28. Folds (II)

I Main Topic: Mechanics of folds above intrusions
A Background
B G.K. Gilbert’s idealization
C Superposition
D Displacements around an opening-mode crack (sill)
E Dimensional analysis of governing eq. for bending
F Idealized form of folds over a laccolith
G Development of laccoliths and saucer-shaped sills
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Laccolith, Montana

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II Mechanics of folds above intrusions

Half-stereogram of Mount Ellsworth

From Gilbert, 1877, Report on the geology of the Henry Mountains

http://www.nps.gov/history/history/online_books/geology/publications/bul/707/images/fig53.jpg
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II Mechanics of folds above intrusions

G.K. Gilbert

David Pollard
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II Mechanics of folds above intrusions (beam theory)

“Ideal Cross-section of a Mountain of Eruption”

“Ideal Cross-section of a Laccolite, showing the typical form and the arching of the overlying strata”

Figures from Gilbert, 1887
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Superposition
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Superposition

- Opening-mode crack modeled by opening-mode displacement discontinuities (dds) of different apertures
- Openings \([X(i)]\) of dds set so that sum of traction changes matches boundary condition \([B(j)]\) on crack walls = \(\Delta \sigma_{yy}^c\)
- \([A_{(ij)}][X(i)] = [B(j)]\), where \(A_{(ij)}\) is effect of unit opening at element i on tractions at element j
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Superposition

- Total stress field around crack equals sum of stress contributions of all dds: \( \sigma^t = \Sigma \sigma_i \)
- Total displacement field around crack equals sum of displacement contributions of all dds: \( u^t = \Sigma u_i \)
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II  Mechanics of folds above intrusions

Displacements arising from opening of a mode-I crack, 2D elastic model (from Pollard and Segall, 1987)

\[ u_x = \frac{\Delta \sigma_I}{2G} \left\{ (1 - 2\nu) \left( R \cos \Theta - r \cos \theta \right) - r \sin \theta \left[ \frac{r}{R} \sin (\theta - \Theta) \right] \right\} \]

“Driving Pressure” (over-pressure)

\[ u_y = \frac{\Delta \sigma_I}{2G} \left\{ 2(1 - \nu) \left( R \sin \Theta - r \sin \theta \right) - r \sin \theta \left[ \frac{r}{R} \cos (\theta - \Theta) - 1 \right] \right\} \]

Shear modulus of host rock

Note: \( rsin\theta = y \)

\[ R = (r_1r_2)^{1/2} \]

\[ \Theta = (\theta_1 + \theta_2)/2 \]
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II Mechanics of folds above intrusions

Displacements arising from opening of a mode-I crack, 2D elastic model (from Pollard and Segall, 1987)

Now specialize to the crack walls ($y = \pm 0$).

$r \sin \theta = y \rightarrow 0$, hence

$$u_x = \frac{\Delta \sigma_i}{2G} \left\{ (1 - 2\nu)(R \cos \Theta - r \cos \theta) - r \sin \theta \left[ \frac{r}{R} \sin(\theta - \Theta) \right] \right\} \rightarrow \frac{\Delta \sigma_i}{2G} \left\{ (1 - 2\nu)(R \cos \Theta - r \cos \theta) \right\}$$

$$u_y = \frac{\Delta \sigma_i}{2G} \left\{ 2(1 - \nu)(R \sin \Theta - r \sin \theta) - r \sin \theta \left[ \frac{r}{R} \cos(\theta - \Theta) - 1 \right] \right\} \rightarrow \frac{\Delta \sigma_i}{2G} \left\{ 2(1 - \nu)(R \sin \Theta) \right\}$$
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**II** Mechanics of folds above intrusions

Displacements arising from opening of a mode-I crack, 2D elastic model (from Pollard and Segall, 1987)

So the remaining key terms are:
- $R$
- $\cos \Theta$
- $\sin \Theta$
- $r \cos \theta$

**Along the crack**, these terms are simple:

$$R = \left( r_1 r_2 \right)^{1/2}$$

$$\Theta = \frac{\theta_1 + \theta_2}{2}$$

$$\Theta = \frac{\pm \pi}{2}, \text{ so } \cos \Theta = 0, \sin \Theta = \pm 1$$

$$r \cos \theta = x$$

$R$ is distance from right end

$r_2 - a + x$

$\cos \theta = x$

$r_1 - a - x$

Along the crack, these terms are simple:
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Displacements arising from opening of a mode-I crack, 2D elastic model (from Pollard and Segall, 1987)

Along the crack, \( R = (r_1 r_2)^{1/2} \), and \( r \cos \theta = x \)

\[
\begin{align*}
 u^c_x (|x| \leq a) &= \frac{\Delta \sigma_I}{2G} \left\{ (1 - 2v)(R \cos \Theta - r \cos \theta) \right\} \\
 &\rightarrow u^c_x = \frac{\Delta \sigma_I}{2G} \left\{ (1 - 2v)(-x) \right\} \\
 u^c_y (|x| \leq a) &= \frac{\Delta \sigma_I}{2G} \left\{ 2(1 - v)(R \sin \Theta) \right\} \\
 &\rightarrow u^c_y = \frac{\Delta \sigma_I}{2G} \left\{ 2(1 - v) \left[ \pm \sqrt{a^2 - x^2} \right] \right\}
\end{align*}
\]
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Displacements arising from opening of a mode-I crack, 2D elastic model (from Pollard and Segall, 1987)

Now consider the displacements normal to the crack:

\[ u_y^c = \frac{\pm \Delta \sigma_I}{2G} \left\{ 2(1 - \nu) \left( \sqrt{a^2 - x^2} \right) \right\} \rightarrow u_{y(y\text{max})}^c (x = 0) = \frac{\pm \Delta \sigma_I}{2G} \left\{ 2(1 - \nu)a \right\} \]

\[ \frac{u_y^c}{u_{y(y\text{max})}^c} = \frac{\pm \left( \sqrt{a^2 - x^2} \right)}{a} = \pm \left( \sqrt{1 - \left( \frac{x}{a} \right)^2} \right) \]
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Displacements arising from opening of a mode-I crack, 2D elastic model (from Pollard and Segall, 1987)

Now consider the displacements parallel to the crack:

\[ u^c_x (|x| \leq a) = \frac{\Delta \sigma_I}{2G} \left\{ (1-2\nu)(-x) \right\}, \text{ and } u^c_{y(\text{max})} (x = 0) = \frac{+\Delta \sigma_I}{2G} \left\{ 2(1-\nu)a \right\} \]

For \( \nu = 0.25 \),

\[ \frac{u^c_x}{u^c_{y(\text{max})}} = \frac{(1/2)(-x)}{2(3/4)a} = \frac{(1/2)(-x)}{(3/2)a} = -\frac{x}{3a} \]
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Displacements arising from opening of a mode-I crack, 2D elastic model (from Pollard and Segall, 1987)

Along the crack (|x/a| ≤ 1)

\[
\frac{u_y^c}{u_y^{c,y(max)}} = \frac{\pm(\sqrt{a^2 - x^2})}{a} = \pm \left(\sqrt{1 - \left(\frac{x}{a}\right)^2}\right)
\]

For v = 0.25, \[
\frac{u_x^c}{u_x^{c,y(max)}} = \frac{(1/2)(-x)}{2\left(\frac{3}{4}\right)a} = \frac{(1/2)(-x)}{\left(\frac{3}{2}\right)a} = -\frac{1}{3} \frac{x}{a}
\]
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II Mechanics of folds above intrusions

Sketch from field notes of Gilbert

http://pangea.stanford.edu/~annegger/images/colorado%20plateau/laccolith_sketch.jpg
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II Mechanics of folds above intrusions

Dimensional analysis of terms in governing equation for bending of an elastic layer (from Pollard and Fletcher, 2005)

\[
\frac{d^4v}{dx^4} = \frac{12p}{BH^3}
\]

v = vertical deflection of mid-plane {Length}
x = horizontal distance {Length}
L = length of flexed part of layer {Length}
p = overpressure {Force/area}
B = stiffness {Force/area}
H = thickness of layer {Length}

Dimensions check
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II  Mechanics of folds above intrusions

Dimensional analysis of terms in governing equation for bending of an elastic layer (from Pollard and Fletcher, 2005)

Find constant length scales and non-dimensionalize

\[ x^* = \frac{x}{L}, v^* = \frac{v}{v_{\text{max}}} \]
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Dimensional analysis ... (cont.)

Now non-dimensionalize the differential operator

\[ x^* = \frac{1}{L} x \rightarrow \frac{dx^*}{dx} = \frac{1}{L} \]

\[ \frac{d}{dx} = \frac{d}{dx^*} \frac{dx^*}{dx} = \frac{d}{dx^*} \frac{1}{L} \]

\[ \frac{d^2}{dx^2} = \left( \frac{d}{dx} \right) \left( \frac{d}{dx} \right) = \left( \frac{d}{dx^*} \frac{1}{L} \right) \left( \frac{d}{dx^*} \frac{1}{L} \right) = \left( \frac{1}{L} \right)^2 \left( \frac{d^2}{dx^* 2} \right) \]

\[ \frac{d^3}{dx^3} = \left( \frac{d}{dx} \right) \left( \frac{d^2}{dx^2} \right) = \left( \frac{d}{dx^*} \frac{1}{L} \right) \left( \left( \frac{1}{L} \right)^2 \left( \frac{d^2}{dx^* 2} \right) \right) = \left( \frac{1}{L} \right)^3 \left( \frac{d^3}{dx^* 3} \right) \]

\[ \frac{d^4}{dx^4} = \left( \frac{d}{dx} \right) \left( \frac{d^3}{dx^3} \right) = \left( \frac{d}{dx^*} \frac{1}{L} \right) \left( \left( \frac{1}{L} \right)^3 \left( \frac{d^3}{dx^* 3} \right) \right) = \left( \frac{1}{L} \right)^4 \left( \frac{d^4}{dx^* 4} \right), \text{etc.} \]
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Dimensional analysis of terms in governing equation for bending of an elastic layer (from Pollard and Fletcher, 2005)

Substitute into governing eq.

\[
\frac{d^4(v)}{dx^4} = \frac{12p}{BH^3}
\]

\[
v = v^* v_{max}
\]

\[
\frac{d^4}{dx^4} = \frac{d^4}{dx^*4} \frac{1}{L^4}
\]

\[
\frac{d^4(v)}{dx^4} = \frac{1}{L^4} \frac{d^4}{dx^*4} (v^* v_{max}) = \frac{1}{L^4} \left( v_{max} \right) \frac{d^4(v^*)}{dx^*4} = \frac{12p}{BH^3}
\]
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Dimensional analysis of terms in governing equation for bending of an elastic layer (from Pollard and Fletcher, 2005)

\[
\frac{d^4 (v)}{dx^4} = \frac{12p}{BH^3}
\]

\[
\frac{d^4 (v)}{dx^4} = \frac{1}{L^4} \frac{v_{\text{max}} d^4 (v^*)}{dx^*} = \frac{12p}{BH^3}
\]

\[
\frac{d^4 (v^*)}{dx^*} = \frac{12p}{B} \frac{L^4}{v_{\text{max}} H^3}
\]

Right side contains only constants

\[v^* \sim L^4\]
\[v^* \sim 1/H^3\]

Long thin layers will deflect much more than short thick layers

Setting up the problem in dimensionless form gives insight into its solution
Theoretical form of solution

\[
\frac{d^4v}{dx^4} = \frac{12p}{BH^3} = C_4
\]

\[
\frac{d^3v}{dx^3} = C_4x + C_3
\]

\[
\frac{d^2v}{dx^2} = \frac{C_4}{2}x^2 + C_3x + C_2
\]

\[
\frac{dv}{dx} = \frac{C_4}{6}x^3 + \frac{C_3}{2}x^2 + C_2x + C_1
\]

The function \( v \) is even: \( v(-x) = v(x) \)

By symmetry, the odd coefficients \( (C_3 \text{ and } C_1) \) must equal zero

\( C_2 \text{ and } C_4 \) are set so that \( v = 0 \) at \( x = \pm L/2 \) and \( v' = 0 \) at \( x = \pm L/2 \)
Bending of layer over laccolith should cause shearing at laccolith perimeter. This suggests laccoliths should propagate up towards the surface as they grow.
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The Golden Valley Sill, South Africa – a saucer-shaped sill

From Polteau et al., 2008
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II  Mechanics of folds above intrusions

1  Dike

2  Sill

3  Bending of overlying layers

4  Laccolith and saucer-shaped sill