6. SCALARS, VECTORS, AND TENSORS
(FOR ORTHOGONAL COORDINATE SYSTEMS)

I Main Topics
A What are scalars, vectors, and tensors?
B Order of scalars, vectors, and tensors
C Linear transformation of scalars and vectors (and tensors)
D Matrix multiplication
6. SCALARS, VECTORS, AND TENSORS

- **What are scalars, vectors, and tensors?**
  - A Quantities with associated directions
  - B Tensors
    1. Broaden our perspectives; geologists unacquainted with them are handicapped
    2. For multi-dimensional thinking and communication
    3. They can be extremely useful
III Order of scalars, vectors, and tensors

A Scalars (magnitudes)
1 Numbers with no associated direction (zero-order tensors)
2 No subscripts in notation
3 Examples: Time, mass, length, volume
4 Matrix representation: 1x1 matrix $[x]$

B Vectors (magnitude and a direction)
1 Quantities with one associated direction (first-order tensors)
2 One subscript in notation (e.g., $u_x$)
3 Examples: Displacement, velocity, acceleration
6. SCALARS, VECTORS, AND TENSORS

III Order of scalars, vectors, and tensors (cont.)
B Vectors (magnitude and a direction) (cont.)
   4 Matrix representation: 1xn row matrix, or nx1 column matrix, with n components
      a Two-dimensional vector (n=2 components):
         \[ \begin{bmatrix} x \\ y \end{bmatrix} \text{ or } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]
         1 row, 2 columns
         2 rows, 1 column
         x = component in x-direction, y = component in y-direction
         x_1 = component in x-direction, x_2 = component in y-direction
      b Three-dimensional vector (n=3 components):
         \[ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ or } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]
         1 row, 3 columns
   5 Don’t confuse the dimensionality of a tensor with its order

C Tensors (magnitude and two directions)
   (for the 2\textsuperscript{nd}-order tensors we will consider)
   1 Quantities with two associated direction
      (second-order tensors)
   2 Two subscripts in notation (e.g., \(\sigma_{xx}\))
   3 Examples: Stress, strain, permeability
6. SCALARS, VECTORS, AND TENSORS

III "Order" of scalars, vectors, and tensors (cont.)
C Tensors (magnitude and two directions) (cont.)
   4 Matrix representation: nxn matrix, with n^2 components
      a Two-dimensional tensor (4 components):
         \[
         \begin{bmatrix}
         \sigma_{xx} & \sigma_{xy} \\
         \sigma_{yx} & \sigma_{yy}
         \end{bmatrix}
         \text{ or }
         \begin{bmatrix}
         \sigma_{11} & \sigma_{12} \\
         \sigma_{21} & \sigma_{22}
         \end{bmatrix}
         \]\n         2 rows, 2 columns
      b Three-dimensional tensor (3 components):
         \[
         \begin{bmatrix}
         \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
         \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
         \sigma_{zx} & \sigma_{zy} & \sigma_{zz}
         \end{bmatrix}
         \text{ or }
         \begin{bmatrix}
         \sigma_{11} & \sigma_{12} & \sigma_{13} \\
         \sigma_{21} & \sigma_{22} & \sigma_{23} \\
         \sigma_{31} & \sigma_{32} & \sigma_{33}
         \end{bmatrix}
         \]\n         3 rows, 3 columns
      5 An n-dimensional 2nd-order tensor consists of n rows of n-dimensional vectors

6. SCALARS, VECTORS, AND TENSORS

IV Linear transformations
A "Transformations" refers to how components change when the coordinate system changes.
B "Linear" means the transformation depends on the length of the components, not, for example, on the square of the component lengths.
C Transformations are used to when we change reference frames in order to present physical quantities from a different (clearer) perspective.
D Transformations of tensors not covered today
6. SCALARS, VECTORS, AND TENSORS

IV Linear transformations (cont.)

E  Linear transformations of scalars
   1  Scalar quantities don’t change in response to a transformation of coordinates; they are invariant
   2  Examples (independent of reference frame orientation)
      a  Mass
      b  Volume
      c  Density

F  Linear transformations of vectors (cont.)
   1  Vector components change with a transformation of coordinates
      a  \( \mathbf{V} = v_x + v_y = v_x \mathbf{i} + v_y \mathbf{j} \)
      b  \( \mathbf{V}' = v_{x'} + v_{y'} = v_x' \mathbf{i}' + v_y' \mathbf{j}' \)
      c  Vector component: \( v_c \)
      d  Scalar component: \( v_c \)
         **Bold:** vector components
         **Unbolded:** Scalar components

Fig. 6.1

I, j, i’, and j’ are unit basis vectors along the x, y, x’, and y’ axes, respectively
IV Linear transformations (cont.)

2 Every component in the unprimed reference frame contributes linearly to each component in the primed reference frame.

\[ v_x' = a_{xx} v_x + a_{xy} v_y \]
\[ v_y' = a_{yx} v_x + a_{yy} v_y \]

3 The direction cosines are weighting factors that specify how much each component in one reference frame contributes to a component in the other reference frame.

\[ a_{xx} = \cos(\theta_{xx}) = \cos(\theta_{yx}) = a_{yx} \]
\[ a_{yy} = \cos(\theta_{yy}) = \cos(\theta_{xy}) = a_{xy} \]
6. SCALARS, VECTORS, AND TENSORS

IV Linear transformations (cont.)

F Linear transformations of vectors (cont.)

4 Transformation rule for vectors

a \( v'_i = a_{ij} v_j \)

b Expanded form

\[
\begin{align*}
v'_i &= a_{ix} v_x + a_{iy} v_y \\
v'_y &= a_{yx} v_x + a_{yy} v_y
\end{align*}
\]

5 Matrix form

\[
[V'] = [A][V]
\]

(Note upper case)

- List what you know
- List what you want to know
- Add the projection terms
6. SCALARS, VECTORS, AND TENSORS

V Matrix Multiplication - Examples
A General Rule: An $n \times m$ matrix times an $m \times p$ matrix gives an $n \times p$ matrix

B Examples
1 A $1 \times 2$ matrix times a $2 \times 1$ matrix gives a $1 \times 1$ matrix

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
3 \\
4
\end{bmatrix} = (1)(3) + (2)(4) = 11
\]

2 A $2 \times 1$ matrix times a $1 \times 2$ matrix gives a $2 \times 2$ matrix

\[
\begin{bmatrix}
3 \\
4
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
1 & 2
\end{bmatrix} =
\begin{bmatrix}
(3)(1) & (3)(2) \\
(4)(1) & (4)(2)
\end{bmatrix} =
\begin{bmatrix}
3 & 6 \\
4 & 8
\end{bmatrix}
\]

3 A $2 \times 2$ matrix times a $2 \times 2$ matrix gives a $2 \times 2$ matrix

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
3 & 4
\end{bmatrix} =
\begin{bmatrix}
(1)(1) + (2)(0) & (1)(0) + (2)(1) \\
(3)(1) + (4)(0) & (3)(0) + (4)(1)
\end{bmatrix} =
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\]