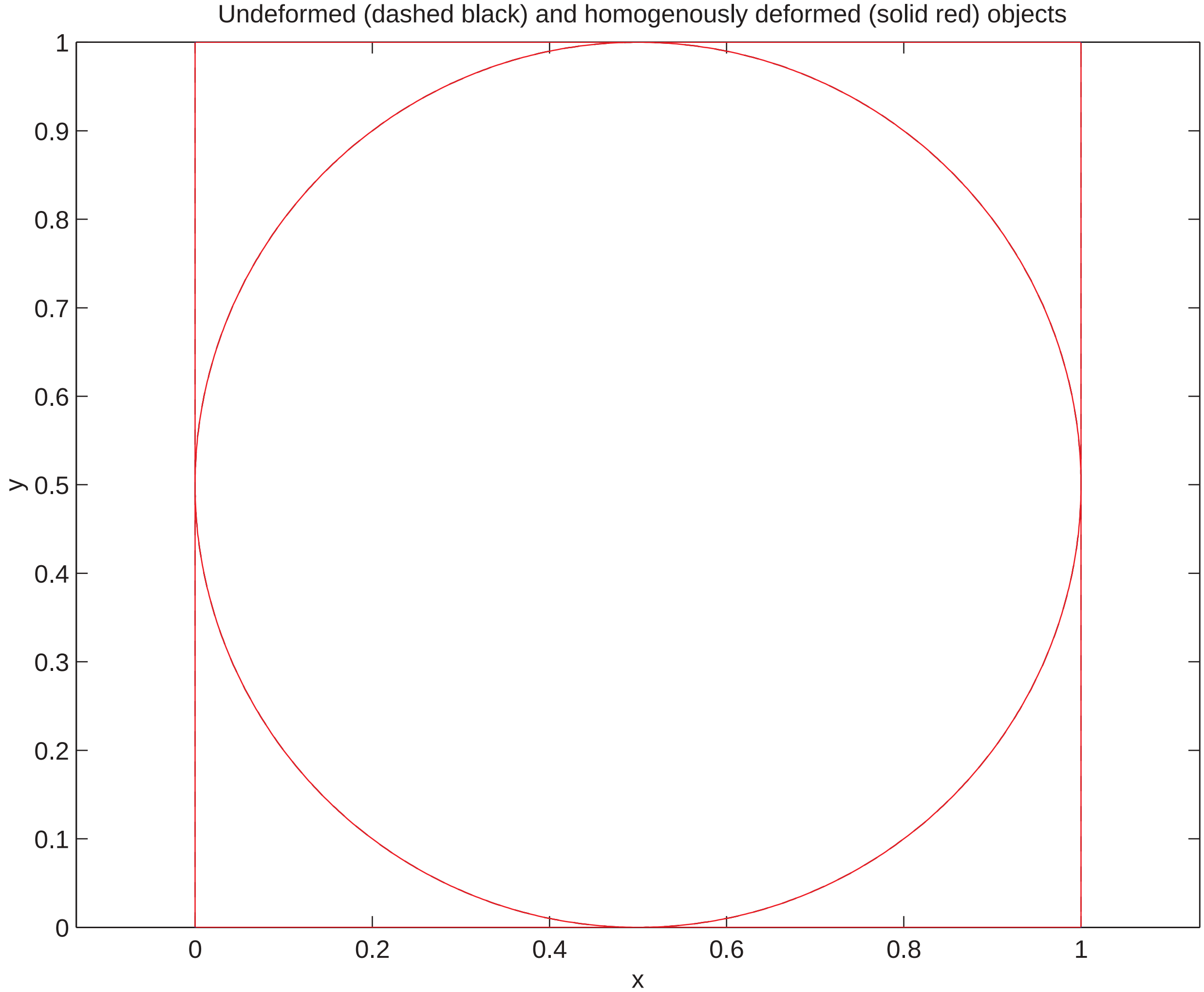
HOMOGENEOUS FINITE STRAIN: DISPLACEMENT & DEFORMATION GRADIENTS

I Main Topics (largely drawn from chapters 18 and 22 of Means, 1976)
  A Homogenous deformation
  B Examples
  C Deformation and displacement and gradients
  D Dependence of net strain on the strain path

II Homogenous (uniform) deformation
  A Straight parallel lines in an initial state (**X=x,y**) remain straight and parallel in the final (current) state (**X'=x',y'**)
  B Parallelograms deform into parallelograms in 2-D, parallelepipeds deform into parallelepipeds in 3-D
  C Considered a useful concept in geology for "small" regions – over large regions deformation is inhomogeneous (non-uniform)
  D Deformed and undeformed positions are described by linear coordinate transformation equations (2-D examples below)
    1 $$x' = ax + by$$ $$y' = cx + dy$$
    2 $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ = deformation gradient matrix [F]
    3 $[X'] = [F][X]$
  E The displacements also are described by linear displacement equations (2-D examples below)
    1 $$u = x' - x = (ax + by) - x = (a-1)x + by$$ $$v = y' - y = (cx + dy) - y = cx + (d-1)y$$
    2 $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, where
    $\begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ = displacement vector gradient matrix $[J_u]$
    3 $[U] = [J_u][X]$
    4 Simple relationship between $[F]$ and $[J_u]$: $J_u = F - I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

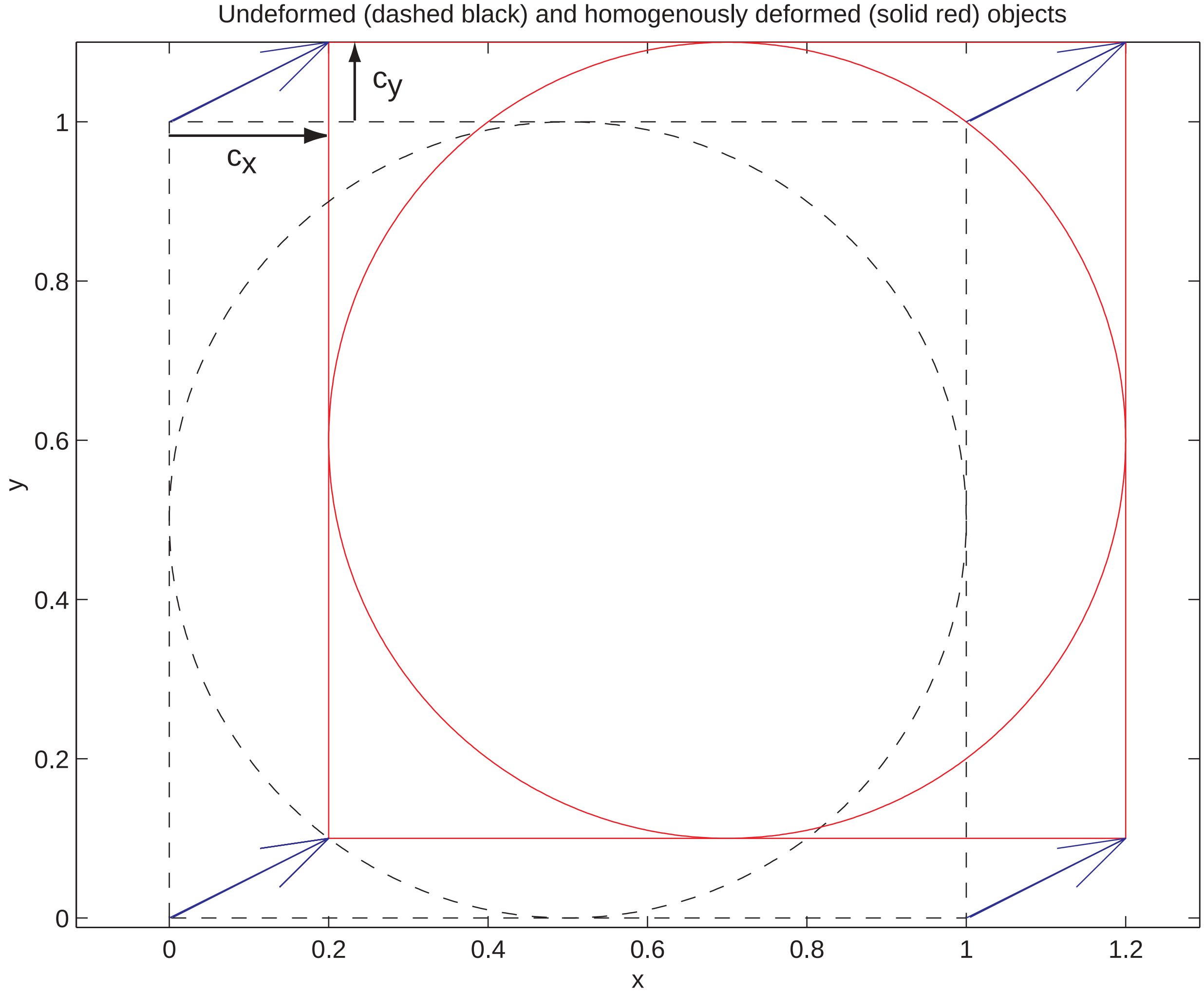
III Examples

A No deformation



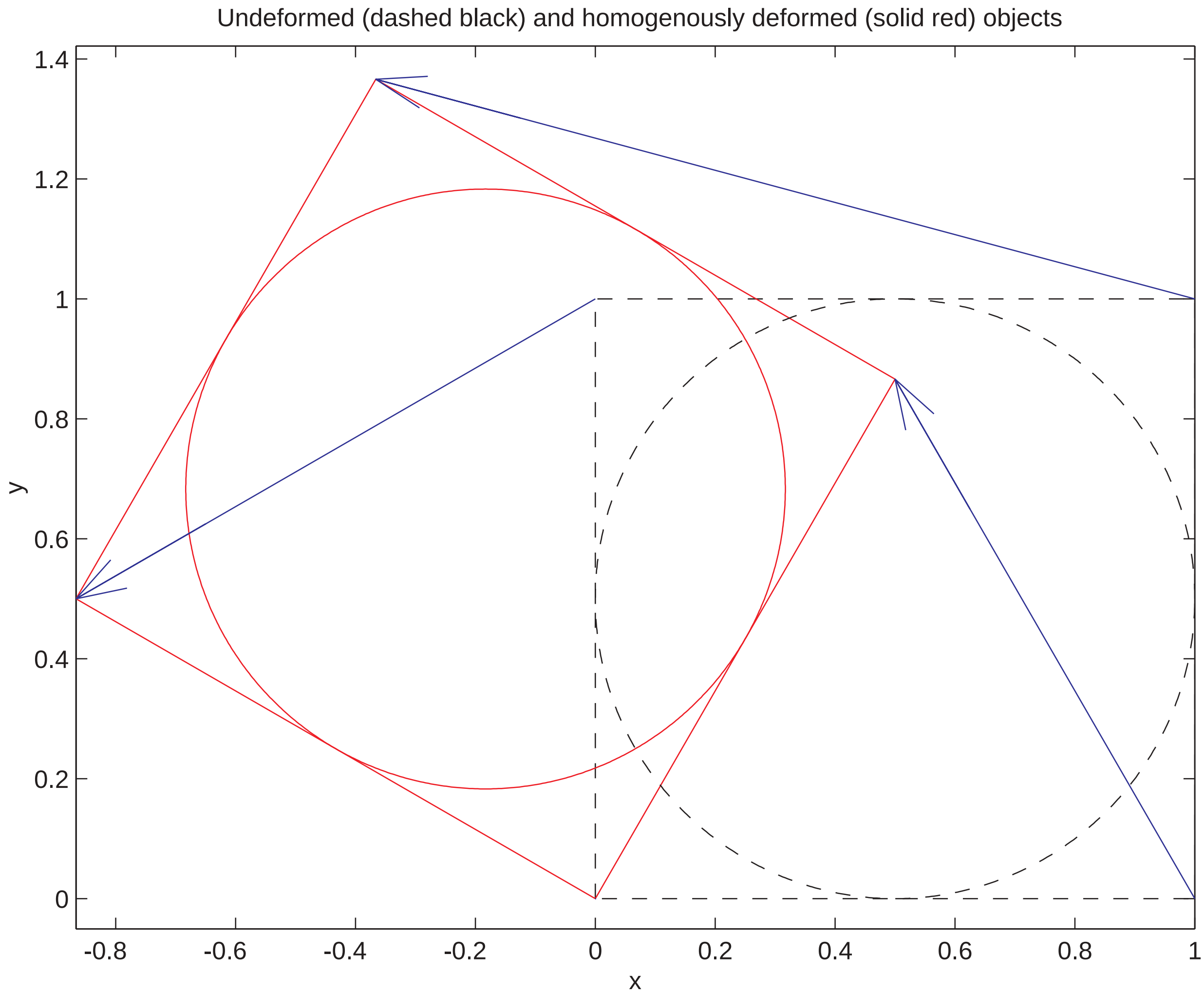
$x' = 1x + 0y$ $y' = 0x + 1y$ Coordinate transformation equations (Lagrangian)	$u_x = 0x + 0y$ $u_y = 0x + 0y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Coordinate transformation equations (matrix form)	$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Displacement gradient tensor J_u

B Rigid body translation



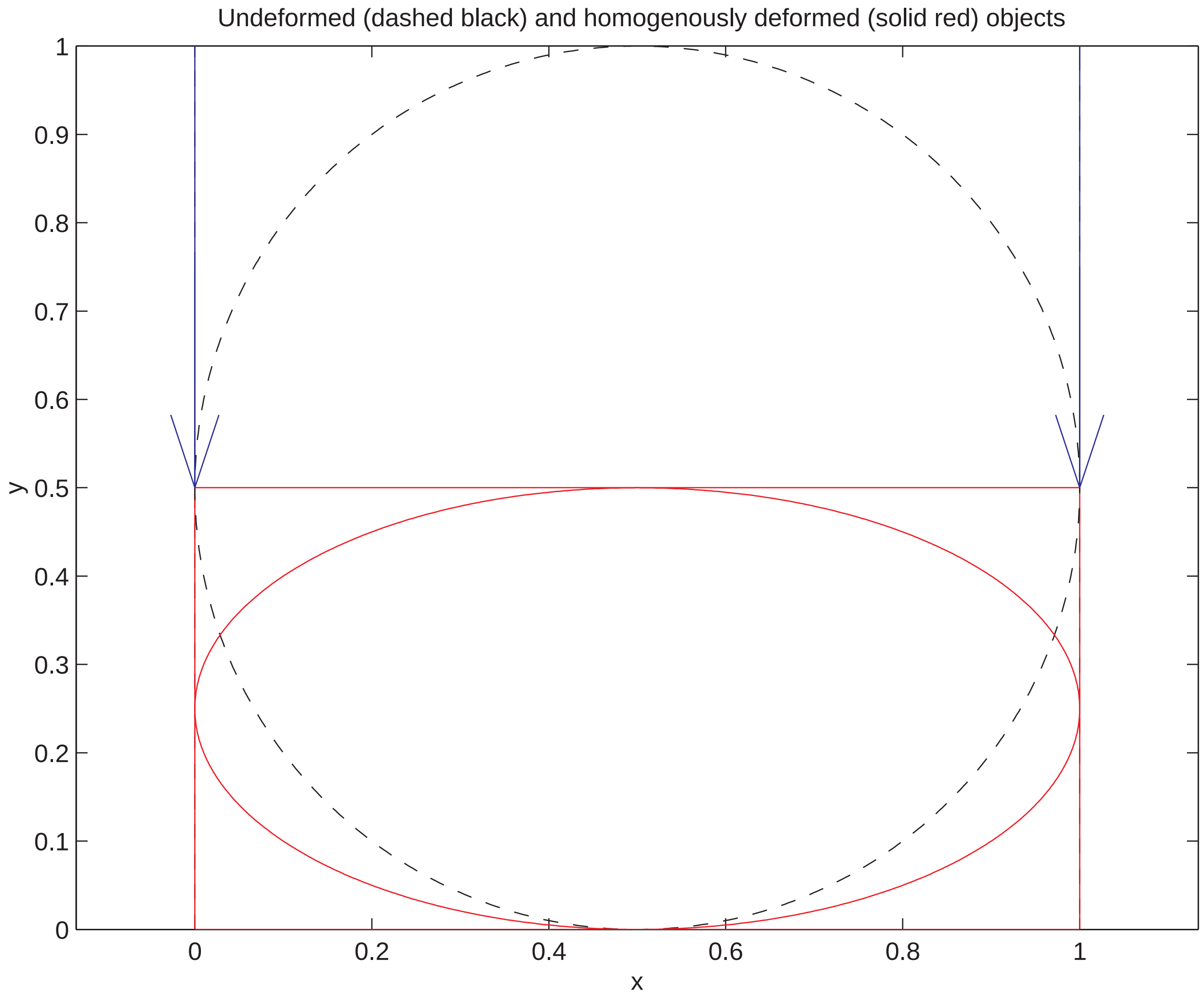
$x' = 1x + 0y + c_x$ $y' = 0x + 1y + c_y$ Coordinate transformation equations (Lagrangian)	$u_x = 0x + 0y$ $u_y = 0x + 0y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}$ Coordinate transformation equations (matrix form)	$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Displacement gradient tensor J_u

C Rigid body rotation



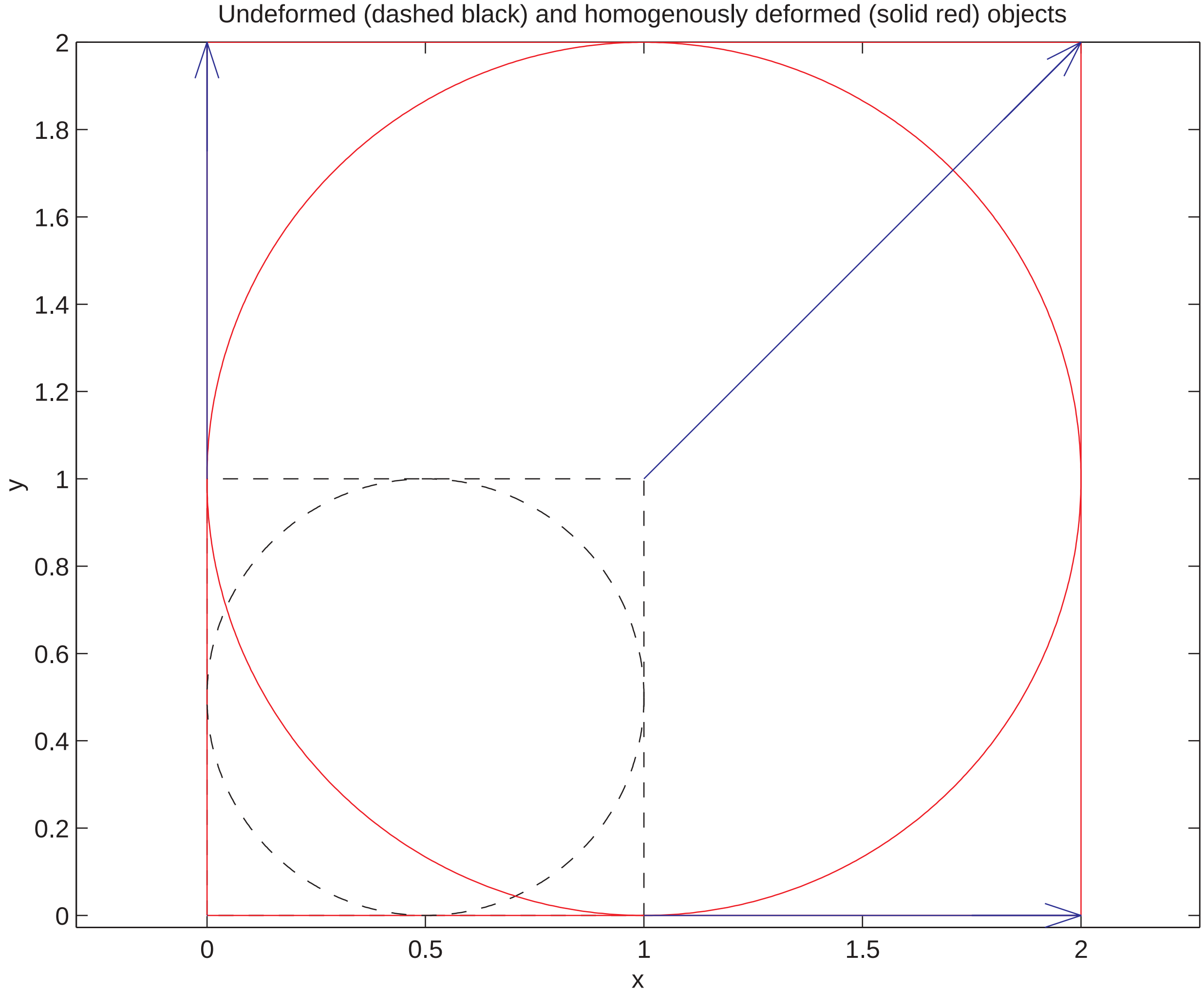
$x' = (\cos 60^\circ)x - (\sin 60^\circ)y$ $y' = (\sin 60^\circ)x + (\cos 60^\circ)y$ Coordinate transformation equations (Lagrangian)	$u_x = (\cos 60^\circ - 1)x - (\sin\cos 60^\circ)y$ $u_y = (\sin 60^\circ)x + (\cos 60^\circ - 1)y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Coordinate transformation equations (matrix form)	$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \cos 60^\circ - 1 & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} \cos 60^\circ - 1 & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ - 1 \end{bmatrix}$ Displacement gradient tensor J_u

D Uniaxial shortening (parallel to y-axis)



$x' = 1x + 0y$ $y' = 0x + 0.5y$ Coordinate transformation equations (Lagrangian)	$u_x = 0x + 0y$ $u_y = 0x - 0.5y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Coordinate transformation equations (matrix form)	$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0 & 0 \\ 0 & -0.5 \end{bmatrix}$ Displacement gradient tensor J_u

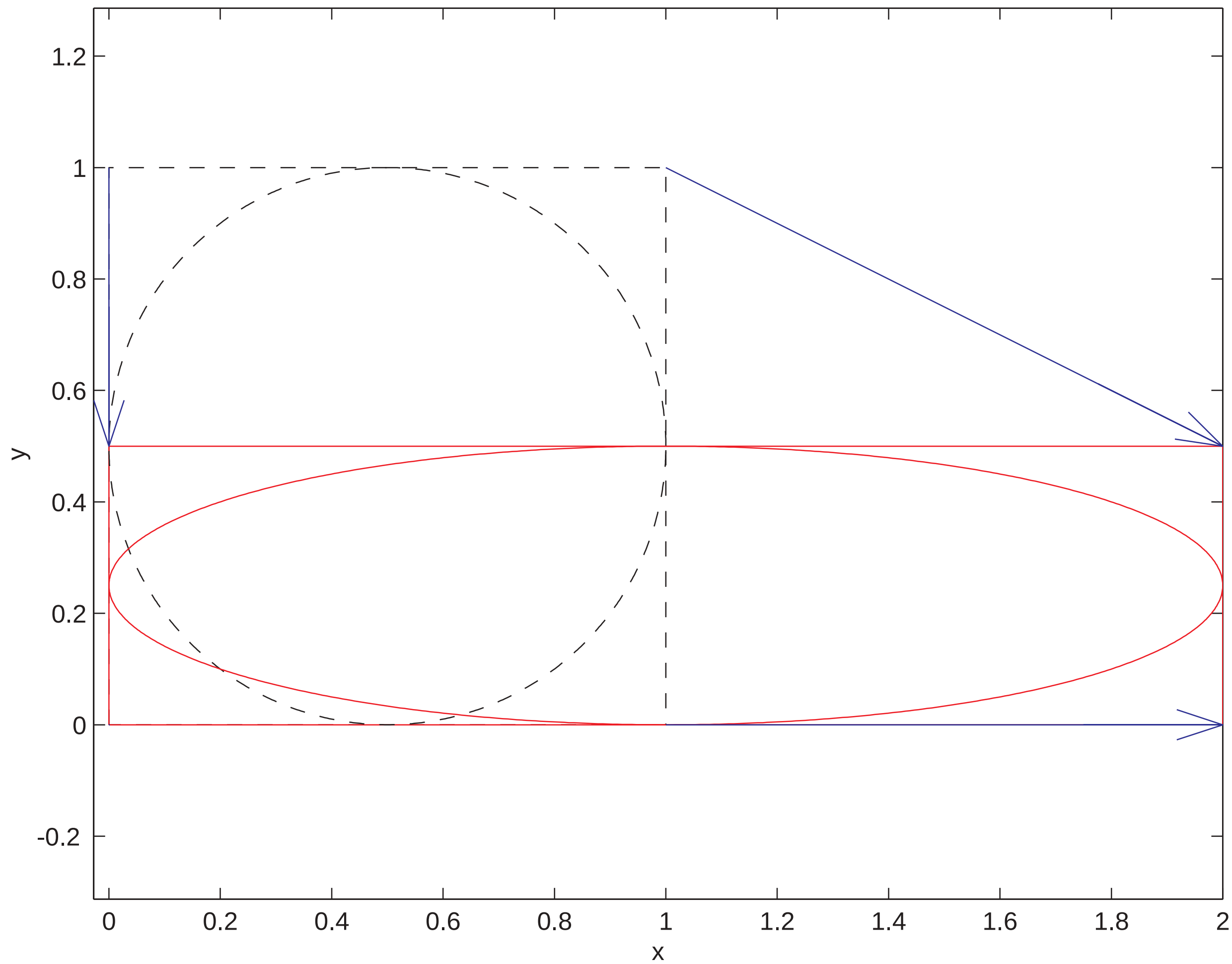
E Dilation



$x' = 2x + 0y$ $y' = 0x + 2y$ Coordinate transformation equations (Lagrangian)	$u_x = 1x + 0y$ $u_y = 0x + 1y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Coordinate transformation equations (matrix form)	$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Displacement gradient tensor J_u

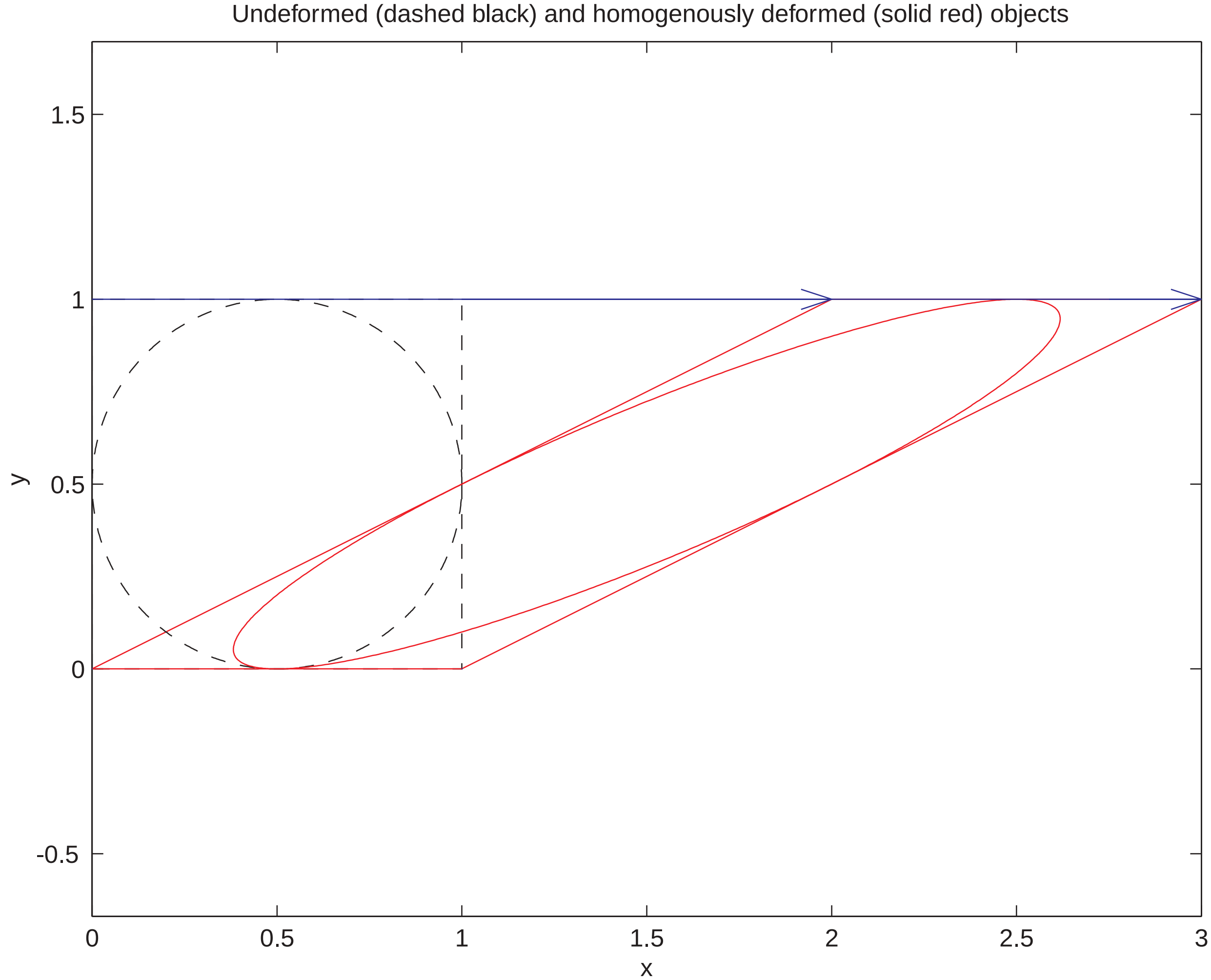
F Pure shear strain (biaxial strain, no dilation)

Undeformed (dashed black) and homogenously deformed (solid red) objects



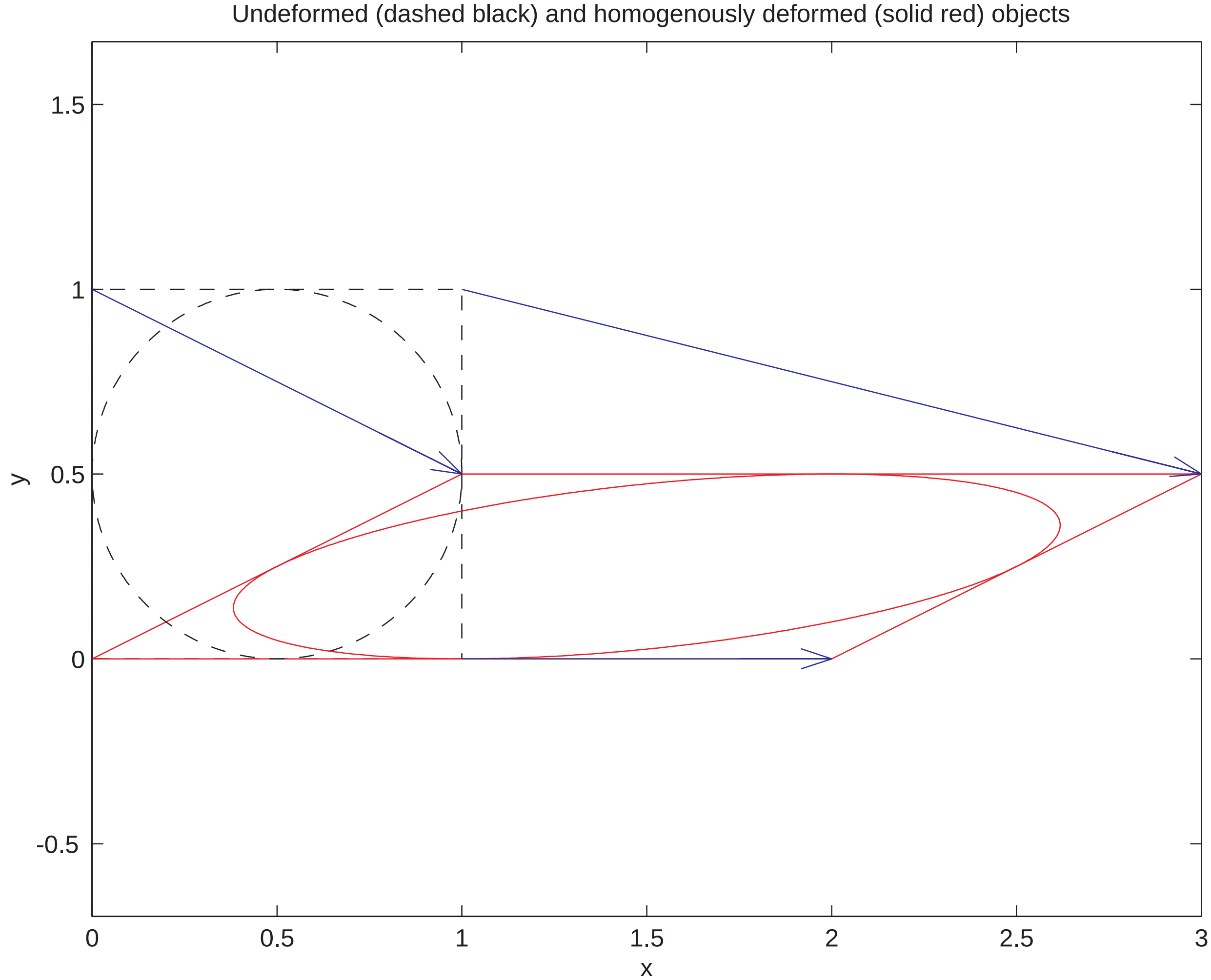
$x' = 2x + 0y$ $y' = 0x + 0.5y$ Coordinate transformation equations (Lagrangian)	$u_x = 1x + 0y$ $u_y = 0x - 0.5y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Coordinate transformation equations (matrix form)	$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}$ Displacement gradient tensor J_u

G Simple shear strain parallel to the x-axis (no dilation)



$x' = 1x + 2y$ $y' = 0x + 1y$ Coordinate transformation equations (Lagrangian)	$u_x = 0x + 2y$ $u_y = 0x + 0y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Coordinate transformation equations (matrix form)	$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ Displacement gradient tensor J_u

H General deformation (plain strain)



$x' = 2x + 1y$ $y' = 0x - 0.5y$ Coordinate transformation equations (Lagrangian)	$u_x = 1x + 1y$ $u_y = 0x - 1.5y$ Displacement equations (Lagrangian)
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Coordinate transformation equations (matrix form)	$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Displacement equations (matrix form)
$\begin{bmatrix} 2 & 1 \\ 0 & -0.5 \end{bmatrix}$ Deformation gradient tensor F	$\begin{bmatrix} 1 & 1 \\ 0 & -1.5 \end{bmatrix}$ Displacement gradient tensor J_u

I General comments on the deformation gradient and displacement gradient tensors

A The tensors that describe deformation of a body depend on derivatives of displacements.

B If the (Lagrangian) deformation gradient and displacement gradient tensors are symmetric about the main diagonal, the deformed body will be symmetric about axes parallel to the axes of the initial reference frame.

C For cases of no deformation and rigid body translation the respective tensors are identical, even though the equations they are derived from differ. The constant displacement terms (c_x and c_y) drop out in forming the tensors. The deformation gradient and displacement gradient tensors therefore provide no information on the presence or absence of a rigid body translation.

D For cases of no deformation and rigid body rotation the respective tensors are different. The deformation gradient and displacement gradient tensors therefore do provide information on the presence or absence of a rigid body rotation; this information is in the off-diagonal terms, which have equal magnitude and opposite sign.

E

IV Deformation and displacement gradients

A Question: what meaning can be attached to the terms in the deformation gradient and displacement vector gradient matrices?

B Deformation gradients

1 Consider two neighboring particles separated by distances $\Delta x'$, $\Delta y'$ in the deformed state, and by distances Δx, Δy in the undeformed state. In a homogeneously deformed body, the ratios between $\Delta x'$, $\Delta y'$ in the deformed state and Δx, Δy in the undeformed state are constant, just as the slope of a line is constant at every point along it. These ratios are called deformation gradients. They describe how $\Delta x'$ and $\Delta y'$ vary as a function of Δx and Δy.

2 To address cases where the deformation is non-uniform and can vary with x and y, the deformation gradients at a point are written using partial derivatives:

a $\frac{\Delta x'}{\Delta x} \Rightarrow \frac{\partial x'}{\partial x}$

b $\frac{\Delta x'}{\Delta y} \Rightarrow \frac{\partial x'}{\partial y}$

c $\frac{\Delta y'}{\Delta x} \Rightarrow \frac{\partial y'}{\partial x}$

d $\frac{\Delta y'}{\Delta y} \Rightarrow \frac{\partial y'}{\partial y}$

3 For any homogeneous deformation, the coefficients on the right side of the (linear) coordinate transformation equations are the deformation gradients.

4 Example: see example H and differentiate

5 In the chain rule of differential calculus:

$$dx' = \frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy$$

The deformation gradients are constant through a body in homogeneous strain; they do not change if dx and dy are small or large. If x and y are measured with respect to the origin, then x = x-0 = dx and y = y-0 = dy: x and y can replace dx and dy, respectively, in the chain rule, so

$$dx' = \frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy \Rightarrow dx' = \frac{\partial x'}{\partial x} x + \frac{\partial x'}{\partial y} y$$

C Displacement gradients

1 Consider two neighboring particles separated by distances Δx, Δy in the undeformed state, and that differ in displacement by Δux and Δuy. In a homogeneously deformed body, the ratios between displacement differences Δu_x, Δu_y and distances Δx, Δy are constant also. These ratios are called displacement gradients. They describe how Δu_x and Δu_y vary as a function of Δx and Δy.

2 To address cases where the deformation is non-uniform and can vary with x and y, the deformation gradients at a point are written using partial derivatives:

a $\frac{\Delta x'}{\Delta x} \Rightarrow \frac{\partial x'}{\partial x}$

b $\frac{\Delta x'}{\Delta y} \Rightarrow \frac{\partial x'}{\partial y}$

c $\frac{\Delta y'}{\Delta x} \Rightarrow \frac{\partial y'}{\partial x}$

d $\frac{\Delta y'}{\Delta y} \Rightarrow \frac{\partial y'}{\partial y}$

3 For any homogeneous deformation, the coefficients on the right side of the (linear) displacement equations are the deformation gradients.

4 Example: see example H and differentiate

5 In the chain rule of differential calculus:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

The displacement gradients are constant through a body in homogeneous strain; they do not change if dx and dy are small or large. If x and y are measured with respect to the origin, then x = x-0 = dx and y = y-0 = dy: x and y can replace dx and dy, respectively, in the chain rule, so

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \Rightarrow du = \frac{\partial u}{\partial x}x + \frac{\partial u}{\partial y}y$$

6 Outstanding issues

a Relationship of elongation and shear strain as defined in the prior lecture to the tensor terms for finite strain. See Malvern (1967) and Ramsay and Huber (1983).

b Relationship of finite strain to forces on (or in) a body

```
function  GG303_lec14b(a,b,c,d)
% Plots undeformed postions (X) of points on a square and a circle
% and deformed positions (X')
% and the displacements (U) relating the undeformed and deformed positions
% given the coefficients of the 2-D coordinate transformation equations
% for homogeneous 2-D plane strain (uz = 0).
% [x'] = [a  b]  [x]
% [y']     [c  d]  [y]
% Malvern (1969, p. 156) and Means (1976, p. 198) call the a,b,c,d
% coordinate transformation matrix the deformation gradient matrix (F).
% Ramsay & Huber (1983) call this the "strain" matrix (p. 71).
%  [X'] = [F][X]
%  Fij = dX'i/dxj

% U = X' - X
% ux = (ax + by) - x = (a-1)x + by = ex + fy
% uy = (cx + dy) - y = cx + (d-1)y = gx + hy
% [ux] = [e  f]  [x]
% [uy]     [g  h]  [y]
%  [U]  =  [Ju][X]  =  [F-I][X]
% Malvern (1969, p. 124) and Means (1976, p. 197) call
% the e,f,g,h matrix the displacement gradient matrix (Ju).
% Ramsay & Huber (1983) call this the "displacement vector gradient" matrix (p. 71).
%  Juij = dui/dxj

% Define initial positions and the coordinate transformation and displacement matrices
x = [0 0 1 1 0];              % x-coordinates of the square, clockwise from origin
y = [0 1 1 0 0];              % y-coordinates of the square, clockwise from origin
X = [x;y]                           %  2x5  matrix
theta  =  0:pi/360:2*pi;
xc  =  0.5+(0.5*cos(theta));
yc  =  0.5+(0.5*sin(theta));
XC = [xc;yc];
F = [a b;c d]                 %  2x2  matrix
Ju = F - eye(size(F))         %  2x2  matrix

% Calculate deformed positions (X') and displacements (U)
Xp = F*X                            % Xp = X' = deformed positions
xp  =  Xp(1,:);               % xp is the first row of Xp
yp  =  Xp(2,:);               % yp is the second row of Xp
XCp = F*XC;                         % XCp = XC' = deformed positions
xcp  =  XCp(1,:);                   % xcp is the first row of XCp
ycp  =  XCp(2,:);                   % ycp is the second row of XCp
U = Ju*X                            % U = displacements
ux  =  U(1,:);                % ux is the first row of U
uy  =  U(2,:);                % uy is the second row of U

% Calculate the volume of the undeformed (V) and deformed (Vp) squares
V   =   abs(det(   [x(4)-x(1),y(4)-y(1);   x(2)-x(1),y(2)-y(1)]   ))
Vp   =   abs(det(   [xp(4)-xp(1),yp(4)-yp(1);   xp(2)-xp(1),yp(2)-yp(1)]   ))
dilation  =  (Vp-V)./V
```

```
% Plot the undeformed square and the deformed "square"
figure(1)
clf
plot(x,y,'--k');                    % Plots the undeformed square in black
hold on
plot(xp,yp,'r');                    % Plots the deformed square in red
hold on

% Plot the undeformed circle and the deformed "circle"
plot(xc,yc,'--k');                  % Plots the undeformed circle in black
hold on
plot(xcp,ycp,'r');                  % Plots the deformed circle in red

% Now plot the displacement vectors with "nice-looking" heads
% The arrow heads produced by the "stock" version of quiver (below) are too big
%quiver(x,y,ux,uy,0);           % Plots the displacement vectors with no scaling
% So first plot lines with no arrowheads connecting the vector tails and heads
for  i=1:4
      line([x(i),xp(i)],[y(i),yp(i)])
end
% and then use quiver to draw arrows of UNIT length with heads where I want heads
dx  =  zeros(size(x));
dy  =  zeros(size(y));
scalefactor  =  sqrt(  (x-xp).^2  +  (y-yp).^2  );
k  =  find(scalefactor);                        % Finds nonzero scalefactors
dx(k)   =   0.25*(xp(k)   -   x(k))./scalefactor(k);
dy(k)   =   0.25*(yp(k)   -   y(k))./scalefactor(k);
quiver   (xp-dx,yp-dy,dx,dy,0);
axis('equal')

% Now label the plots
xlabel('x')
ylabel('y')
title('Undeformed (dashed black) and homogenously deformed (solid red) objects')
```

References

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* These are particularly good for undergraduate students