15. FINITE STRAIN & INFINITESIMAL STRAIN (AT A POINT)

I Main Topics
A The finite strain tensor \([E]\)
B Deformation paths for finite strain
C Infinitesimal strain and the infinitesimal strain tensor \(\varepsilon\)

15. FINITE STRAIN & INFINITESIMAL STRAIN

- Elongate horizontally (double horizontal dimensions)
- Rotate 45° about axis pointing out

- Rotate 45° about axis pointing out
- Elongate horizontally (double horizontal dimensions)

The sequence of deformation matters for large strains
15. FINITE STRAIN & INFINITESIMAL STRAIN

The sequence of deformation does not matter for small strains.

II The finite strain tensor $[E]$

A Used to find the changes in the squares of distances $(ds)^2$ between points in a deformed body based on differences in their initial positions.

B Displacements by themselves don’t give changes in size or shape.
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II The finite strain tensor \([E]\)

C Derivation of \([E]\)

1 \((ds)^2 = (dx)^2 + (dy)^2\)

2 \((ds)^2 = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}\)

3 \(\begin{bmatrix} dx \\ dy \end{bmatrix} = [dX]\)

4 \(\begin{bmatrix} dx & dy \end{bmatrix} = [dX]^T\)

5 \((ds)^2 = [dX]^T [dX] = [dX]^T [I] [dX], \quad \text{where} \quad [I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\)

6 \((ds')^2 = \begin{bmatrix} dx' & dy' \end{bmatrix} \begin{bmatrix} dx' \\ dy' \end{bmatrix} = [dX']^T [dX']\)

7 \([dX'] = [F] [dX]\)

8 \((ds')^2 = [F] [dX]^T [F] [dX]\)

From lecture 14:
15. FINITE STRAIN & INFINITESIMAL STRAIN

C Derivation of [E] (cont.)


\[ (ds')^2 - (ds)^2 = [dX]^T \left( [F]^T [F] - [I] \right) [dX] \]  

\[ \frac{1}{2} \left( (ds')^2 - (ds)^2 \right) = \frac{1}{2} [dX]^T \left[ E \right] [dX] \]

\[ [E] = \frac{1}{2} \left[ [F]^T [F] - [I] \right] \]

D Meaning of [E]

\[ \frac{1}{2} \left( (ds')^2 - (ds)^2 \right) = \frac{1}{2} [dX]^T \left[ E \right] [dX] \]

Given [E] and [dX] (the difference in initial positions) of points, then one can find the difference in the squares of the lengths of lines connecting the points before and after deformation.

\[ [E] = \frac{1}{2} \left( [F]^T [F] - [I] \right) \]

\[ [E] = \frac{1}{2} \left( \left[ J_u + I \right]^T [J_u + I] - [I] \right) \]

But what do the terms of [E] mean?
15. FINITE STRAIN & INFINITESIMAL STRAIN

E Expansion of $[E]$

1. $J_u = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$

2. $[E] = \frac{1}{2} \left[ (J_u + I)^T (J_u + I) - I \right]$ 

3. $[E] = \frac{1}{2} \left[ \begin{bmatrix} \frac{\partial u}{\partial x} + 1 & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + 1 \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + 1 \end{bmatrix} \right] - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $[E] = \frac{1}{2} \left[ \begin{bmatrix} \frac{\partial u}{\partial x} + 1 & \frac{\partial v}{\partial x} + 1 \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + 1 \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + 1 \end{bmatrix} \right] - \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + 1 \end{bmatrix}$

The meaning of the terms in $E$ is not intuitive. Imagine what the products of multiple finite deformations yield.
III

If the displacement derivatives are <<1, then their products are tiny and can be neglected.

\[ [E] = \frac{1}{2} \left[ \begin{array}{ccc}
\frac{\partial u}{\partial x} + 1 & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} + 1 \\
\frac{\partial v}{\partial y} & \frac{\partial v}{\partial y} + 1 & \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y} + 1
\end{array} \right]^{-1} \]

Conversions of \([E]\) to \([\varepsilon]\) for small strains:

\[ [\varepsilon] = \frac{1}{2} \left[ \begin{array}{ccc}
\frac{\partial u}{\partial x} + 1 & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} + 1 \\
\frac{\partial v}{\partial y} & \frac{\partial v}{\partial y} + 1 & \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y} + 1
\end{array} \right]^{-1} + \left[ \begin{array}{ccc}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} + 1 \\
\frac{\partial v}{\partial y} & \frac{\partial v}{\partial y} + 1 & \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y} + 1
\end{array} \right]^{-1} \]

15. FINITE STRAIN & INFINITESIMAL STRAIN

III

Deformation paths for finite strain:

Consider two different deformations:

A Deformation 1:

\[ [F_1] = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \]

B Deformation 2:

\[ [F_2] = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \]
15. FINITE STRAIN & INFINITESIMAL STRAIN

III Deformation paths for finite strain

Consider two different deformations

A Deformation 1

\[
[F_1] = \begin{bmatrix}
  a_1 & b_1 \\
  c_1 & d_1 \\
\end{bmatrix}
\]

B Deformation 2

\[
[F_2] = \begin{bmatrix}
  a_2 & b_2 \\
  c_2 & d_2 \\
\end{bmatrix}
\]

C F2 acts on F1

\[
[F_2][F_1] = \begin{bmatrix}
  a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\
  c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \\
\end{bmatrix}
\]

D F1 acts on F2

\[
[F_1][F_2] = \begin{bmatrix}
  a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\
  c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \\
\end{bmatrix}
\]

E The sequence of finite deformations matters – unless off-diagonal terms are small

\[
[F_2][F_1] \neq [F_1][F_2]
\]
15. FINITE STRAIN & INFINITESIMAL STRAIN

IV Infinitesimal strain and the infinitesimal strain tensor \([\varepsilon]\)

A Infinitesimal strain

Deformation where the displacement derivatives in \([J_s]\) are small relative to one so that the products of the derivatives are very small and can be ignored.

\[\varepsilon = \frac{1}{2} \left[ \begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right] \]

B An approximation to finite strain
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IV Infinitesimal strain and the infinitesimal strain tensor \([\varepsilon]\) (cont.)

C Why consider \([\varepsilon]\) if it is an approximation?

1 Relevant to important geologic deformations
   A Fracture
   B Earthquake deformation
   C Volcano deformation

C Why consider \([\varepsilon]\) if it is an approximation? (cont.)

2 Terms of the infinitesimal strain tensor \([\varepsilon]\) have clear geometric meaning

3 Can apply principal of superposition (addition)

4 Infinitesimal deformation is essentially independent of the deformation sequence

5 Amenable to sophisticated mathematical treatment (e.g., elasticity theory)

6 Quantitative predictive ability

http://www.spacegrant.hawaii.edu/class acts/WebImg/gelatinVolcano.gif

Gelatin Volcano Experiment

Hawai‘i Space Grant Consortium

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Gelatin Volcano Experiment

Hawai‘i Space Grant Consortium
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Why consider $[\varepsilon]$ if it is an approximation? (cont.)

7 Infinitesimal strain example

\[ F_1 = \begin{bmatrix} 1.02 & 0.01 \\ 0 & 1.01 \end{bmatrix} \rightarrow J_{\varepsilon(1)} = \begin{bmatrix} 0.02 & 0.01 \\ 0 & 0.01 \end{bmatrix} \]

\[ F_2 = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.02 \end{bmatrix} \rightarrow J_{\varepsilon(2)} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix} \]

\[ [F_1 F_1] = \begin{bmatrix} 1.01 & 0 \\ 0 & 1.02 \end{bmatrix} \begin{bmatrix} 1.02 & 0.01 \\ 0 & 1.01 \end{bmatrix} = \begin{bmatrix} 1.0302 & 0.0100 \\ 0.0000 & 1.0302 \end{bmatrix} \]

Sequence results can be obtained regardless of the order of events, but also by superposition.

IV Infinitesimal strain and the infinitesimal strain tensor $[\varepsilon]$

D Taylor series expansion

We seek $U_2$ given $U_1$ and $dX$

1 $u_2 = u_1 + du = u_1 + \left( \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + \ldots$

2 $v_2 = v_1 + dv = v_1 + \left( \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right) + \ldots$

3 $\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} + \ldots \Rightarrow [U_2] = [U_1] + [dU_1] = [U_1] + [J_u] [dX]$
15. FINITE STRAIN & INFINITESIMAL STRAIN

D Taylor series expansion (cont.)

Now split $[J_0]$ into two matrices: the infinitesimal strain matrix $[\varepsilon]$ and the anti-symmetric rotation matrix $[\omega]$

$$[\varepsilon] = \frac{1}{2} \left( [J_0] + [J_0]^T \right) = \frac{1}{2} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} \end{bmatrix}$$

$$[\omega] = \frac{1}{2} \left( [J_0] - [J_0]^T \right) = \frac{1}{2} \begin{bmatrix} 0 & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & -\frac{\partial u}{\partial x} \\ -\frac{\partial v}{\partial x} & 0 & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{bmatrix}$$

$$[\varepsilon] + [\omega] = [J_0]$$

IV Infinitesimal strain and the infinitesimal strain tensor $[\varepsilon]$ 

D Taylor series expansion

$$[U_2] = [U_1] + [J_0] [dX] = [U_1] + [[\varepsilon] + [\omega]] [dX]$$

The deformation can be decomposed into a translation, a strain, and a rotation
15. FINITE STRAIN & INFINITESIMAL STRAIN

\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \]
\[ \varepsilon_{xy} = \frac{1}{2} (\Psi_1 - \Psi_2) = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]
\[ \omega_{xy} = \frac{1}{2} (\Psi_1 + \Psi_2) = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \]

For small angles, \( \Psi = \tan \Psi \)

Positive angles are measured about the z-axis using a right hand rule. In (c) the angle \( \Psi_2 \) is clockwise (negative), but \( du \) is positive. In (d) \( \Psi_2 \) is counter-clockwise, and \( du < 0 \).
15. FINITE STRAIN & INFINITESIMAL STRAIN

Infinitesimal Deformation of Line Elements AB and AO (modified from Chou and Pagano, 1967)

Line length changes
\[ \ell_{\text{new}} = \ell_{\text{old}} + \frac{\Delta \ell}{\ell_{\text{old}}} \]
\[ \Delta \ell = \frac{A_0}{A_1} \]
\[ \Delta \ell = \frac{A_2}{A_3} \]

Angle changes
\[ \tan \theta_1 = \frac{\Delta \gamma_1}{\Delta \ell} = \frac{\gamma_1}{\ell} \text{ is small} \]
\[ \tan \theta_2 = \frac{\Delta \gamma_2}{\Delta \ell} = \frac{\gamma_2}{\ell} \text{ is small} \]

\( \gamma_1 - \gamma_2 \) = change in right angle

\[ \gamma = \gamma_1 + \gamma_2 = (1/2) \tan \theta \]
\[ \gamma = \text{engineering shear strain} \]
\[ \tau = \text{tensile shear strain} \]