16. STRESS AT A POINT

I Main Topics
   A Stress vector (traction) on a plane
   B Stress at a point
   C Principal stresses
   D Transformation of principal stresses to tractions in 2D
16. STRESS AT A POINT

16. STRESS AT A POINT

I Stress vector (traction) on a plane
A \( \vec{\tau} = \lim_{A \to 0} \frac{\vec{F}}{A} \)
B Traction vectors can be added as vectors
C A traction vector can be resolved into normal \((\tau_n)\) and shear \((\tau_s)\) components
  1 A normal traction \((\tau_n)\) acts perpendicular to a plane
  2 A shear traction \((\tau_s)\) acts parallel to a plane
D Local reference frame
  1 The n-axis is normal to the plane
  2 The s-axis is parallel to the plane
16. STRESS AT A POINT

III Stress at a point (cont.)

A Stresses refer to balanced internal "forces (per unit area)". They differ from force vectors, which, if unbalanced, cause accelerations

B "On -in convention": The stress component $\sigma_{ij}$ acts on the plane normal to the $i$-direction and acts in the $j$-direction

1 Normal stresses: $i=j$
2 Shear stresses: $i\neq j$
16. STRESS AT A POINT

III Stress at a point

C Dimensions of stress: force/unit area

D Convention for stresses
1 Tension is positive
2 Compression is negative
3 Follows from on-in convention
4 Consistent with most mechanics books
5 Counter to most geology books
16. STRESS AT A POINT

III Stress at a point

C \[ \sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \] 2-D
4 components

D \[ \sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \] 3-D
9 components

E In nature, the state of stress can (and usually does) vary from point to point

F For rotational equilibrium,
\[ \sigma_{xy} = \sigma_{yx}, \sigma_{xz} = \sigma_{zx}, \sigma_{yz} = \sigma_{zy} \]
16. STRESS AT A POINT

IV Principal Stresses (these have magnitudes and orientations)

A Principal stresses act on planes which feel no shear stress

B The principal stresses are normal stresses.

C Principal stresses act on perpendicular planes

D The maximum, intermediate, and minimum principal stresses are usually designated $\sigma_1$, $\sigma_2$, and $\sigma_3$, respectively.

E Principal stresses have a single subscript.
16. STRESS AT A POINT

IV Principal Stresses (cont.)

F Principal stresses represent the stress state most simply

\[
\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}
\]

2-D 2 components

H \[
\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}
\]

3-D 3 components
V Transformation of principal stresses to tractions in 2D

A Description of terms

1 Three planes $A, A_1, \text{ and } A_2$ form the sides of a triangular prism; these have normals in the $n$-, $1$-, and $-2$-directions, respectively.

2 Plane $A_1$ is acted on by known normal stress $\sigma_1$.

3 Plane $A_2$ is acted on by known normal stress $\sigma_2$.

4 The $n$-direction is at angle $\theta_1$ (=\(\theta\)) with respect to the $1$-direction, and at angle $\theta_2$ with respect to the $2$-direction.

5 The $s$-direction is at a counter-clockwise $90^\circ$ angle relative to the $n$-direction (like $y$ and $x$).

\[ \theta_1 + \theta_2 = 90^\circ \]
\[ \cos \theta_1 = \cos \theta \]
\[ \cos \theta_2 = \sin \theta \]
16. STRESS AT A POINT

V Transformation of principal stresses to tractions in 2D

B Approach

Find weighting factors that determine contributions of known stresses to desired tractions and sum contributions

1. \( \tau_n = w_{n1} \sigma_1 + w_{n2} \sigma_2 \)
2. \( \tau_s = w_{s1} \sigma_1 + w_{s2} \sigma_2 \)
16. STRESS AT A POINT

C Contribution of $\sigma_1$ (on face $A_1$ of area $A_1$) to $\tau_n$
(on face A of area A)

Start with the definition of traction:

1 $\tau_n^{(1)} = \frac{F_n^{(1)}}{A}$

Find unknowns $F_n^{(1)}$ and $A$ from knowns $\sigma_1$ and $\theta$.

First find the force $F_1$ associated with $\sigma_1$

2 $F_1 = \sigma_1 A_1$  Force = (stress)(area)

Find $F_n^{(1)}$, the component of $F_1$ in the n-direction

3 $F_n^{(1)} = F_1 \cos \theta_1$

Find $A$ in terms of $A_1$

4 $A = A_1 / \cos \theta_1$

Contribution of $\sigma_1$ to $\tau_n$:

5a $\tau_n^{(1)} = \frac{F_n^{(1)}}{A} = \frac{F_1 \cos \theta_1}{(A_1 / \cos \theta_1)}$

5b $\tau_n^{(1)} = (F_1 / A_1) \cos \theta_1 \cos \theta_1 = \sigma_1 \cos \theta_1 \cos \theta_1$

Weighting factor $w_{n1}$

6 $w_{n1} = \cos \theta_1 \cos \theta_1 = \cos \theta \cos \theta$
16. STRESS AT A POINT

D Contribution of $\sigma_2$ (on face $A_2$ of area $A_2$) to $\tau_n$
(on face $A$ of area $A$)

Start with the definition of traction:

1 $\tau_n^{(2)} = F_n^{(2)} / A$

Find unknowns $F_n^{(2)}$ and $A$ from knowns $\sigma_2$ and $\theta$.

First find the force $F_2$ associated with $\sigma_2$

2 $F_2 = \sigma_2 A_2$  
Force = (stress)(area)

Find $F_n^{(2)}$, the component of $F_2$ in the $n$-direction

3 $F_n^{(2)} = F_2 \cos \theta_2$

Find $A$ in terms of $A_2$

4 $A = A_2 / \cos \theta_2$

Contribution of $\sigma_2$ to $\tau_n$:

5a $\tau_n^{(2)} = F_n^{(2)} / A = F_2 \cos \theta_2 / (A_2 / \cos \theta_2)$

5b $\tau_n^{(2)} = (F_2 / A_2) \cos \theta_2 \cos \theta_2 = \sigma_2 \cos \theta_2 \cos \theta_2$

Weighting factor $w_{n2}$

6 $w_{n2} = \cos \theta_2 \cos \theta_2 = \sin \theta \sin \theta$

Fig. 16.2

\[ \theta_1 + \theta_2 = 90^\circ \]

\[ \cos \theta_1 = \cos \theta \]

\[ \cos \theta_2 = \sin \theta \]
16. STRESS AT A POINT

E Contribution of \( \sigma_1 \) (on face \( A_1 \) of area \( A_1 \)) to \( \tau_s \) (on face \( A \) of area \( A \))

Start with the definition of traction:

1 \[ \tau_s^{(1)} = \frac{F_s^{(1)}}{A} \]

Find unknowns \( F_s^{(1)} \) and \( A \) from knowns \( \sigma_1 \) and \( \theta \).

First find the force \( F_1 \) associated with \( \sigma_1 \)

2 \[ F_1 = \sigma_1 A_1 \quad \text{Force} = (\text{stress})(\text{area}) \]

Find \( F_s^{(1)} \), the component of \( F_1 \) in the s-direction

3 \[ F_s^{(1)} = -F_1 \cos \theta_2 \]

Find \( A \) in terms of \( A_2 \)

4 \[ A = A_1 / \cos \theta_1 \]

Contribution of \( \sigma_1 \) to \( \tau_n \) of \( \sigma_1 \):

5a \[ \tau_s^{(1)} = \frac{F_s^{(1)}}{A} = \frac{-F_1 \cos \theta_2}{(A_1 / \cos \theta_1)} \]

5b \[ \tau_s^{(1)} = -\left( \frac{F_1}{A_1} \right) \cos \theta_2 \cos \theta_1 = -\sigma_1 \cos \theta_2 \cos \theta_1 \]

Weighting factor \( w_{s1} \)

6 \[ w_{s1} = -\cos \theta_2 \cos \theta_1 = -\sin \theta \cos \theta \]
16. STRESS AT A POINT

F Contribution of $\sigma_2$ (on face $A_1$ of area $A_1$) to $\tau_s$ (on face $A$ of area $A$)

Start with the definition of traction:

1 $\tau_s^{(2)} = F_s^{(2)}/A$

Find unknowns $F_s^{(2)}$ and $A$ from knowns $\sigma_2$ and $\theta$.

First find the force $F_2$ associated with $\sigma_2$

2 $F_2 = \sigma_2 A_2$ Force = (stress)(area)

Find $F_s^{(2)}$, the component of $F_2$ in the s-direction

3 $F_s^{(2)} = F_2 \cos \theta_1$

Find $A$ in terms of $A_1$

4 $A = A_2/\cos \theta_2$

Contribution of $\sigma_1$ to $\tau_n$ of $\sigma_1$:

5a $\tau_s^{(2)} = F_s^{(2)}/A = F_2 \cos \theta_1 / (A_2/\cos \theta_2)$

5b $\tau_s^{(2)} = (F_2/A_2) \cos \theta_1 \cos \theta_2 = \sigma_2 \cos \theta_1 \cos \theta_2$

Weighting factor $w_{s2}$

6 $w_{s2} = \cos \theta_1 \cos \theta_2 = \cos \theta \sin \theta$
16. STRESS AT A POINT

V Transformation of principal stresses to tractions in 2D

G Original equations
1. \( \tau_n = w_{n1} \sigma_1 + w_{n1} \sigma_2 \)
2. \( \tau_s = w_{s1} \sigma_1 + w_{s1} \sigma_2 \)

H Original equations
1. \( \tau_n = \cos\theta\cos\theta \sigma_1 + \sin\theta\sin\theta \sigma_2 \)
2. \( \tau_s = -\sin\theta\cos\theta \sigma_1 + \sin\theta\cos\theta \sigma_2 \)

Weighting factors are products of two direction cosines.