal stress  $\sigma_2$ . The n-direction is at angle  $\theta_1 (= \theta)$  with respect to the 1-direction, and at angle  $\theta_2$  with respect to the 2-direction. The s-direction is at a counter-clockwise  $90^\circ$  angle relative to the n-direction (like y and x).



To find the normal traction,  $\tau_n$  that acts on plane A, we determine how much  $\sigma_1$  and  $\sigma_2$  each contribute individually to  $\tau_n$  and then sum the contributions (Figures 16.1, 16.2). Similarly, to find the shear traction,  $\tau_s$  that acts on plane A, we determine how much  $\sigma_1$  and  $\sigma_2$  each contribute to  $\tau$  and then sum those contributions (Figures 16.3, 16.4). The results are:

- 1  $\tau_n = \sigma_1 \cos^2 \theta + \sigma_2 \sin^2 \theta$
- 2  $\tau_s = (\sigma_2 - \sigma_1) \sin \theta \cos \theta$



What does  $\sigma_1$  on face  $A_1$  of area  $A_1$  contribute to  $\tau_n$  on face  $A$  of area  $A$ ?

Start with the definition of stress:  $\tau_n^{(1)} = F_n^{(1)} / A$ .

The unknown quantities  $F_n$  and  $A$  must be found from the known quantities  $\sigma_1$  and  $\theta$ .

To do this we first find the force  $F_1$  associated with  $\sigma_1$ :

Force = (stress)(area)

$$F_1 = \sigma_1 A_1$$

The component of force  $F_1$  that acts along the  $n$ -direction is  $F_1 \cos \theta_1$ .

$$F_n^{(1)} = F_1 \cos \theta_1$$

As can be seen from the diagram atop the page  $A_1 = A \cos \theta_1$ , so

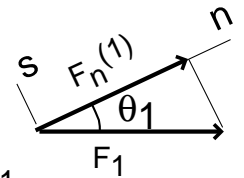
$$A = A_1 / \cos \theta_1$$

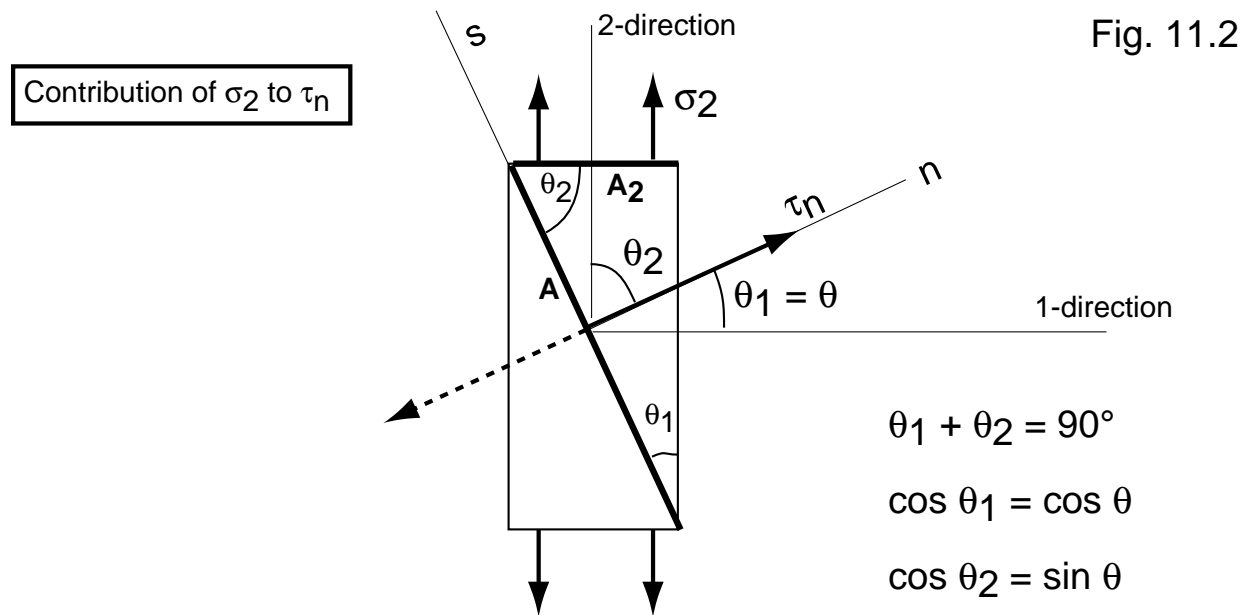
So the contribution to  $\sigma_n$  by  $\sigma_1$  is:

$$\tau_n^{(1)} = F_n^{(1)} / A = F_1 \cos \theta_1 / (A_1 / \cos \theta_1) = (F_1 / A_1) \cos \theta_1 \cos \theta_1 = \sigma_1 \cos \theta_1 \cos \theta_1$$

$$\tau_n^{(1)} = \sigma_1 \cos \theta \cos \theta$$

← First contribution to  $\tau_n$





What does  $\sigma_2$  on face  $A_2$  of area  $A_2$  contribute to  $\tau_n$  on face  $A$  of area  $A$ ?

Start with the definition of stress:  $\tau_n^{(2)} = F_s^{(2)} / A$ .

The unknown quantities  $F_s$  and  $A$  must be found from the known quantities  $\sigma_2$  and  $\theta$ .

To do this we first find the force  $F_2$  associated with  $\sigma_2$ :

Force = (stress)(area)

$$F_2 = \sigma_2 A_2$$

The component of force  $F_2$  that acts along the  $n$ -direction is  $F_2 \cos \theta_2$ .

$$F_n^{(2)} = F_2 \cos \theta_2$$

As can be seen from the diagram atop the page  $A_2 = A \cos \theta_2$ , so

$$A = A_2 / \cos \theta_2$$

So the contribution to  $\sigma_n$  by  $\sigma_2$  is:

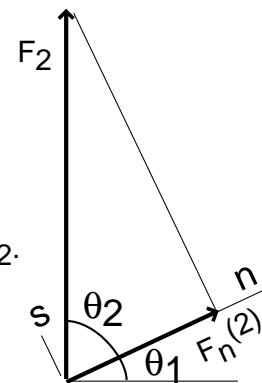
$$\tau_n^{(2)} = F_n^{(2)} / A = F_2 \cos \theta_2 / (A_2 / \cos \theta_2) = (F_2 / A_2) \cos \theta_2 \cos \theta_2 = \sigma_2 \cos \theta_2 \cos \theta_2$$

$$\tau_n^{(2)} = \sigma_2 \sin \theta \sin \theta$$

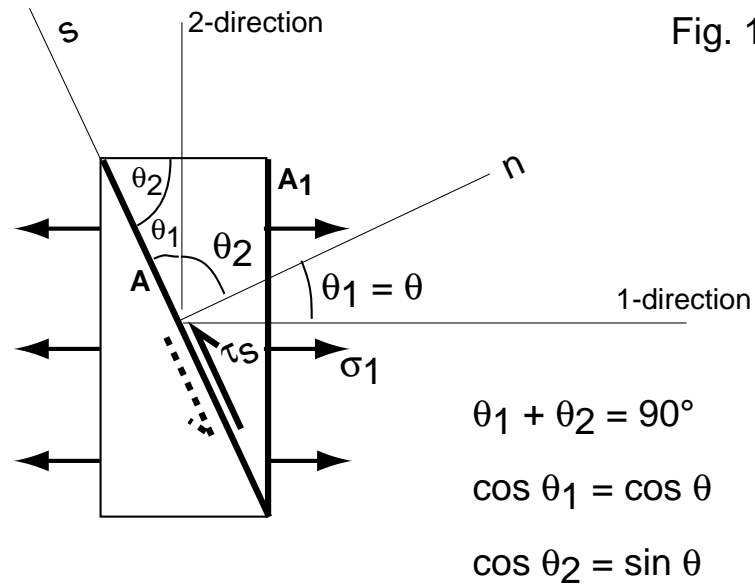
← Second contribution to  $\sigma_n$

$$\tau_n = \tau_n^{(1)} + \tau_n^{(2)} = \sigma_1 \cos^2 \theta + \sigma_2 \sin^2 \theta$$

← Total value of  $\sigma_n$



Contribution of  $\sigma_1$  to  $\tau_s$



What does  $\sigma_1$  on face  $A_1$  of area  $A_1$  contribute to  $\tau_s$  on face  $A$  of area  $A$ ?

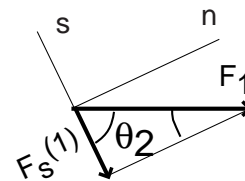
Start with the definition of stress:  $\tau_s^{(1)} = F_s^{(1)} / A$

The unknown quantities  $F_s$  and  $A$  must be found from the known quantities  $\sigma_1$  and  $\theta$ .

To do this we first find the force  $F_1$  associated with  $\sigma_1$ :

Force = (stress)(area)

$$F_1 = \sigma_1 A_1$$



The component of force  $F_1$  that acts along the  $s$ -direction is  $-F_1 \cos \theta_2$ .

$$F_s^{(1)} = -F_1 \cos \theta_2$$

As can be seen from the diagram atop the page  $A_1 = A \cos \theta_1$ , so

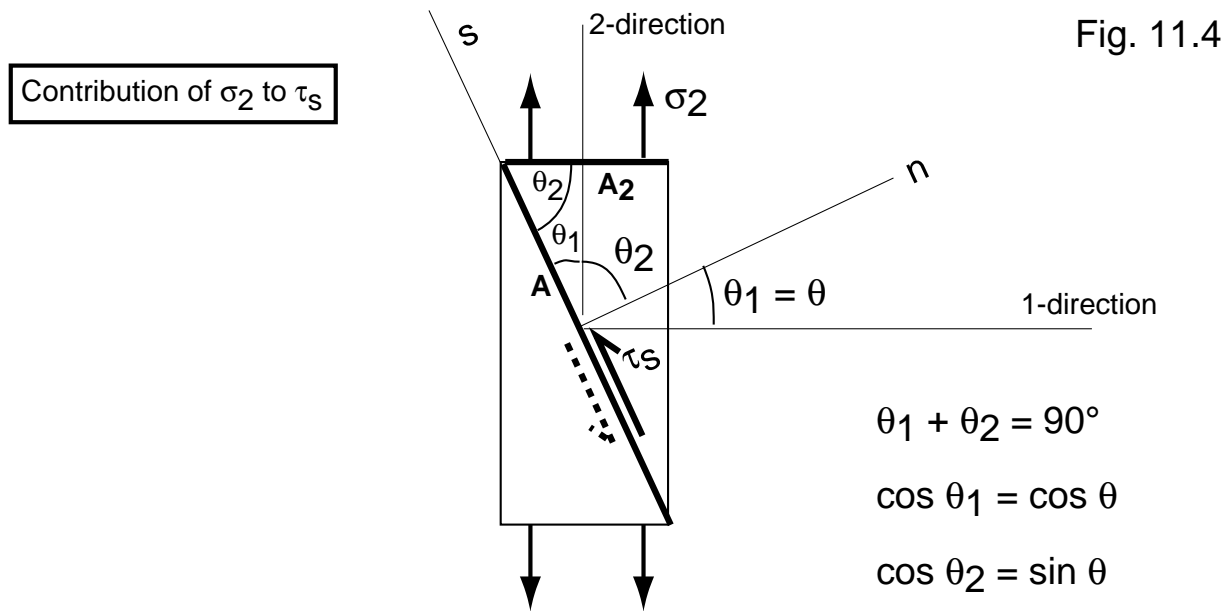
$$A = A_1 / \cos \theta_1$$

So the contribution to  $\tau$  by  $\sigma_1$  is:

$$\tau_s^{(1)} = F_s^{(1)} / A = -F_1 \cos \theta_2 / (A_1 / \cos \theta_1) = (-F_1 / A_1) \cos \theta_2 \cos \theta_1 = -\sigma_1 \cos \theta_2 \cos \theta_1$$

$$\tau_s^{(1)} = -\sigma_1 \sin \theta \cos \theta$$

← First contribution to  $\tau_s$



What does  $\sigma_2$  on face  $A_2$  of area  $A_2$  contribute to  $\tau_s$  on face  $A$  of area  $A$ ?

Start with the definition of stress:  $\tau_s^{(2)} = F_s^{(2)} / A$

The unknown quantities  $F_s$  and  $A$  must be found from the known quantities  $\sigma_2$  and  $\theta$ .

To do this we first find the force  $F_2$  associated with  $\sigma_2$ :

Force = (stress)(area)

$$F_2 = \sigma_2 A_2$$

The component of force  $F_2$  that acts along the  $s$ -direction is  $F_2 \cos \theta_1$ .

$$F_s^{(2)} = F_2 \cos \theta_1$$

As can be seen from the diagram at the top of the page  $A_2 = A \cos \theta_2$ , so

$$A = A_2 / \cos \theta_2$$

So the contribution to  $\tau$  by  $\sigma_2$  is:

$$\tau_s^{(2)} = F_s^{(2)} / A = F_2 \cos \theta_1 / (A_2 / \cos \theta_2) = (F_2 / A_2) \cos \theta_1 \cos \theta_2 = \sigma_2 \cos \theta_2 \cos \theta_1$$

$$\tau_s^{(2)} = \sigma_2 \sin \theta \cos \theta$$

← Second contribution to  $\tau_s$

$$\tau_s = \tau_s^{(1)} + \tau_s^{(2)} = (\sigma_2 - \sigma_1)(\sin \theta \cos \theta)$$

← Total value of  $\tau_s$

