22. Stresses Around a Hole (II)

Main Topics

A General solution for a plane strain case
B Boundary conditions
C Solution that honors boundary conditions
D Significance of solution
E Superposition
F Stress concentrations

21. Stresses Around a Hole (I)

Ship Rock, New Mexico

Hydraulic Fracture

http://jencarta.com/images/aerial/ShipRock.jpg

From Wu et al., 2007
22. Stresses Around a Hole (II)

II General solution for a plane strain case

Start with the governing equation

\[ 0 = \frac{d^2 u_r}{dr^2} + \frac{1}{r} \frac{du_r}{dr} - \frac{1}{r^2} u_r \]

Consider a power series solution for \( u_r \) and its derivatives

\[ u_r = \cdots - 3C_3 r^{-3} + 2C_2 r^{-2} - C_1 r^{-1} + C_0 r^0 + C_1 r^1 + C_2 r^2 + C_3 r^3 + \cdots \]

\[ \frac{du_r}{dr} = \cdots - 3C_3 r^{-4} - 2C_2 r^{-3} - 1C_1 r^{-2} + 0C_0 r^{-1} + 1C_1 r^0 + 2C_2 r^1 + 3C_3 r^2 + \cdots \]

\[ \frac{d^2 u_r}{dr^2} = \cdots 12C_3 r^{-5} + 6C_2 r^{-4} + 2C_1 r^{-3} + 0C_0 r^{-2} + 0C_1 r^{-1} + 2C_2 r^0 + 2C_3 r^1 + 6C_4 r^2 + \cdots \]

Now substitute the series solutions into the governing equation

\[ 0 = \frac{d^2 u_r}{dr^2} + \frac{1}{r} \frac{du_r}{dr} - \frac{1}{r^2} u_r \]

\[ 0 = \cdots \]

\[ \frac{1}{r} \left( \cdots - 3C_3 r^{-4} - 2C_2 r^{-3} - 1C_1 r^{-2} + 0C_0 r^{-1} + 1C_1 r^0 + 2C_2 r^1 + 3C_3 r^2 + \cdots \right) \]

\[ - \frac{1}{r^2} \left( \cdots C_3 r^{-3} + C_2 r^{-2} + C_1 r^{-1} + C_0 r^0 + C_1 r^1 + C_2 r^2 + C_3 r^3 + \cdots \right) \]

\[ 0 = \cdots \]

\[ \left( \cdots - 3C_3 r^{-5} - 2C_2 r^{-4} - 1C_1 r^{-3} + 0C_0 r^{-2} + 1C_1 r^{-1} + 2C_2 r^0 + 3C_3 r^1 + \cdots \right) \]

\[ - \left( \cdots C_3 r^{-4} + C_2 r^{-3} + C_1 r^{-2} + C_0 r^{-1} + C_1 r^0 + C_2 r^1 + C_3 r^2 + \cdots \right) \]
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II General Solution for a plane strain case

\[ 0 = \ldots 12C_3 r^{-5} + 6C_4 r^{-4} + 2C_5 r^{-3} + 0C_6 r^{-2} + 0C_7 r^{-1} + 2C_8 r^0 + 6C_9 r^1 + \ldots \]
\[ + (\ldots - 3C_3 r^{-5} - 2C_4 r^{-4} - 1C_5 r^{-3} + 0C_6 r^{-2} + 1C_7 r^{-1} + 2C_8 r^0 + 3C_9 r^1 + \ldots) \]
\[ + (\ldots - 1C_4 r^{-5} - 1C_5 r^{-4} - 1C_6 r^{-3} - 1C_7 r^{-2} - 1C_8 r^{-1} - 1C_9 r^0 - 1C_{10} r^1 - \ldots) \]

Now collect terms of the same powers

\[ 0 = \ldots 8C_3 r^{-5} + 3C_4 r^{-4} + 0C_5 r^{-3} - 1C_6 r^{-2} + 0C_7 r^{-1} + 3C_8 r^0 + 8C_9 r^1 + \ldots \]

For this to hold for all values of \( r \), the product of each leading coefficient and constant must equal 0 because the powers of \( r \) are linearly independent. All coefficients except \( C_1 \) and \( C_1 \) thus must be zero.

\[ u_r = \ldots C_4 r^{-3} + C_5 r^{-2} + C_6 r^{-1} + C_7 r^0 + C_8 r^2 + C_9 r^3 + \ldots \]
\[ u_r = C_4 r^{-1} + C_1 r^1 \]

- General solution for radial displacements
- Solve for constants via boundary conditions
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III Boundary conditions

A Two boundary conditions must be specified to solve our problem because our general solution has two unknown coefficients:

\[ u_r = C_{-1} r^{-1} + C_{1} r^{1} \]

B \( u_r = u_0 \) at the wall of the hole: \( u_r \mid_{r=a} = u_0 \)

C \( u_r = 0 \) at infinity: \( u_r \mid_{r=b=\infty} = 0 \)
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IV Solution that honors boundary conditions
Gov. eq.: \[ u_r = C_1 r^{-1} + C_2 r \]
BC 1: \[ u_r|_{r=a} = u_0 \]
BC 2: \[ u_r|_{r=b=\infty} = 0 \]

As \( r \to \infty \), \( u_r \to C_2 r \)

BC 2 requires \( C_2 = 0 \), so
\[ u_r = C_1 r^{-1} \]

By BC 1, \( u_r|_{r=a} = u_0 = C_1 a^{-1} \)
So, \( C_1 = a u_0 \)

\[ u_r = (\alpha/r) u_0 \]
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IV Solution that honors boundary conditions (cont.)

\[ u_r = \left( \frac{a}{r} \right) u_0 \]

- The hole radius \( a \) provides a scale
- The displacements decay with distance \( r \) from the hole (as suspected)
- The displacements scale with \( u_0 \)
- Problems with different boundary conditions have different solutions

\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{\partial (u_0ar^{-1})}{\partial r} = u_0a \frac{\partial (r^{-1})}{\partial r} = \frac{-u_0a}{r^2} = -u_0ar^{-2} \]

\[ \varepsilon_{\theta\theta} = \frac{u_\theta}{r} = \frac{u_0ar^{-1}}{r} = \frac{u_0a}{r^2} = u_0ar^{-2} \]

\[ \varepsilon_{r\theta} = 0 \]
22. Stresses Around a Hole (II)

IV Solution that honors boundary conditions (cont.)

Stresses (in terms of $u_0$)

Hooke’s Law

Strains

\[
\sigma_r = \frac{E}{1 + v} \left[ \varepsilon_r + \frac{v}{(1 - 2v)} (\varepsilon_\theta + \varepsilon_{th}) \right]
\]

\[
= \frac{E}{1 + v} \left[ \frac{-u_0 a}{r^2} + \frac{v}{(1 - 2v)} \left( \frac{-u_0 a}{r^2} + \frac{u_0 a}{r^2} \right) \right]
\]

\[
= \frac{E}{1 + v} \left[ \frac{-u_0 a}{r^2} \right]
\]

\[
\sigma_\theta = \frac{E}{1 + v} \left[ \varepsilon_\theta + \frac{v}{(1 - 2v)} (\varepsilon_\theta + \varepsilon_{th}) \right]
\]

\[
= \frac{E}{1 + v} \left[ \frac{u_0 a}{r^2} + \frac{v}{(1 - 2v)} \left( \frac{u_0 a}{r^2} + \frac{-u_0 a}{r^2} \right) \right]
\]

\[
= \frac{E}{1 + v} \left[ \frac{u_0 a}{r^2} \right]
\]

\[
\sigma_{th} = \frac{E}{1 + v} \left[ \varepsilon_{th} \right]
\]

\[
= \frac{E}{1 + v} \left[ \varepsilon_{th} \right]
\]

\[
= \frac{E}{1 + v} \left[ \frac{u_0 a}{r^2} \right]
\]

\[
\sigma_{th} = 2G\varepsilon_{th} = 0
\]
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IV Solution that honors boundary conditions (cont.)

Stresses (in terms of tractions)

First evaluate the radial stress on the wall of the hole \((r=a)\), which equals traction \(T\), and from that solve for \(u_0\)

\[
\sigma_n|_{r=a} = \frac{E}{(1+v)} \left[ \frac{-u_0 a}{a^2} \right] = \frac{E}{(1+v)} \left[ \frac{-u_0}{a} \right] = T
\]

\[u_0 = -aT \frac{E}{(1+v)}\]

Now substitute for \(u_0\) in the general expression for \(\sigma_n\) on previous page

\[
\sigma_n = \frac{E}{(1+v)} \left[ \frac{aT}{(1+v)} \right] \left( \frac{a}{r} \right)^2 = T \left( \frac{a}{r} \right)^2
\]

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V Significance

For a pressure in a hole with no remote load at \(r = \infty\):

A The radial normal stress \(\sigma_r\) is a principal stress because \(\sigma_\theta = 0\); \(\sigma_r\) is the most compressive stress.

B The circumferential normal stress \(\sigma_\theta\) is a principal stress because \(\sigma_\theta = 0\); \(\sigma_\theta\) is the most tensile stress.

C A high pressure could cause radial cracking (e.g., radial dikes around a magma chamber).
VI. Superposition

\[ \sigma_{rr} = -T(\alpha h)^2 \]
\[ \sigma_{\theta\theta} = -T(\alpha h)^2 \]

\[ \sigma_{rr} = T \]
\[ \sigma_{\theta\theta} = T \]

\[ \sigma_{rr} = T - T(\alpha h)^2 \]
\[ \sigma_{\theta\theta} = T + T(\alpha h)^2 \]

Stress concentration!

VII. Stress concentrations

A. The hole causes a doubling of the normal stress far from the hole (i.e., a stress concentration).

B. The magnitude of the circumferential stress at the hole is independent of the size of the hole, so a tiny cylindrical hole causes the same stress concentration as a large one.

C. A tiny hole near the wall of a larger hole might be expected to have an even larger stress concentration (why?)
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VI Stress concentrations

D Stress concentrations explain myriad phenomena, such as

1. Why paper doesn’t explode when pulled upon hard
2. Why paper tears along “the dotted line”
3. Why cracks in riveted steel plates start from the rivet holes
4. Why cracks in drying mudflats originate from the where grass stems have poked through the mud etc.
5. Why “strong” rocks can fail under “low” stresses
6. Why dikes can propagate through the Earth’s crust
7. Why seismic ruptures grow so large