Eigenvectors, Eigenvalues, and Finite Strain

GG303, 2013
“Lab 9”

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

I Main Topics
A Elementary linear algebra relations
B Equations for an ellipse
C Equation of homogeneous deformation
D Eigenvalue/eigenvector equation
E Solutions for symmetric homogeneous deformation matrices
F Solutions for general homogeneous deformation matrices
G Rotations in homogeneous deformation
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Examples of 2D homogeneous deformation
Note that the symmetry of the displacement fields (or lack thereof) in the examples corresponds to the symmetry (or lack thereof) in the deformation gradient matrix \([F]\).

What is a simple way to describe homogeneous deformation that is geometrically meaningful?
What is the geologic relevance?

II Elementary linear algebra relations

A Inverse \([A]^{-1}\) of a real matrix \(A\)

1 \([A][A]^{-1} = [A]^{-1}[A] = [I]\),

where \([I]\) = identity matrix (e.g., \([I]\) = \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\])

2 \([A]\) and \([A]^{-1}\) must be square nxn matrices

3 Inverse \([A]^{-1}\) of a 2x2 matrix

\[
[A] = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \quad \text{then} \quad [A]^{-1} = \frac{1}{ad-bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix} = \frac{1}{|A|} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}
\]

4 Inverse \([A]^{-1}\) of a 3x3 matrix also requires determinant \(|A|\) to be non-zero
II Elementary linear algebra relations

B Determinant |A| of a real matrix A

1 A number that provides scaling information on a square matrix

2 Determinant of a 2x2 matrix

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad |A| = ad - bc \]

Akin to: Cross product (an area)
Scalar triple product (a volume)

3 Determinant of a 3x3 matrix

\[ A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad |A| = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \]

C Transpose

For \( [A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), \( [A]^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \)

D Transpose of a matrix product

If \( [A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( [B] = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \), then \( [A]^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \) and \( [B]^T = \begin{bmatrix} e & g \\ f & h \end{bmatrix} \)

\[ [A][B] = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}, \quad ([A][B])^T = \begin{bmatrix} ae + bg & ce + dg \\ af + bh & cf + dh \end{bmatrix} \]

\[ [B]^T[A]^T = \begin{bmatrix} ea + gb & ec + gd \\ fa + hb & fc + hd \end{bmatrix} = ([A][B])^T \]

This is true for any real nxn matrices
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II Elementary linear algebra relations

E Representation of a dot product using matrix multiplication and the matrix transpose

\[ \mathbf{a} \cdot \mathbf{b} = \langle a_x, a_y, a_z \rangle \cdot \langle b_x, b_y, b_z \rangle = a_x b_x + a_y b_y + a_z b_z \]

\[ = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = [\mathbf{a}]^T [\mathbf{b}] \]

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III Equations for an ellipse

A Equation of a unit circle

1 \[ x^2 + y^2 = \mathbf{X} \cdot \mathbf{X} = 1 \]

2 \[ \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [\mathbf{X}]^T [\mathbf{X}] = 1 \]

3 \[ x = \cos \theta \]

\[ y = \sin \theta \]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Equations for an ellipse

B Ellipse centered at (0,0), aligned along x, y axes

1 Standard form
\[
\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1
\]

2 General form
\[Ax^2 + Dy^2 + F = 0\]

3 Matrix form
\[
\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax \\ Dy \end{bmatrix} = -F
\]
A, D, and F are constants here, not matrices

B Ellipse centered at (0,0), aligned along x, y axes

4 Parametric form
\[
x = a \cos \theta \\
y = b \sin \theta
\]

5 Vector form
\[
\vec{r} = a \cos \theta \hat{i} + b \sin \theta \hat{j}
\]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

III Equations for an ellipse

C Ellipse centered at (0,0), arbitrary orientation

1. General form
   \[ Ax^2 + (B+C)xy + Dx^2 + F = 0 \]
   provided \( 4AD > (B+C)^2 \)

2. Matrix form
   \[
   \begin{bmatrix}
   x & y \\
   \end{bmatrix}
   \begin{bmatrix}
   A & B \\
   C & D \\
   \end{bmatrix}
   \begin{bmatrix}
   x \\
   y \\
   \end{bmatrix}
   =
   \begin{bmatrix}
   Ax + By \\
   Cx + Dy \\
   \end{bmatrix}
   =
   -F
   \]
   A, B, C, D, and F are constants here, not matrices

D Position vector for an ellipse

\[
\mathbf{r} = a \cos \theta \mathbf{i} + b \sin \theta \mathbf{j}
\]

E Derivative of position vector for an ellipse (\(d\mathbf{r}/d\theta\))

\[
\frac{d\mathbf{r}}{d\theta} = -a \sin \theta \mathbf{i} + b \cos \theta \mathbf{j}
\]

F Dot product of \(\mathbf{r}\) and \(d\mathbf{r}/d\theta\)

\[
\mathbf{r} \cdot \frac{d\mathbf{r}}{d\theta} = (b^2 - a^2) \sin \theta \cos \theta
\]

G The position vector and its tangent are perpendicular if and only if

1. \(a=b\), or
2. \(\theta = 0^\circ\), or
3. \(\theta = 90^\circ\)

We will use these results shortly

So the axes of an ellipse/ellipsoid are perpendicular, and the tangents to an ellipse/ellipsoid at the ends of the axes are perpendicular. Those tangents parallel the axes.

Along axes of ellipse

\[
r \cdot dr/d\theta = -a^2 \sin \theta \cos \theta \]

\[
r \cdot dr/d\theta = (b^2 - a^2) \sin \theta \cos \theta
\]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

I. Equation of homogeneous deformation

A. \[ X' = [F][X] \]

B. 2D

\[
\begin{bmatrix}
\frac{dx'}{dx} & \frac{dx'}{dy} \\
\frac{dy'}{dx} & \frac{dy'}{dy}
\end{bmatrix}
\begin{bmatrix}
dx \\
dy
\end{bmatrix}
= 
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
= 
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
F_{x1} & F_{x2} \\
F_{y1} & F_{y2}
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

C. 3D

\[
\begin{bmatrix}
\frac{dx'}{dx} & \frac{dx'}{dy} & \frac{dx'}{dz} \\
\frac{dy'}{dx} & \frac{dy'}{dy} & \frac{dy'}{dz} \\
\frac{dz'}{dx} & \frac{dz'}{dy} & \frac{dz'}{dz}
\end{bmatrix}
\begin{bmatrix}
dx \\
dy \\
dz
\end{bmatrix}
= 
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix}
= 
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

For homogeneous strain, the derivatives are uniform (constants), and \( dx, dy \) can be small or large.

D. Critical matter: Understanding the geometry of the deformation

E. In homogeneous deformation, a unit circle transforms to an ellipse (and a sphere to an ellipsoid)

F. Proof

\[
[X]' = [F][X]
\]

Now solve for \([X]\)

\[
[F]^{-1}[X]' = [F]^{-1}[F][X] = [I][X] = [X]
\]

Now solve for \([X]'\)

\[
[X]' = [F]^{-1}[X]' = [F]^{-1}[[F]^{-1}]'[X]' = [X]'[F][F]^{-1}
\]

Now substitute for \([X]'\) and \([X]\) in first equation

\[
[X]' = [X]'[F][F]^{-1}[F][F]^{-1}[X]' = 1
\]

Equation of ellipse
See slide 11
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IV Equation of homogeneous deformation \([X'] = [F][X]\)

G \([F]\) transforms a unit circle to a "strain ellipse"

H "Strain ellipse" geometrically represents \([F][X]\)

I \([F]^{-1}\) transforms a "strain ellipse" back to a unit circle

J \([F]^{-1}\) transforms a unit circle to a "reciprocal strain ellipse"

K \([F]^{-1}\) transforms a "reciprocal strain ellipse" back to a unit circle

L "Reciprocal strain" ellipse geometrically represents \([F]^{-1}[X]\)

\[
[F] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad [F]^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues

A The eigenvalue matrix equation \([A][X] = \lambda[X]\)

1 \([A]\) is a (known) square matrix \((nxn)\)

2 \([X]\) is a non-zero directional eigenvalue \((nx1)\)

3 \(\lambda\) is a number, an eigenvalue

4 \(\lambda[X]\) is a vector \((nx1)\) parallel to \([X]\)

5 \([A][X]\) is a vector \((nx1)\) parallel to \([X]\)
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

A The eigenvalue matrix equation \([A][X] = \lambda[X]\) (cont.)
6 The vectors \([A][X]\), \(\lambda[X]\), and \([X]\) share the same direction if \([X]\) is an eigenvector
7 If \([X]\) is a unit vector, \(\lambda\) is the length of \([A][X]\)
8 Eigenvectors \([X_i]\) have corresponding eigenvalues \([\lambda_i]\), and vice-versa
9 In Matlab, \([vec, val] = eig(A)\), finds eigenvectors (vec) and eigenvalues (val)

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)
B Examples
1 Identity matrix \([I]\) \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \lambda
\begin{bmatrix}
x \\
y
\end{bmatrix}
= I
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

All vectors in the xy-plane maintain their orientation when operated on by the identity matrix, so all vectors are eigenvectors, and all vectors maintain their length, so all eigenvalues of \([I]\) equal 1. The eigenvectors are not uniquely determined but could be chosen to be perpendicular.
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

B Examples (cont.)

2 A matrix for rotations in the xy plane

\[
\begin{bmatrix}
\cos \omega & \sin \omega \\
-\sin \omega & \cos \omega
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \lambda
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

All non-zero real vectors rotate; a 2D rotation matrix has no real eigenvectors and hence no real eigenvalues

3 A 3D rotation matrix

a The only unit vector that is not rotated is along the axis of rotation

b The real eigenvector of a 3D rotation matrix gives the orientation of the axis of rotation

c A rotation does not change the length of vectors, so the real eigenvalue equals 1
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

B Examples (cont.)

4 \( A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \)

\[
A \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = 2 \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}
\]

\[
A \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} = -2 \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}
\]

5 \( A = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \)

\[
A \begin{bmatrix} -3\sqrt{0.1} \\ -\sqrt{0.1} \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -3\sqrt{0.1} \\ -\sqrt{0.1} \end{bmatrix} = \begin{bmatrix} -30\sqrt{0.1} \\ -10\sqrt{0.1} \end{bmatrix} = 10 \begin{bmatrix} -3\sqrt{0.1} \\ -\sqrt{0.1} \end{bmatrix}
\]

\[
A \begin{bmatrix} \sqrt{0.1} \\ -3\sqrt{0.1} \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{0.1} \\ -3\sqrt{0.1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} \sqrt{0.1} \\ -3\sqrt{0.1} \end{bmatrix}
\]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

E Geometric meanings of the real matrix equation $[A][X] = [B] = 0$

1. $|A| \neq 0$;
   a. $[A]^{-1}$ exists
   b. Describes two lines (or 3 planes) that intersect at the origin
   c. $X$ has a unique solution

2. $|A| = 0$;
   a. $[A]^{-1}$ does not exist
   b. Describes two co-linear lines that pass through the origin (or three planes that intersect in a line or a plane through the origin)
   c. $[X]$ has no unique solution

F Alternative form of an eigenvalue equation

1. $[A][X] = \lambda[X]$

   Subtracting $\lambda[I][X] = \lambda[X]$ from both sides yields:

2. $[A-I\lambda][X] = 0$ (same form as $[A][X] = 0$)

G Solution conditions and connections with determinants

1. Unique trivial solution of $[X] = 0$ if and only if $|A-I\lambda| \neq 0$

2. Eigenvector solutions ($[X] \neq 0$) if and only if $|A-I\lambda| = 0$

* See previous slide
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

H Characteristic equation: \(|A-I\lambda|=0\)

1 The roots of the characteristic equation are the eigenvalues

\[
\begin{align*}
\lambda_1 + \lambda_2 &= \text{tr}(A) \\
\lambda_1 \lambda_2 &= |A|
\end{align*}
\]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

V Eigenvectors and eigenvalues (cont.)

I To solve for eigenvectors, substitute eigenvalues back into $AX = \lambda X$ and solve for $X$

J See notes of lecture 19 for details of analytic solution for eigenvectors of 2D matrices

K Matlab solution: $[vec, val] = \text{eig}(M)$

1 $M =$ matrix to solve for

2 $vec =$ matrix of unit eigenvectors (in columns)

3 $val =$ matrix of eigenvalues (in columns)

L Example: $>> [vec, val] = \text{eig}([2 2; 2 2])$

$$vec =$$

$$\begin{pmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{pmatrix}$$

$$val =$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Solutions for symmetric matrices

A Eigenvalues of a symmetric 2x2 matrix

1. \[ \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - b^2)}}{2} \]
2. \[ \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a + 2ad + d)^2 - 4ad + 4b^2}}{2} \]
3. \[ \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a - 2ad + d)^2 + 4b^2}}{2} \]
4. \[ \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a - d)^2 + 4b^2}}{2} \]

Replace “c” by “b” in eqns. Of slide 26
Radical term cannot be negative. Eigenvalues are real.

B Any distinct eigenvectors \((X_1, X_2)\) of a symmetric nxn matrix are perpendicular \((X_1 \cdot X_2 = 0)\)

1a. \[ AX_1 = \lambda_1 X_1 \]
1b. \[ AX_2 = \lambda_2 X_2 \]
AX_1 parallels X_1, AX_2 parallels X_2 (property of eigenvectors)

Dotting AX_1 by X_2 and AX_2 by X_1 can test whether X_1 and X_2 are orthogonal.

2a. \[ X_2 \cdot AX_1 = X_2 \cdot \lambda_1 X_1 = \lambda_1 (X_2 \cdot X_1) \]
2b. \[ X_1 \cdot AX_2 = X_1 \cdot \lambda_2 X_2 = \lambda_2 (X_1 \cdot X_2) \]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

B’ Distinct eigenvectors \((\mathbf{X}_1, \mathbf{X}_2)\) of a symmetric 2x2 matrix are perpendicular \((\mathbf{X}_1 \cdot \mathbf{X}_2 = 0)\) (cont.)

The material below shows \(\mathbf{X}_1 \cdot \mathbf{AX}_2 = \mathbf{X}_2 \cdot \mathbf{AX}_1\) for the 2D case:

\[
\begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}
\cdot
\begin{bmatrix}
  a & b \\
  b & d
\end{bmatrix}
\begin{bmatrix}
  x_2 \\
  y_2
\end{bmatrix}
= 
\begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}
\cdot
\begin{bmatrix}
  ax_2 + by_2 \\
  bx_2 + dy_2
\end{bmatrix}
= 
ax_1 x_2 + bx_1 y_2 + by_1 x_2 + dy_1 y_2
\]

\[
\begin{bmatrix}
  x_2 \\
  y_2
\end{bmatrix}
\cdot
\begin{bmatrix}
  a & b \\
  b & d
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}
= 
\begin{bmatrix}
  ax_1 + by_1 \\
  bx_1 + dy_1
\end{bmatrix}
= 
ax_1 x_2 + by_1 y_2 + bx_2 y_2 + dy_2 y_2
\]

The sums on the right sides are scalars, but the ordering of the terms in the sums look like the elements of transposed matrices

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

B” Distinct eigenvectors \((\mathbf{X}_1, \mathbf{X}_2)\) of a symmetric 3x3 matrix are perpendicular \((\mathbf{X}_1 \cdot \mathbf{X}_2 = 0)\) (cont.)

The material below shows \(\mathbf{X}_1 \cdot \mathbf{AX}_2 = \mathbf{X}_2 \cdot \mathbf{AX}_1\) for the 3D case:

\[
\begin{bmatrix}
  x_1 \\
  y_1 \\
  z_1
\end{bmatrix}
\cdot
\begin{bmatrix}
  a & b & c \\
  b & d & e \\
  c & e & f
\end{bmatrix}
\begin{bmatrix}
  x_2 \\
  y_2 \\
  z_2
\end{bmatrix}
= 
\begin{bmatrix}
  x_1 \\
  y_1 \\
  z_1
\end{bmatrix}
\cdot
\begin{bmatrix}
  ax_2 + by_2 + cz_2 \\
  bx_2 + dy_2 + ez_2 \\
  cx_2 + ey_2 + fz_2
\end{bmatrix}
= 
ax_1 x_2 + by_1 y_2 + cz_1 z_2 + bx_1 y_2 + dy_1 z_2 + ez_1 z_2 + cx_1 z_2 + ey_1 y_2 + fz_1 z_2
\]

\[
\begin{bmatrix}
  x_2 \\
  y_2 \\
  z_2
\end{bmatrix}
\cdot
\begin{bmatrix}
  a & b & c \\
  b & d & e \\
  c & e & f
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  y_1 \\
  z_1
\end{bmatrix}
= 
\begin{bmatrix}
  x_2 \\
  y_2 \\
  z_2
\end{bmatrix}
\cdot
\begin{bmatrix}
  ax_1 + by_1 + cz_1 \\
  bx_1 + dy_1 + ez_1 \\
  cx_1 + ey_1 + fz_1
\end{bmatrix}
= 
ax_1 x_2 + by_1 y_2 + cz_1 z_2 + bx_1 y_2 + dy_1 z_2 + ez_1 z_2 + cx_1 z_2 + ey_1 y_2 + fz_1 z_2
\]

Again, the sums on the right sides are scalars, but the ordering of the terms in the sums look like the elements of transposed matrices
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

B”” Distinct eigenvectors \((X_1, X_2)\) of a symmetric \(n \times n\) matrix are perpendicular \((X_1 \cdot X_2 = 0)\) (cont.)

The 2D and 3D results suggest matrix transposes could test whether \(X_1 \cdot A X_2 = X_2 \cdot A X_1\) in general

\[
X_1 \cdot A X_2 = [X_1]^T [A] [X_2] \\
X_2 \cdot A X_1 = [X_2]^T [A] [X_1] = [X_2]^T [A] [X_2] \\
= [X_1]^T [A] [X_2] \\
= [X_2]^T [A] [X_1] \\
= [X_1]^T [A] [X_2] \\
= [X_2]^T [A] [X_1]
\]

Are these equal?  The transpose of a scalar is the same scalar

This step and the next invoke \([BC]^T = [C]^T [B]^T\)

\[
= [X_1]^T [A] [X_2] \\
= [X_2]^T [A] [X_2] \\
= [X_1]^T [A] [X_2] \\
= [X_2]^T [A] [X_1]
\]

\[\text{If } [A] \text{ is symmetric, } [A]^T = [A] \quad \text{Yes!}\]

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

B  Distinct eigenvectors \((X_1, X_2)\) of a symmetric \(n \times n\) matrix are perpendicular (cont.)

Since the left sides of (2a) and (2b) are equal, the right sides must be equal too. Hence,

\[
4 \quad \lambda_1 (X_2 \cdot X_1) = \lambda_2 (X_1 \cdot X_2)
\]

Now subtract the right side of (4) from the left

\[
5 \quad (\lambda_1 - \lambda_2) (X_2 \cdot X_1) = 0
\]

• The eigenvalues generally are different, so \(\lambda_1 - \lambda_2 \neq 0\).
• This means for (5) to hold that \(X_2 \cdot X_1 = 0\).
• The eigenvectors \((X_1, X_2)\) of a symmetric \(n \times n\) matrix are perpendicular (or can be chosen to be perpendicular)
• We can pick reference frames with orthogonal axes to simplify problems and gain insight into their solutions
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Solutions for symmetric matrices (cont.)

C Maximum and minimum squared lengths

Set derivative of squared lengths to zero

\[ X' \cdot X' = (AX) \cdot (AX) = L_j^2 \]

\[ \frac{d(X' \cdot X')}{d \theta} = X' \cdot \frac{dX'}{d \theta} + \frac{dX'}{d \theta} \cdot X' = 0 \]

\[ 2 \begin{pmatrix} X' \cdot \frac{dX'}{d \theta} \\ \frac{dX'}{d \theta} \cdot X' \end{pmatrix} = 0 \]

D Position vectors (X') with maximum and minimum (squared) lengths are those that are perpendicular to tangent vectors (dX') along ellipse

E AX=λX

F Since eigenvectors of symmetric matrices are mutually perpendicular, so too are the parallel transformed vectors λX

G At the point identified by the transformed vector λX, the other eigenvector(s) is (are) perpendicular and hence must parallel dX' and be tangent to the ellipse
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VI Solutions for symmetric matrices (cont.)

H Recall that position vectors \( \mathbf{X}' \) with maximum and minimum (squared) lengths are those that are perpendicular to tangent vectors \( (d\mathbf{X}') \) along ellipse. Hence, the smallest and largest transformed vectors \( \lambda \mathbf{X} \) for a symmetric matrix give the minimum and maximum distances to an ellipse from its center and the directions of the ellipse axes.

I The \( \lambda \) values are the principal stretches associated with a symmetric \([F]\) matrix.

J These conclusions extend to three dimensions and ellipsoids.

9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VII Solutions for general homogeneous deformation matrices

A Eigenvalues

1 Start with the definition of quadratic elongation \( Q \), which is a scalar.

\[
\frac{L_f^2}{L_0^2} = Q
\]

2 Express using dot products.

\[
\frac{\mathbf{X}' \cdot \mathbf{X}'}{\mathbf{X} \cdot \mathbf{X}} \equiv Q
\]

3 Clear the denominator. Dot products and \( Q \) are scalars.

\[
\mathbf{X}' \cdot \mathbf{X}' = (\mathbf{X} \cdot \mathbf{X}) Q
\]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VII Solutions for general homogeneous deformation matrices

A. Eigenvalues

4. Replace $X'$ with $[FX]$

5. Re-arrange both sides

6. Both sides of this equation lead off with $[X]^T$, which cannot be a zero vector, so it can be dropped from both sides to yield an eigenvector equation

7. $[F^TF]$ is symmetric: $[F^TF]=[F^TF]^T$

8. The eigenvalues of $[F^TF]$ are the principal quadratic elongations $Q = (L_i/L_0)^2$

9. The eigenvalues of $[F^TF]^{1/2}$ are the principal stretches $S = (L_i/L_0)$

B. Special Case: $[F]$ is symmetric

1. $[F^TF] = [F^2]$ because $F = F^T$

2. The principal stretches ($S$) again are the square roots of the principal quadratic elongations ($Q$) (i.e., the square roots of the eigenvalues of $[F^2]$)

3. The principal stretches ($S$) also are the eigenvalues of $[F]$, directly

4. The directions of the principal stretches ($S$) are the eigenvectors of $[F]$, and of $[F^TF] = [F^2]$!

5. The axes of the principal (greatest and least) strain do not rotate

$\mathbf{Q} = \text{Rot}(\mathbf{S}, \mathbf{Q})$

$\mathbf{S} = \sqrt{\mathbf{Q}}$

$\mathbf{F} = \text{Sym}(\mathbf{Q})$

$\mathbf{F} = \text{Sym}(\mathbf{Q})$

$\mathbf{F} = \text{Sym}(\mathbf{Q})$

$\mathbf{F} = \text{Sym}(\mathbf{Q})$
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

A Just getting the size and shape of the “strain” (stretch) ellipse is not enough. Need to consider points on the ellipse

B F=VR (which “R”?)
   1 R = rotation matrix
   2 V = stretch matrix

C F=RU (which “U”? “R”?)
   1 U = stretch matrix
   2 R = rotation matrix

D The choices narrow if the stretch matrices are symmetric

---

E If an ellipse is transformed to a unit circle, the axes of the ellipse are transformed too.

F In the diagram, the axes of the ellipses do not maintain their orientation when the ellipse is transformed back to a unit circle

G If F is not symmetric, the axes of the red ellipse and the retro-deformed (black) axes will have a different absolute orientation

H The transformation from the retro-deformed (black) axes to the orientation of the principal axes gives the rotation of the axes
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

I We know how to find the principal stretch magnitudes: they are the square roots of the eigenvalues of the symmetric matrix $[F^T][F]$.

J The eigenvectors of $[F^T][F]$ give some of the information needed to find the direction of the principal stretch axes. The rotation describes the orientation difference between the principal strain (stretch) axes and their retro-deformed counterparts.

K To find the rotation of the principal axes, start with the parametric equation for an ellipse and its tangent, and the requirement that the position vectors for the semi-axes of the ellipse are perpendicular to the tangent.

\[
\begin{align*}
\mathbf{\hat{x}}' &= (a\cos\theta + b\sin\theta)\mathbf{i} + (c\cos\theta + d\sin\theta)\mathbf{j} \\
\frac{d\mathbf{\hat{x}}'}{d\theta} &= (-a\sin\theta + b\cos\theta)\mathbf{i} + (-c\sin\theta + d\cos\theta)\mathbf{j} \\
\mathbf{\hat{x}}' \cdot \frac{d\mathbf{\hat{x}}'}{d\theta} &= 0
\end{align*}
\]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogenous deformation

Now solve for \( \theta \)

\[
X' = (\cos \theta + \sin \theta) i + (\cos \theta + \sin \theta) j
\]

\[
\frac{dX'}{d\theta} = (-\sin \theta + \cos \theta) i + (-\sin \theta + \cos \theta) j
\]

\[
X' \cdot \frac{dX'}{d\theta} = 0
\]

\[
= -a^2 \sin \theta - ab \cos \theta + ab \sin \theta + b^2 \sin \theta + c^2 \cos \theta
\]

\[
- c^2 \sin \theta - ab \cos \theta + c^2 \cos \theta + d^2 \sin \theta - \sin \theta
\]

\[
= -(a^2 - b^2 + c^2 - d^2) \sin \theta + (ab + cd) \cos \theta - (ab + cd) \sin \theta
\]

\[
= -(a^2 - b^2 + c^2 - d^2) \sin \theta + (ab + cd) \cos \theta - (ab + cd) \sin \theta
\]

\[
= -(a^2 - b^2 + c^2 - d^2) \sin 2\theta + (ab + cd) \cos 2\theta
\]

\[
= \frac{(a^2 - b^2 + c^2 - d^2)}{2} \sin (-2\theta) + (ab + cd) \cos (-2\theta) = 0
\]

Continuing....

\[
\frac{(a^2 - b^2 + c^2 - d^2)}{2} \sin (-2\theta) + (ab + cd) \cos (-2\theta) = 0
\]

\[
\tan(-2\theta) = \frac{-2(ab + cd)}{a^2 - b^2 - c^2 - d^2}
\]

\[
\theta_1 = \frac{1}{2} \tan^{-1} \left( \frac{2(ab + cd)}{a^2 - b^2 - c^2 - d^2} \right), \theta_2 = \frac{1}{2} \tan^{-1} \left( \frac{2(ab + cd)}{a^2 - b^2 - c^2 - d^2} \right) + 90^\circ
\]

So \( \theta_1 \) and \( \theta_2 \) are 90° apart

Recall that two angles that differ by 180° have the same tangent
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Example 1

\[ F = \begin{bmatrix} 2 & 2 \\ 0.5 & 1 \end{bmatrix} \]

\[ [F] = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{bmatrix} \]

First, symetrically stretch the unit circle using \([U]\)

Eigenvalues of \([U]\) give principal stretch magnitudes

\[ X' = [F]X \]

\[ [F] = \begin{bmatrix} 2 & 2 \\ 0.5 & 1 \end{bmatrix} \]

\[ [X'] = [F'][X] \]

\[ [F'] = \begin{bmatrix} 2.5 & 2.5 \\ 4.5 & 5 \end{bmatrix} \]

\[ [U'] = [F']^{-1} \]

\[ [R] = [F'][U'] \]

\[ [R] = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \]

\[ \lambda_1 = 1.56 \]

\[ \lambda_2 = 1.34 \]

\[ \mu_1 = 1.34 \]

\[ \mu_2 = 1.79 \]

Eigenvectors of \([U]\) are along axes of blue ellipses. Rotated eigenvectors of \([U]\) give principal stretch directions

Second, rotate the ellipse (not the reference frame) using \([R]\)
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

Example 2

First, rotate the unit circle using $[R]$

Second, stretch the rotated unit circle symmetrically using $[V]$

Eigenvectors of $[V]$ also give principal stretch magnitudes

Unrotated eigenvectors of $[V]$ give principal stretch directions directly

VIII Rotations in homogeneous deformation

- Decomposition of $F = VR$ by method of Ramsay and Huber (for 2D). Consider the effect of an irrotational (symmetric) strain $[V]$ that follows a pure rotation $[R]$ of an object (not a rigid rotation of the reference frame)

$$ F = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{bmatrix} = VR $$
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

- Key fact about rotation matrices: 
  \[ [R]^{-1} = [R]^T \]

\[
R(\omega) = \begin{bmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{bmatrix}
\]

\[
R^{-1} = R(-\omega) = \begin{bmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{bmatrix}
\]

\[
R^T = \begin{bmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{bmatrix}
\]

3D treatment: rotating a reference frame does not change the length of a vector, so \( \mathbf{x} \cdot \mathbf{x} = \mathbf{x}' \cdot \mathbf{x}' \). This also leads to \( [R]^{-1} = [R]^T \):

\[
\mathbf{x}' = [R][\mathbf{x}]
\]

\[
\mathbf{x} \cdot \mathbf{x} = \mathbf{x}' \cdot \mathbf{x}'
\]

\[
\mathbf{x} = [R^T][R][\mathbf{x}]
\]

\[
[R^T][R] = [I], \text{ but }
\]

\[
[R]^{-1}[R] = [I]
\]

\[
\therefore [R]^T = [R]^{-1}
\]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

VIII Rotations in homogeneous deformation

1 \[ F = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{bmatrix} = VR \]

2 \[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} A \cos \omega + B \sin \omega & -A \sin \omega + B \cos \omega \\ B \cos \omega + D \sin \omega & -B \sin \omega + D \cos \omega \end{bmatrix} \]

By inspection, \( c-b = (A+D)\sin \omega \) and \( a+d = (A+D)\cos \omega \)

3 \[ \frac{c-b}{a+d} = \tan \omega \]

If \( c=b \), then \( F \) is symmetric and \( \omega = 0! \)

From 3 one can obtain \( \omega \) and hence \( R \).

Post-multiplying both sides of (1) by \([R]^{-1} = R^T\) yields \( V \), the symmetric “part” of \( F \).

\[ F = VR \Rightarrow F[R]^{-1} = VR[R]^{-1} = VR[R]^T = V \]
9. EIGENVECTORS, EIGENVALUES, AND FINITE STRAIN

IX Closing comments
1 Our solutions so far depend on knowing the displacement field.
2 With satellite imaging we can get an approximate value for the displacement field at the surface of the Earth for current deformations.
3 Evaluating strains for past deformations require certain assumptions about initial sizes and shapes of bodies, the original locations of point, and/or the displacement field.
4 Alternative approach: formulation and solution of boundary value problems to solve for the displacement and strain fields.
5 The deformation gradient matrix $F$ has strain and rotation intertwined; the two can be separated using matrix multiplication. In the infinitesimal strain matrix $\varepsilon$, the rotation is already separated.
6 References