4. INTERSECTIONS OF PLANES

I Main Topics
A Equation of a plane
B Pole to a plane using cross-products
C Intersection of two planes in a line (apparent dip problem)
D Intersection of three planes in a point (solution of simultaneous linear equations)

II Equation for plane: \( \mathbf{n} \cdot \mathbf{v} = d \)
A Meaning
1 The distance from a known reference point to a plane, as measured in the direction of a unit vector \( \mathbf{n} \) normal to the plane, is \( d \).
2 \( d > 0 \) if \( \mathbf{n} \) points from the reference point towards the plane;
3 \( d < 0 \) if \( \mathbf{n} \) points from the plane towards the reference point.
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II Equation for plane: \( \mathbf{n} \cdot \mathbf{v} = d \) (cont.)

B Solution for \( \mathbf{n} = \alpha \mathbf{i} + \beta \mathbf{j} + \gamma \mathbf{k} = < \alpha, \beta, \gamma > \)

1. If \( x = \text{north}, y = \text{east}, z = \text{down} \)
   \( \alpha = \cos \phi \cos \theta, \beta = \cos \phi \sin \theta, y = \sin \phi \)

2. If \( x = \text{east}, y = \text{north}, z = \text{up} \)
   \( \alpha = \cos \phi \sin \theta, \beta = \cos \phi \cos \theta, y = -\sin \phi \)
   where \( \theta \) is the pole trend and \( \phi \) is the pole plunge

3. Matlab: \( \{\alpha, \beta, \gamma\} = \text{sph2cart}(\text{trend}, \text{plunge}, 1) \)
   Trend and plunge must be in radians.

4. Matlab: \( \{\text{trend}, \text{plunge}, R\} = \text{cart2sph}(\alpha, \beta, \gamma) \)

5. **Good for \( z = \text{down} \);** trend and plunge must be in radians.

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II Equation for plane: \( \mathbf{n} \cdot \mathbf{v} = d \) (cont.)

C \( \mathbf{v} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = < x, y, z > \)

D Solution for \( d \)

1. \( d = \alpha x + \beta y + \gamma z \)
   (Normal form)

2. \( d = \text{dot}(n, v) \)
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III Pole to a plane using cross-products
A Consider three non-collinear points A, B, C that form a clockwise circuit as viewed from above
B \((B-A) \times (C-A)\) is normal to the plane
C \(n = \frac{(B-A) \times (C-A)}{|(B-A) \times (C-A)|}\), unit normal to the plane.

IV Intersection of two planes in a line
A Two planes A and B intersect in a line. The equations of those two planes define the line.
B The direction of intersection is along the vector \(\mathbf{a} \times \mathbf{b}\), where \(\mathbf{a}\) is normal to plane A, and \(\mathbf{b}\) is normal to plane B.
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IV Intersection of two planes in a line (cont.)

C Apparent dip

1) How a geologic plane appears to dip in a vertical cross section of arbitrary strike
2) Plunge of the line of intersection between a geologic plane and a vertical cross section plane of arbitrary strike
3) Plunge of the cross product of two vectors
   a) Vector normal to a geologic plane
   b) Vector normal to vertical cross section plane of arbitrary strike
   c) Make sure cross product points down to get a pole

Apparent Dip

1) \( \frac{a}{b} = \tan(\text{true dip}) \).
2) \( a = b \tan(\text{true dip}) = c \tan(\text{app. dip}) \). So
3) \( b \tan(\text{true dip}) = c \tan(\text{app. dip}) \)

Now consider the angle \( \psi \) between the line of strike of the cross section and the line of strike for the dipping plane.

4) \( \frac{b}{c} = \sin\psi \), so
5) \( c = \frac{b}{\sin\psi} \).

Now insert (5) into the right side of (3)

6) \( b \tan(\text{true dip}) = (b/\sin\psi) \tan(\text{app. dip}) \)
7) \( \text{app. dip} = \tan^{-1}\{[\tan(\text{true dip})][\sin\psi]\} \)

Since \( \sin\psi \leq 1 \), app. dip \( \leq \) true dip.
If \( \psi = 90^\circ \), then app. dip = true dip.
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IV Intersection of two planes in a line (cont.)

D Apparent dip and orthographic projection example

Find the apparent dip of plane ABC in a vertical cross section that strikes N45W.

C Apparent dip and orthographic projection example

(1) In the top view, intersect the cross section plane with plane ABC. Call the line of intersection DE. Point D is on line AC. Point E is on line BC.

Next we find the plunge of line DE.

(2) Project the line of intersection DE onto an auxiliary view that is taken parallel to the line of cross section. The plunge of line DE is the apparent dip of plane ABC as seen in the cross section.
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V Intersection of features in a point

A Intersection of two lines (L₁ and L₂) at a point

1 Intersection criteria
   a Vectors along the lines are not parallel (i.e., \( \vec{L}_1 \times \vec{L}_2 \neq 0 \))
   b Vectors along the lines (and normals to the lines) lie in the same plane:
      \( (\vec{n}_1 \times \vec{L}_1) \times (\vec{n}_2 \times \vec{L}_2) = 0 \)

A Intersection of two lines (cont.)

2 Write equations for two lines (n•v=d)
   \[ \vec{n}_1 \cdot \vec{v}_1 = a_{n_1}x + a_{n_2}y = d_i \]
   \[ \vec{n}_2 \cdot \vec{v}_2 = a_{n_2}x + a_{n_3}y = d_i \]

3 Find \( n_1, n_2, d_1, d_2 \)

Example
   \( 0)(-1) + (1)(2) = 2 = d_i \)
   \( 1)(1.5) + (0)(-1) = 1.5 = d_2 \)
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A Intersection of two lines (cont.)

3 Write equations for lines in matrix form

\[
\begin{bmatrix}
  a_{n,x} & a_{n,y} \\
  a_{n,x} & a_{n,y}
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= 
\begin{bmatrix}
  d_1 \\
  d_2
\end{bmatrix}
\]

\[
[A] [X] = [B]
\]

We seek the vector \([X]\) that satisfies both equations (i.e., a point \((x,y)\) on both lines).

Example

\[
\begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= 
\begin{bmatrix}
  2 \\
  1.5
\end{bmatrix}
\]

\[
[A] [X] = [B]
\]

4 Solve for \([X]\)

\[
[A][X]=[B]
\]

\[
[A]^{-1}[A][X]=[A]^{-1}[B]
\]

\[
[X]=[A]^{-1}[B]
\]

a Matlab

\[
>> A = [0 1; 1 0] \quad >> X = A \backslash B
\]

\[
A =
\begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix}
\]

\[
B = [2; 1.5]
\]

\[
>> X = \text{inv}(A) * B
\]

\[
X =
\begin{bmatrix}
  1.5000 \\
  2.0000
\end{bmatrix}
\]

b Cramer’s Rule

\[
x = \frac{B_1 A_2 - B_2 A_1}{A_1 A_2 - A_1 A_2}
\]

\[
\begin{bmatrix}
  1.5 & 1 \\
  2 & 0
\end{bmatrix}
\]

\[
y = \frac{A_1 B_2 - A_2 B_1}{A_1 A_2 - A_1 A_2}
\]

\[
\begin{bmatrix}
  A_1 & B_1 \\
  A_2 & B_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1.5 \\
  2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  A_1 & B_1 \\
  A_2 & B_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  A_1 & A_2 \\
  A_3 & A_2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  -1 & -1.5
\end{bmatrix}
\]

\[
\begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  -22 = 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
  -1.5 = 1.5
\end{bmatrix}
\]
4. INTERSECTIONS OF PLANES

B Intersection of three plane \( (P_1, P_2, \text{and } P_3) \) at a point

1 Intersection criteria
   Normals of any planes are not parallel; scalar triple product of normals to planes \( \neq 0 \)
   (i.e., \( n_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0 \))

2 Write equations for three planes
   \( (n \cdot v = d) \)

   \[\begin{align*}
   \vec{n}_1 \cdot \vec{v}_1 &= a_{n_1x}x + a_{n_1y}y + a_{n_1z}z = d_1 \\
   \vec{n}_2 \cdot \vec{v}_2 &= a_{n_2x}x + a_{n_2y}y + a_{n_2z}z = d_2 \\
   \vec{n}_3 \cdot \vec{v}_3 &= a_{n_3x}x + a_{n_3y}y + a_{n_3z}z = d_3
   \end{align*}\]

3 Find \( n_1, n_2, n_3, d_1, d_2, d_3 \)

   Example
   \[\begin{align*}
   n_1 &= 1\hat{i} + 0\hat{j} + 0\hat{k} \\
   n_2 &= 0\hat{i} + 1\hat{j} + 0\hat{k} \\
   n_3 &= 0\hat{i} + 0\hat{j} + 1\hat{k}
   \end{align*}\]

Example:
\[\begin{align*}
(1)(3) + (0)(1) + (0)(0) &= 3 = d_1 \\
(0)(x) + (1)(2) + (0)(0) &= 2 = d_2 \\
(0)(x) + (0)(0) + (1)(1) &= 1 = d_3
\]
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B Intersection of three planes (cont.)

3 Write equations for lines in matrix form

\[
\begin{bmatrix}
    a_{nx} & a_{ny} & a_{nz} \\
    a_{nx} & a_{ny} & a_{nz} \\
    a_{nx} & a_{ny} & a_{nz}
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
=
\begin{bmatrix}
    d_1 \\
    d_2 \\
    d_3
\end{bmatrix}
\]

We seek the vector \([X]\) that satisfies each equation (i.e., one point \((x,y,z)\) on all three planes)

\[
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}
=
\begin{bmatrix}
    3 \\
    2 \\
    1
\end{bmatrix}
\]

Example

\[
\begin{bmatrix}
    A \\
    B
\end{bmatrix}
\begin{bmatrix}
    X
\end{bmatrix}
=
\begin{bmatrix}
    B
\end{bmatrix}
\]

4 Solve for \([X]\)

\[
[A][X]=[B]
\]

\[
[A]^{-1}[A][X]=[A]^{-1}[B]
\]

\[
[X]=[A]^{-1}[B]
\]

a Matlab

```matlab
>> A = [0 1;1 0] >> X = A\B
A =
    1.0000    0.0000
    0.0000    1.0000
>> B = [3;2;1] >> X = inv(A)*B
B =
    3.0000
    2.0000
    1.0000
>> X =
    3.0000
    2.0000
    1.0000
```

b Cramer’s Rule

Could use Cramer’s rule, but it is cumbersome.
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C Intersection of a line and a plane
1 A line (L₁) and a plane (P₁)
   intersect at a point
2 Point of intersection can be
   viewed as the intersection of 3
   planes
   a Plane P₁ (white)
   b Plane P₂ (blue)
      P₂ intersects plane P₁ at line
      L₁. Plane P₂ is a vertical plane
      containing L₁, so P₂ strikes
      parallel to the trend of L₁.
   c Plane P₃ (gray)
      P₃ intersects plane P₁ at line
      L₁. Plane P₃ is an inclined
      plane that contains L₁. The
      dip of P₃ equals the plunge of
      L₁, and the strike of P₃ is 90°
      from the trend of L₁.

Two Math Handbook References
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Kästner, H., 1977, The VNR concise
encyclopedia of mathematics: Van Nostrand
Tuma, J.J., 1979, Engineering mathematics