FINITE STRAIN AND INFINITESIMAL STRAIN

I Main Topics (on infinitesimal strain)
  A The finite strain tensor [E]
  B Deformation paths for finite strain
  C Infinitesimal strain and the infinitesimal strain tensor ε

II The finite strain tensor [E]
  A Used to find the changes in the squares of lengths of line segments in a deformed body.
  B Definition of [E] in terms of the deformation gradient tensor [F]
  Recall the coordinate transformation equations:

1 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } [X'] = [F][X]$

2 $\begin{bmatrix} dx' \\ dy' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \text{ or } [dX'] = [F][dX]$

If $\begin{bmatrix} dx \\ dy \end{bmatrix} = [dX]$, $\text{then } [dx \quad dy] = [dX]^T$; transposing a matrix is switching its rows and columns

3 $(ds)^2 = (dx)^2 + (dy)^2 = [dx \quad dy]\begin{bmatrix} dx \\ dy \end{bmatrix} = [dX]^T[dX] = [dX]^T[I][dX]$,

where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

4 $(ds')^2 = (dx')^2 + (dy')^2 = [dx' \quad dy']\begin{bmatrix} dx' \\ dy' \end{bmatrix} = [dX']^T[dX']$

Now dX' can be expressed as [F][dX] (see eq. II.B.2). Making this substitution into eq. (4) and proceeding with the algebra

5 $(ds')^2 = [[F][dX]]^T[[F][dX]] = [dX]^T[F]^T[F][dX]$

6 $(ds')^2 - (ds')^2 = [dX]^T[F]^T[F][dX] - [dX]^T[I][dX]$

7 $(ds')^2 - (ds')^2 = [dX]^T\left[[F]^T[F] - I\right][dX]$

8 $\frac{1}{2}\left\{(ds')^2 - (ds')^2\right\} = \left(\frac{1}{2}\right)[dX]^T\left[[F]^T[F] - I\right][dX] \equiv [dX]^T[E][dX]$

9 $[E] \equiv \left(\frac{1}{2}\right)\left[[F]^T[F] - I\right] = \textit{finite strain tensor}$

III Deformation paths

Consider two different finite strains described by the following two coordinate transformation equations:

A $$\begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 x + b_1 y \\ c_1 x + d_1 y \end{bmatrix} = [F_1][X]$$ Deformation 1

B $$\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_2 x + b_2 y \\ c_2 x + d_2 y \end{bmatrix} = [F_2][X]$$ Deformation 2

Now consider deformation 3, where deformation1 is acted upon (followed) by deformation 2 (i.e., deformation gradient matrix F1 first acts on [X], and then F2 acts on [F1][X])

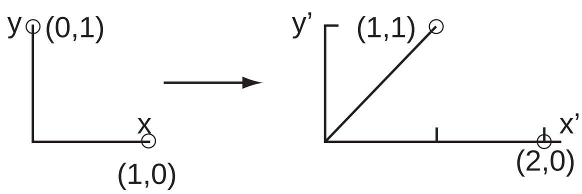
C $$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_2 a_1 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ Deformation 3

Next consider deformation 4, where deformation 2 is acted upon (followed) by deformation 1 (i.e., deformation gradient matrix F2 first acts on [X], and then F1 acts on [F2][X].

D $$\begin{bmatrix} x''' \\ y''' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ Deformation 4

E A comparison of the net deformation gradient matrices in C and D shows they generally are different. Hence, <u>the net deformation in a sequence of finite strains depends on the order of the deformations</u>. (If the b and c terms [the off-diagonal terms] are small, then the deformations are similar)

Coordinate Transformation 1

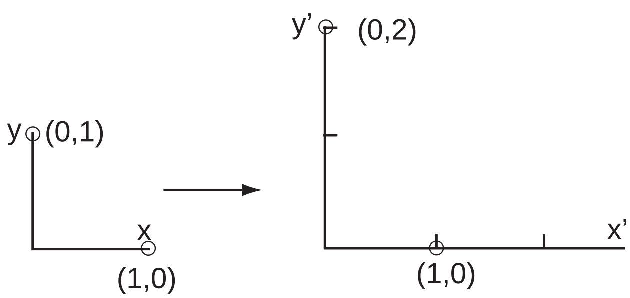


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Coordinate Transformation 2

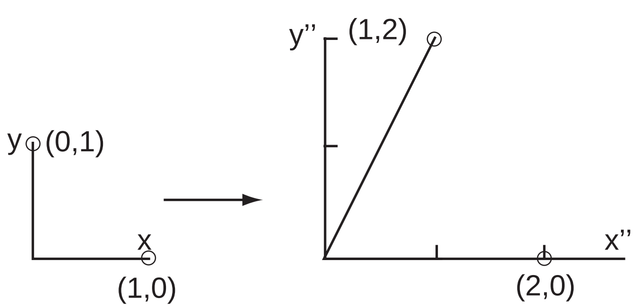


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Coordinate transformation 3 (transformation 1 followed by transformation 2)

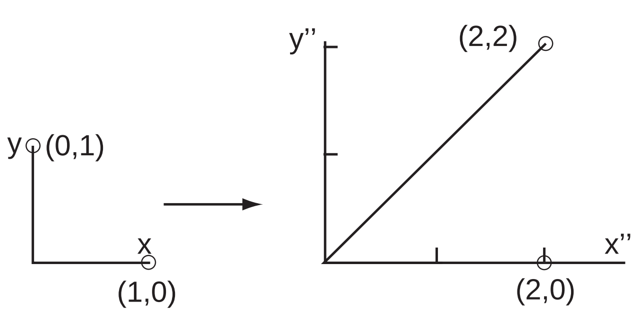


$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left[\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Coordinate transformation 4 (transformation 2 followed by transformation 1)



$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \left[\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right] \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left[\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}\right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\left[\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}\right] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

IV Infinitesimal strain and the infinitesimal strain tensor [ε]

A What is infinitesimal strain?

Deformation where the displacement derivatives are small relative to one (i.e., the terms in the corresponding displacement gradient matrix $[J_u]$ are very small), so that the products of the derivatives are very small and can be ignored.

B Why consider infinitesimal strain if it is an approximation?

1 Many important geologic deformations are small (and largely elastic) over short time frames (e.g., fracture earthquake deformation, volcano deformation).

2 The terms of the infinitesimal strain tensor [ε] have clear geometric meaning (clearer than those for finite strain)

3 Infinitesimal strain is much more amenable to sophisticated mathematical treatment than finite strain (e.g., elasticity theory).

4 The net deformation for infinitesimal strain is independent of the deformation sequence.

5 Example

$$F5 = \begin{bmatrix} 1.02 & 0.01 \\ 0 & 1.01 \end{bmatrix} \quad F6 = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.02 \end{bmatrix} \quad J_u5 = \begin{bmatrix} 0.02 & 0.01 \\ 0 & 0.01 \end{bmatrix} \quad J_u6 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}$$

Deformation 5 followed by deformation 6 gives deformation 7:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1.01 & 0.00 \\ 0.00 & 1.02 \end{bmatrix} \begin{bmatrix} 1.02 & 0.01 \\ 0.00 & 1.01 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.0302 & 0.0100 \\ 0.0000 & 1.0302 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Deformation 6 followed by deformation 5 gives deformation "7a":

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1.02 & 0.01 \\ 0.00 & 1.01 \end{bmatrix} \begin{bmatrix} 1.01 & 0.00 \\ 0.00 & 1.02 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.0302 & 0.0101 \\ 0.0000 & 1.0302 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The net deformation is essentially the same in the two cases.

C The infinitesimal strain tensor (Taylor series derivation)

Consider the displacement of two neighboring points, where the distance from point 0 to point 1 initially is given by dx and dy. Point 0 is displaced by an amount u^0, and we wish to find u^1. We use a truncated Taylor series; it is linear in dx and dy (dx and dy are only raised to the first power).

1 $$u_x^{\,1} = u_x^{\,0} + \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \ldots$$

2 $$u_y^{\,1} = u_y^{\,0} + \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \ldots$$

These can be rearranged into a matrix format:

3 $$\begin{bmatrix} u_x^{\,1} \\ u_y^{\,1} \end{bmatrix} = \begin{bmatrix} u_x^{\,0} \\ u_y^{\,0} \end{bmatrix} + \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \left[U^0\right] + \left[J_u\right]\left[dX\right]$$

Now split $\left[J_u\right]$ into two matrices: the symmetric infinitesimal strain matrix [ε], and the anti-symmetric rotation matrix [ω] by using $\left[J_u\right]^T$,

$$\left[J_u\right] = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, \qquad \left[J_u\right]^T = \begin{bmatrix} e & g \\ f & h \end{bmatrix},$$

$$\left[J_u + J_u^{\,T}\right] = \begin{bmatrix} e+e & f+g \\ g+f & h+h \end{bmatrix}, \qquad \left[J_u - J_u^{\,T}\right] = \begin{bmatrix} 0 & f-g \\ g-f & 0 \end{bmatrix}$$

4 $$\left[J_u\right] = \frac{1}{2}\left[J_u + J_u\right] + \frac{1}{2}\left[J_u^{\,T} - J_u^{\,T}\right] = \frac{1}{2}\left[J_u + J_u^{\,T}\right] + \frac{1}{2}\left[J_u - J_u^{\,T}\right] = [\varepsilon] + [\omega]$$

Now the displacement expression can be expanded using [ε] and [ω]

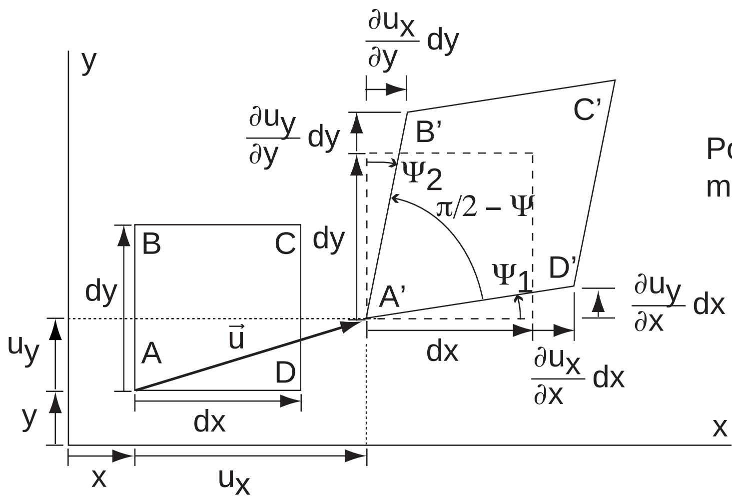
5 $$[\varepsilon] = \frac{1}{2}\begin{bmatrix} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\right) & \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\ \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y}\right) \end{bmatrix}, \quad [\omega] = \frac{1}{2}\begin{bmatrix} 0 & -\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) \\ -\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) & 0 \end{bmatrix}$$

Equations (3) and (5) show that the deformation can be decomposed into a translation, a strain, and a rotation.

D Geometric interpretation of the infinitesimal strain components

Infinitesimal Deformation of Line Elements AB and AD

(modified from Chou and Pagano, 1967)



Positive angles
measured counterclockwise

Line length changes

$$\varepsilon_{xx} = \frac{A'D'-AD}{AD} \approx \frac{\left(dx + \frac{\partial u_x}{\partial x} dx\right) - dx}{dx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{A'B'-AB}{AB} \approx \frac{\left(dy + \frac{\partial u_y}{\partial y} dy\right) - dy}{dy} = \frac{\partial u_y}{\partial y}$$

Angle changes

$$\tan\Psi_1 = \frac{\frac{\partial u_x}{\partial y} dy}{dy} = \frac{\partial u_x}{\partial y} \approx \Psi_1 \text{ if } \frac{\partial u_x}{\partial y} \text{ is small}$$

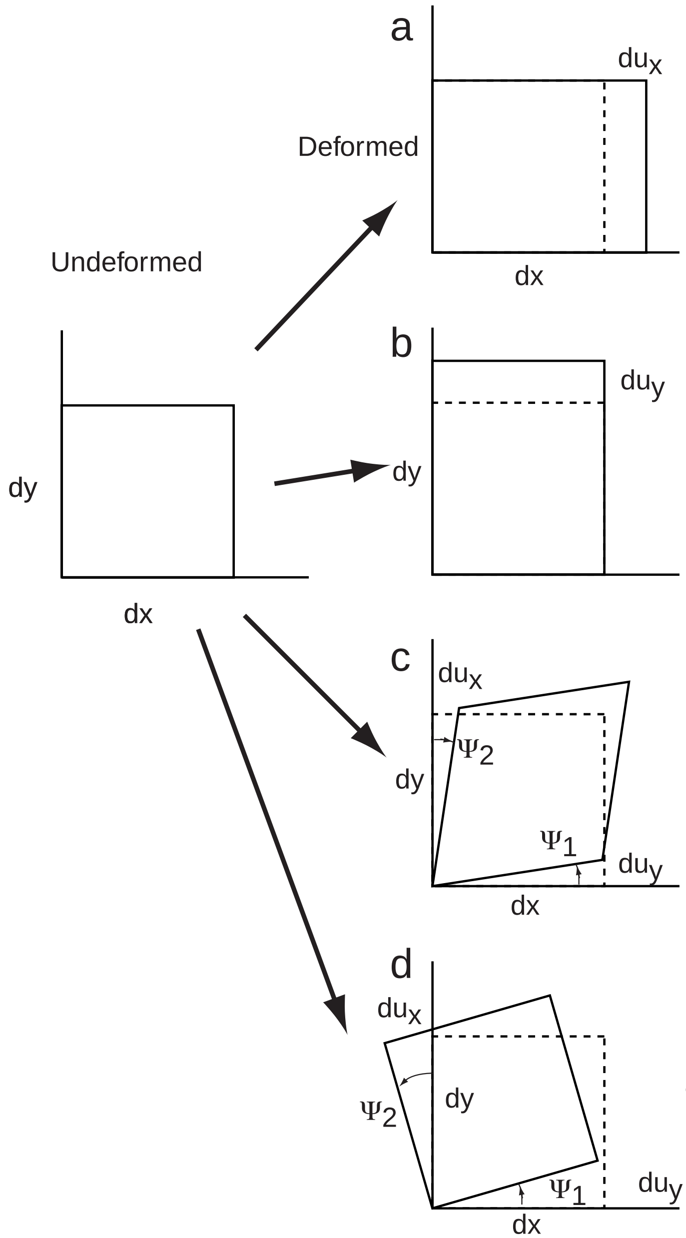
$$\tan\Psi_2 = \frac{\frac{\partial u_y}{\partial x} dx}{dx} = \frac{\partial u_y}{\partial x} \approx \Psi_2 \text{ if } \frac{\partial u_y}{\partial x} \text{ is small}$$

$\Psi = \Psi_1 - \Psi_2$ = change in right angle (the minus sign accounts for Ψ_2 being negative)

$\gamma = \tan\Psi$ $\varepsilon_{xy} = \varepsilon_{yx} = (1/2) \tan\Psi$

γ = engineering shear strain ε = tensor shear strain

Infinitesimal Strains



Normal strain e_{xx}

$e_{xx} = \partial u_x/\partial x$

Normal strain e_{yy}

$e_{yy} = \partial u_y/\partial y$

Shear strains e_{xy}, e_{yx} †

$e_{xy} = e_{yx}$ = 1/2(change in right angle)

= $1/2(\Psi_1 - \Psi_2)$*

But for small angles, $\Psi = \tan\Psi$

$e_{xy} = e_{yx}$ = $1/2(\partial u_x/\partial y + \partial u_y/\partial x)$
(This is pure shear; there is no rotation)

Rotations ω_{yx}, ω_{xy} *

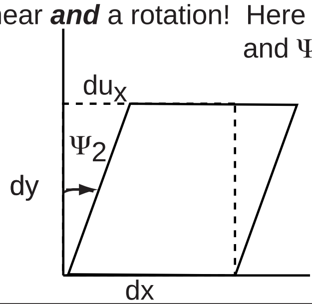
$\omega_{xy} = \omega_{yx}$ = (change in right angle)

= $1/2(\Psi_1 + \Psi_2)$*

But for small angles, $\Psi = \tan\Psi$

$\omega_{xy} = -\omega_{yx}$ = $1/2(\partial u_y/\partial x - \partial u_x/\partial y)$

Note that "simple shear strain" involves a shear ***and*** a rotation! Here Ψ_1 is zero and Ψ_2 is negative.



† The shear strain $e_{xy} = e_{yx}$ is half the shear strain γ

* Positive angles are measured about the z-axis using a right hand rule. In (b) the angle Ψ_2 is clockwise (negative), but du$_x$ is positive. In (d) Ψ_2 is counter-clockwise, and du$_x$< 0.

E Relationship between [ε] and [E]

From eq. II.B.9, [E] is defined in terms of deformation gradients:

1 $[E] \equiv \left(\frac{1}{2}\right)\left[[F]^T[F] - I\right] = \textit{finite strain tensor}$

The tensor [E] also can be solved for in terms of displacement gradients because $F = J_u + I$.

2 $[E] = \left(\frac{1}{2}\right)\left[[J_u + I]^T[J_u + I] - I\right]$

3 $$[E] = \left(\frac{1}{2}\right)\left[\left[\begin{bmatrix} \frac{\partial u_x}{dx} & \frac{\partial u_x}{dy} \\ \frac{\partial u_y}{dx} & \frac{\partial u_y}{dy} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]^T \left[\begin{bmatrix} \frac{\partial u_x}{dx} & \frac{\partial u_x}{dy} \\ \frac{\partial u_y}{dx} & \frac{\partial u_y}{dy} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

4 $$[E] = \left(\frac{1}{2}\right)\left[\begin{bmatrix} \frac{\partial u_x}{dx}+1 & \frac{\partial u_y}{dx} \\ \frac{\partial u_x}{dy} & \frac{\partial u_y}{dy}+1 \end{bmatrix}\begin{bmatrix} \frac{\partial u_x}{dx}+1 & \frac{\partial u_x}{dy} \\ \frac{\partial u_y}{dx} & \frac{\partial u_y}{dy}+1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

5 $$[E] = \left(\frac{1}{2}\right)\begin{bmatrix} \left(\frac{\partial u_x}{dx}+1\right)\left(\frac{\partial u_x}{dx}+1\right) + \left(\frac{\partial u_y}{dx}\right)\left(\frac{\partial u_y}{dx}\right) - 1 & \left(\frac{\partial u_x}{dx}+1\right)\left(\frac{\partial u_x}{dy}\right) + \left(\frac{\partial u_y}{dx}\right)\left(\frac{\partial u_y}{dy}+1\right) \\ \left(\frac{\partial u_x}{dy}\right)\left(\frac{\partial u_x}{dx}+1\right) + \left(\frac{\partial u_y}{dy}+1\right)\left(\frac{\partial u_y}{dx}\right) & \left(\frac{\partial u_x}{dy}\right)\left(\frac{\partial u_x}{dy}\right) + \left(\frac{\partial u_y}{dy}+1\right)\left(\frac{\partial u_y}{dy}+1\right) - 1 \end{bmatrix}$$

If the displacement gradients are small relative to 1, then the products of the displacements are very small relative to 1, and in infinitesimal strain theory they can be dropped, yielding [ε]:

6 $$[\varepsilon] \approx \left(\frac{1}{2}\right)\begin{bmatrix} \left(\frac{\partial u_x}{dx}\right) + \left(\frac{\partial u_x}{dx}\right) & \left(\frac{\partial u_x}{dy}\right) + \left(\frac{\partial u_y}{dx}\right) \\ \left(\frac{\partial u_x}{dy}\right) + \left(\frac{\partial u_y}{dx}\right) & \left(\frac{\partial u_y}{dy}\right) + \left(\frac{\partial u_y}{dy}\right) \end{bmatrix} = \frac{1}{2}\left[\left[J_u\right] + \left[J_u\right]^T\right]$$

This suggests that for multiple deformations, infinitesimal strains might be obtained by matrix addition (i.e., linear superposition) rather than by matrix multiplication; the former is simpler. Also see equation IV.C.5.

7 Example of IV.B.5: [ε] from superposed vs. sequenced deformations

$$F5=\begin{bmatrix}1.02 & 0.01\\ 0 & 1.01\end{bmatrix}\quad J_u5=\begin{bmatrix}0.02 & 0.01\\ 0 & 0.01\end{bmatrix}\quad F6=\begin{bmatrix}1.01 & 0\\ 0 & 1.02\end{bmatrix}\quad J_u6=\begin{bmatrix}0.01 & 0\\ 0 & 0.02\end{bmatrix}$$

a Linear superposition, assuming infinitesimal strain (approx.)

»F5 = [1.02 0.01;0.00 1.01] F5 = 1.0200 0.0100 0 1.0100	»F6 = [1.01 0.00;0.00 1.02] F6 = 1.0100 0 0 1.0200
$E5=\frac{1}{2}\left[[F5]^T[F5]-I\right]$	$E6=\frac{1}{2}\left[[F6]^T[F6]-I\right]$
»E5 = 0.5*(F5'*F5-eye(2)) E5 = 0.0202 0.0051 0.0051 0.0101	» E6 = 0.5*(F6'*F6-eye(2)) E6 = 0.0101 0 0 0.0202
$\left(\approx\frac{1}{2}\left[[J_u5]+[J_u5]^T\right]\right)$	$\left(\approx\frac{1}{2}\left[[J_u6]+[J_u6]^T\right]\right)$

```
»E7 = E5 + E6
E7 =
   0.0302    0.0051
    0.0051   0.0303
```

Linear superposition of strains
(Infinitesimal approximation)

b Sequenced deformation (exact) $[E_7]\equiv\left(\frac{1}{2}\right)\left[[F_7]^T[F_7]-I\right]$

»F7 = F6*F5 See eq. IV.B.5

```
F7 =
1.0302    0.0101
     0    1.0302
```

»E7 = 0.5*(F7'*F7-eye(2)) Convert def. gradients to strain

E7 = Good match with approximation

```
   0.0307    0.0052
   0.0052    0.0307
```

8 Recap

The infinitesimal strain tensor can be used to find the change in the square of the length of a deformed line segment connecting two nearby points separated by distances dx and dy,

$$\frac{1}{2}\left\{(ds')^2 - (ds')^2\right\} = [dX]^T[\varepsilon][dX]$$

and, with the rotation tensor, to find the change in displacement of two points in a deformed medium that are initially separated by distances dx and dy:

$$[\Delta U] = \frac{1}{2}[\varepsilon][dX] + \frac{1}{2}[\omega][dX]$$

9 For infinitesimal strains the displacements are essentially the same no matter whether the pre- or post-deformation positions are used.