KIRCHHOFF-HELMHOLTZ SYNTHETIC SEISMOGRAMS

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ABSTRACT

Most of the existing methods of computing theoretical seismograms assume that the elastic properties in the earth vary only vertically. However, in many areas such as subduction zones, spreading ridges etc., the structures are two dimensional or even three dimensional. This warrants calculation of synthetic seismograms for media in which elastic properties vary in more than one spatial direction. The non-asymptotic techniques such as finite difference and finite element methods become prohibitively expensive for long range and high frequency calculations even for use on a supercomputer. On the other hand, the asymptotic methods such as Gaussian beams and WKBJ theory do not compute the frequency dependent diffractions originating from corners in any of the interfaces of the medium.

The Kirchhoff-Helmholtz (KH) integral has been used successfully by exploration seismologists for modeling small offset seismic data assuming the medium to be acoustic. This dissertation introduces an extension of the KH integral for the case of an elastic medium. Rays are traced from source and receiver to each point on the interface. One KH integral is evaluated for each generalized ray either in the frequency domain by using a generalized Filon method (GFM) quadrature formula or directly in the time domain. This method works even when the receiver is located at a caustic and computes diffraction arrivals correctly from any corner on the surface of integration. No head wave arrivals are included in the KH synthetics. Since geometrical ray theory (GRT) is used to evaluate the integrand of the KH integral, any caustic(s) on the surface of integration gives rise to singularities in the KH integrand. Methods have been devised to partially overcome this breakdown. The KH synthetics are used here for modeling seismic refraction data recorded by ocean bottom seismometers at the Middle America Trench offshore Guatemala. The analysis shows the presence of a low velocity zone in the subduction complex.

The KH integral can be considered to be a special case of a more general multifold path integral (MFPI). The MFPIs' can be reduced to a single fold KH integral by successive application of stationary-phase methods. MFPI is essentially a multiple Huygen’s construction; it uses geometrical ray theory, generalized plane
wave reflection and transmission coefficients and an elastic KH integral. The MFPI method includes diffractions from any corner present in the medium. Since GRT Green's function is used in the integrand, the method breaks down at the layer pinchouts and the head waves are not included. The GFM quadrature formula is extended here for use in the case of multifold integrals. MFPIs' are evaluated in the time domain for many complex models. Comparison of MFPI synthetic seismograms with those calculated by finite difference method, shows that the method works well.

Acousticians have long been interested in studying the acoustic wave propagation in an oceanic wedge. High frequency ocean acoustics data collected by DREP off the west coast of Vancouver Island are examined here. These data have been studied in the time domain first by ray tracing and then by waveform modeling. Ray tracing by Hubley's algorithm shows different ray paths in different regions of the ocean. The different ray paths explain, to a great extent, the variation of propagation loss with range and frequency. Waveform modeling of the data in the region in which bathymetry is flat was done by a modified reflectivity algorithm. Waveform modeling resulted in an improved geoacoustic model of the ocean bottom indicating the presence of a thin and fairly homogeneous sediment column.
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INTRODUCTION

Synthetic seismograms have now become an important tool for the routine analysis of field reflection and refraction seismograms. There exist numerous ways of computing synthetics based on the model used and the approximations used in solving the elastodynamic wave equation. To the first approximation, the elastic properties in the earth may be assumed to vary in only one spatial direction such as depth. The methods for computing synthetics for such stratified earth models have reached a very advanced state of development. For example the reflectivity method (non-asymptotic method) and the WKBJ method (asymptotic method) are now used for routine analysis of both land and marine reflection and refraction data. Unfortunately, in many regions the earth is not stratified and synthetics for one dimensional earth models can not be used for modeling data collected from these areas. Hence, in the last decade a lot of effort has gone into calculating synthetic seismograms for laterally variable media.

Amongst the non-asymptotic methods both finite difference and finite element methods can handle models of any complexity but these methods become very expensive for long range and high frequency calculations. Thus when one is interested in synthesizing a single seismic phase, an asymptotic method of any form is best suited for the purpose. Geometrical ray theory (GRT) has been the most popular high frequency method. Even though GRT is valid at infinite frequency, it works well even at frequencies much lower than the theory strictly allows. In fact most of what we know today about the structure of the interior of the earth has been derived from GRT results only. However, the GRT breaks down at caustics and many other regions of the ray-field. Other asymptotic methods such as extended WKBJ or Maslov theory and Gaussian beams work at caustics but are unable to compute frequency dependent diffractions from corners in the medium.

Kirchhoff-Helmholtz integrals in their acoustic form have been in use in the exploration industry for quite sometime now. In this dissertation, I introduce new methods based on an elastic form of the Kirchhoff-Helmholtz integral. KH integrals
are found to be useful because such methods work even when the receiver is located at a caustic and correctly compute diffractions originating from any sharp corner at any of the interfaces of the medium.

The first chapter discusses the methods for calculating synthetic seismograms using multifold path integrals (MFPI). In the second chapter, I discuss the analysis of high frequency ocean acoustics data set collected over the west coast of Vancouver island. In appendices A and B, I discuss the theory and computational details of elastic Kirchhoff-Helmholtz method.
CHAPTER I
CALCULATION OF SYNTHETIC SEISMOGRAMS USING MULTIFOLD PATH INTEGRALS

Introduction

Since exact analytical solutions of the elastodynamic wave equation rarely exist for an inhomogeneous medium of geologic interest, the solution must usually be obtained numerically or by a combination of analytical and numerical methods. In the last decade, computation of synthetic seismograms for a laterally variable multilayered elastic medium has been an active area of research. Methods for computing seismograms for such media can be broadly classified into two groups:

(a) Non-asymptotic or more exact methods in which the complete response of the model is computed.

(b) Asymptotic or high frequency methods in which each seismic phase is synthesized separately. Such methods are essentially based on generalized ray approaches.

A third category includes the laterally varying reflectivity method (Frazer and McCoy, 1985) which is based on a hybrid factorized Helmholtz equation - invariant embedding technique. Although the factorization of the Helmholtz equation is exact, a high frequency approximation for the symbol of the square root operator is used for computation. In this formulation the interfaces, i.e., surfaces of material discontinuity, are horizontal while the velocities are allowed to vary continuously within a layer.

Among the non-asymptotic methods the finite difference (Kelly et al., 1976; Stephen, 1983), finite element (Marfurt, 1985) and boundary integral equation (Cole, 1980) methods are the most popular. The finite difference method requires at least 8 to 10 grid points per smallest wavelength to avoid grid dispersion. This makes the method very expensive for high frequency and long range computations. Although the invention of fast computers with vector array processors has considerably reduced the time of computation by such methods, the results are often as difficult to interpret as the real data, since all such methods compute the complete response of the model. Thus, when a geophysical interpreter is interested in identi-
fying the arrival of a particular phase by studying its variation in arrival time and waveform with offset, a generalized ray method is usually best.

Several asymptotic methods have been developed in the recent past for modelling seismic reflection and refraction in complex media of geologic interest. The range of validity of some asymptotic methods has been discussed by Ben-Menahem and Beydoun (1985). Geometrical ray theory (GRT) (e.g., Červený et al., 1977) has been the most popular because of its simplicity and speed of computation. GRT can essentially be applied to smoothly varying media and, although it is an infinite frequency approximation, it seems to work fairly well at much lower frequencies than the theory would allow. The main limitation of the theory is the breaking down of the theory itself in certain regions of the ray field such as caustics, critical regions and shadow zones (e.g., Chapman, 1985). Other asymptotic methods, such as extended WKBJ (Frazer and Phinney, 1982), Maslov theory (Chapman and Drummond, 1982), Gaussian beams (Červený, 1983), Kirchhoff methods (Hilterman, 1970, 1975; Trörey, 1977; Scott and Helberger, 1982; Frazer and Sen, 1985a; Sen and Frazer, 1985a) have attempted to repair the breakdown of GRT in some of these regions. All such methods work when the receiver is located at caustics but only those based on the Kirchhoff approach can model the frequency dependent diffractions originating from any sharp corners present in any of the interfaces of the medium. The other asymptotic methods give incorrect arrival times for these diffractions.

In reflection seismology the Kirchhoff-Helmholtz integral has found extensive use in modelling zero offset to near offset seismic reflection data from a single reflector assuming the propagation medium to be acoustic. However, in many experiments, such as marine reflection/refraction studies, the source-receiver offset is often too large and the data contains both precritical and postcritical reflection arrivals. In such a situation a substantial amount of energy is lost due to conversion from P to S and vice versa at the discontinuities and the propagation medium cannot be assumed to be acoustic anymore. Taking this point into consideration, Frazer and Sen (1985a) extended the KH integral to the elastic case and applied it to compute synthetic seismograms in a laterally variable multilayered elastic medium (Sen and Frazer, 1985a). Frazer (1983) and Frazer and Sen (1985a) have shown that their
single-fold elastic KH integral can be treated as a special case of a more general multifold KH configuration space path integral (MFPI). Whereas a single-fold elastic KH integral has integrable singularities at caustics on the surface(s) of integration, an MFPI is more robust in nature. The theory of the MFPI method has been discussed in detail in the theory paper (Frazer, 1987). In this paper we are mainly concerned with the numerical evaluation of such multifold KH configuration space path integrals.

Theory

The single-fold elastic KH formulation of Frazer and Sen (1985a) and Sen and Frazer (1985a) works well when the receiver is located at a caustic and can correctly compute the diffraction arrivals from corners present on the surface of integration. Unfortunately this formulation breaks down when the source or receiver ray-fields have caustics on the surface of integration, a defect which can be partially overcome by methods suggested in the above papers (Sen and Frazer, 1983). A more significant defect is that diffractions from corners present on intermediate surfaces of material discontinuity are not correctly treated in this formulation. Multifold Kirchhoff-Helmholtz path integrals (MFPI) are a generalization of the single-fold KH integral designed to overcome these problems. MFPI makes use of geometrical ray theory, plane wave reflection and transmission coefficients, and an elastodynamic form of the KH integral. Generalized transmission coefficients are used to propagate the field across an interface and the KH integral is used to propagate the field between two consecutive interfaces. Geometrical ray theory is used to evaluate the integrand of the KH integral. Thus the approximations made are those used in geometrical ray theory and the Kirchhoff or tangent plane approximation. In essence, an MFPI is a multiple Huygen's construction and the resulting integral resembles the configuration space path integral used in quantum mechanics (e.g., Schulmann, 1981). In this formulation, the contributions of all possible paths from source to receiver via the interfaces are summed in computing the field at the receiver.
Figure 1.1 A four-layered medium showing a generalized ray path of the primary reflection from the deepest interface.
A detailed derivation of the formulae for different generalized rays is given in Frazer (1987). Here, for completeness, we briefly review the theory and reproduce a few of the definitions and results necessary for our computation.

As an example, we consider the four-layered medium shown in Figure 1.1. The MFPI for the generalized P ray for primary reflection from the deepest interface $S_3$ can be written as

$$u(x_r) \approx \frac{(i\omega)^5}{\rho_s c_z} \int dS_1 \int dS_2 \int dS_3 \int dS_4 \int dS_5 \exp \left[ i\omega \left( T^P(x_s, x_1) + T^P(x_1, x_2) + T^P(x_2, x_3) + T^P(x_3, x'_2) + T^P(x'_2, x'_1) + T^P(x'_1, x_r) \right) \right] \frac{B^P(x_1^-, x_s) B^P(x_2^-, x_1^-) B^P(x_3^-, x_2^+) B^P(x'_2^+, x'_3^+) B^P(x'_1^+, x'_2^-) B^P(x_r, x'_1^-)}{B^P(x_1^-, x_s) B^P(x_2^-, x_1^-) B^P(x_3^-, x_2^+) B^P(x'_2^+, x'_3^+) B^P(x'_1^+, x'_2^-) B^P(x_r, x'_1^-)}$$

$$\alpha_s \cdot \hat{t}(x_s)$$

$$T_{12}^{PP} (\hat{t}(x_1^-)) \cdot M_2 \left( \tilde{T}_{12}^{PP} (\hat{t}(x_1^-)), -\hat{n}_1(x_1), \hat{t}(x_1^+), x_1^+ \right) \cdot \hat{t}(x_1^+)$$

$$T_{23}^{PP} (\hat{t}(x_2^-)) \cdot M_3 \left( \tilde{T}_{23}^{PP} (\hat{t}(x_2^-)), -\hat{n}_2(x_2), \hat{t}(x_2^+), x_2^+ \right) \cdot \hat{t}(x_2^+)$$

$$R_{34}^{PP} (\hat{t}(x_3^-)) \cdot M_3 \left( \tilde{R}_{34}^{PP} (\hat{t}(x_3^-)), \hat{n}_3(x_3), \hat{t}_R(x_3), x_3^- \right) \cdot \hat{t}_R(x_3^-)$$

$$T_{32}^{PP} (\hat{t}(x_2^+)) \cdot M_2 \left( \tilde{T}_{32}^{PP} (\hat{t}(x_2^+)), -\hat{n}_2(x_2), \hat{t}(x_2^-), x_2^- \right) \cdot \hat{t}(x_2^-)$$

$$T_{21}^{PP} (\hat{t}(x_1^+)) \cdot M_1 \left( \tilde{T}_{21}^{PP} (\hat{t}(x_1^+)), -\hat{n}_1(x_1), \hat{t}(x_1^-), x_1^- \right) \cdot \hat{t}(x_1^-)$$

$$\hat{t}(x_r)$$

If we make the substitutions

$$T_{12}^{PP} = \tilde{T}_{12}^{PP} (\hat{t}(x_1^-)) \cdot M_2 \left( \tilde{T}_{12}^{PP} (\hat{t}(x_1^-)), -\hat{n}_1(x_1), \hat{t}(x_1^+), x_1^+ \right) \cdot \hat{t}(x_1^+)$$

$$R_{34}^{PP} (\hat{t}(x_3^-)) \cdot M_3 \left( \tilde{R}_{34}^{PP} (\hat{t}(x_3^-)), \hat{n}_3(x_3), \hat{t}_R(x_3^-), x_3^- \right) \cdot \hat{t}_R(x_3^-)$$

etc.

Then equation (1) reduces to the form

$$u(x_r) \approx \frac{(i\omega)^5}{\rho_s c_z} \int dS e^{i\omega T^P} \int dS \cdot a_s \cdot \hat{t}(x_s) T_{12}^{PP} T_{23}^{PP} R_{34}^{PP} T_{32}^{PP} T_{21}^{PP} \hat{t}(x_r).$$
Here we have assumed a source at $x_s$ with equivalent body force density

$$f = a_s \delta(x - x_s)$$  \hspace{1cm} (4)

where $\delta$ is the Dirac-delta function.

For homogeneous layers

$$T^P = \frac{||x_s - x_1||}{\alpha_1} + \frac{||x_1 - x_2||}{\alpha_2} + \frac{||x_2 - x_3||}{\alpha_3} + \frac{||x_3 - x'_2||}{\alpha_3} + \frac{||x'_2 - x'_1||}{\alpha_2} + \frac{||x'_1 - x_r||}{\alpha_1}.$$  \hspace{1cm} (5)

In 3-D we have

$$B^P = 4\pi \alpha_1 ||x_s - x_1|| 4\pi \alpha_2 ||x_1 - x_2|| 4\pi \alpha_3 ||x_2 - x_3|| 4\pi \alpha_3 ||x_3 - x'_2|| 4\pi \alpha_2 ||x'_2 - x'_1|| 4\pi \alpha_1 ||x'_1 - x_r||$$  \hspace{1cm} (6)

whereas in 2-D

$$B^P = \left(e^{-i\pi/4} 8\pi \omega\right)^5 \left(\alpha_1 ||x_s - x_1||\right)^{1/2} \left(\alpha_2 ||x_1 - x_2||\right)^{1/2} \left(\alpha_3 ||x_2 - x_3||\right)^{1/2}
\left(\alpha_3 ||x_3 - x'_2||\right)^{1/2} \left(\alpha_2 ||x'_2 - x'_1||\right)^{1/2} \left(\alpha_1 ||x'_1 - x_r||\right)^{1/2}.$$  \hspace{1cm} (7)

Each $T$ is a generalized transmission coefficient, $R$ is a generalized reflection coefficient (Frazer and Sen, 1985a) and

$$M^P (\hat{\ell}_1, \hat{n}, \hat{\ell}_2, x_1^-) = \frac{\lambda (\hat{n}\hat{\ell}_1 + \hat{\ell}_2\hat{n}) + 2\mu (\hat{\ell}_1\hat{n} + \hat{n}\hat{\ell}_2)}{(\lambda + 2\mu)}.$$  \hspace{1cm} (8)

Similar formulae for the pressure response in an acoustic medium can be found in Frazer (1986).

We may note here that each interaction of a ray with an interface corresponds to one fold of the integral in the MFPI. The individual layers need not be homogeneous. The formulation is robust as long as the rays from source/receiver to the shallowest interface and rays from points on one interface to the next do not develop any caustics. In case the rays drawn from points on one interface to another do develop caustics, one can always choose additional intermediate surface(s) of integration which do not have caustics. Such intermediate interfaces need not coincide with any physical interfaces, i.e., velocity may be continuous across them. In such a
case the Snell transmission coefficient across such an interface equals 1 at all angles of incidence, but the generalized transmission coefficient given by (2) will still be a function of both angle of incidence and angle of transmission. MFPI's for S-waves and for generalized rays with conversions or multiples can be written out in the same fashion.

Computations

The MFPI algorithm requires tracing rays from both source and receiver down to the shallowest interface and then from an array of points on each interface to an array of points on the next interface. Thus the first step in computation is ray tracing. For homogeneous layers, this is straightforward since the rays are piecewise straight lines. For inhomogeneous layers we use Hubley's (1971) algorithm for ray tracing. We parameterized the interfaces by using arc tangent functions (Gebrande, 1976; Sen and Frazer, 1985a) since these are very easy to code and many geologic structures can be correctly modelled with them. However, sharp corners on interfaces cannot be modelled with arc tangent functions and in such cases we specified the interfaces by piece-wise straight lines with discontinuous slopes at the corners.

Equation (1) is a five-fold MFPI for a generalized ray which interacts with different interfaces at five points (Fig. 1.1), each interaction giving rise to one fold of the integral. This integral can be reduced to the form

\[ u(\omega) = (i\omega)^5 \int \int \int \int dS_1 dS_2 dS_3 dS_4 dS_5 \]
\[ \mathbf{f}(x_1, x_2, x_3, x_2', x_1') e^{i\omega T(x_1, x_2, x_3, x_2', x_1')} \]

where \( \mathbf{f} \) contains the spreading factors, generalized reflection and transmission coefficients, etc. and \( T \) is the total travel time for a ray drawn from the source to the receiver via each one of the interfaces. The form (9) is quite general; thus for a generalized ray with \( n \) interactions with the surfaces of integration, the MFPI is

\[ u(\omega) = (i\omega)^n \int \int \int \ldots \int dS_1 dS_2 \ldots dS_n \]
\[ \mathbf{f}(x_1, x_2, \ldots, x_n) e^{i\omega T(x_1, x_2, \ldots, x_n)}. \]
**Frequency Domain Computations:** The simplest approach to computing integrals of the form (10) is to evaluate them in the frequency domain once for each frequency and then obtain the time domain response by taking an inverse Fourier transform of the frequency series. At large values of frequency, the phase function is highly oscillatory and use of a multidimensional version of the trapezoidal rule or similar formulae requires a step size inversely proportional to ω and hence the computation will be very expensive. Frazer (1977) and Frazer and Gettrust (1984) used a generalized Filon method (GFM) quadrature formula for numerical evaluation of similar oscillatory one-fold integrals. In the GFM approach the step size is proportional to ω^ε where ε ≈ 0.5. In appendix A, we derive a GFM quadrature formula for the evaluation of multifold oscillatory integrals. It is obvious that frequency domain computations will require more computation time than any time domain approach since the MFPIs have to be evaluated as many times as the number of frequencies. However in the frequency domain, one can easily include attenuation and dispersion by making the velocity frequency-dependent and complex (Frazer and Sen, 1985a).

**Time Domain Methods:** An inverse Fourier transform of (10) gives

\[ \tilde{u}(t) \approx \left(-\partial / \partial t\right)^n \int \int \ldots \int dS_1 dS_2 \ldots dS_n \]

\[ f(x_1, x_2, \ldots, x_n) \delta \left(t - T(x_1, x_2, \ldots, x_n)\right) \]  

(11)

where δ is the Dirac-delta function.

The integral in (11) is singular wherever \( \partial T / \partial x = 0 \), i.e., for points corresponding to Snell paths. To get around this difficulty we may note here that we never really make observations of \( \tilde{u}(t) \) but instead we observe \( \tilde{u}(t) * A(t) \) where \( A(t) \) is the impulse response of a more or less narrow band physical system. Thus instead of trying to synthesize \( \tilde{u}(t) \), it is better to compute a smooth or filtered form of \( \tilde{u}(t) \). The smoothing function \( A(t) \) can be chosen in a convenient way (Chapman, 1978). One possible choice of \( A(t) \) is

\[ A_1(t) = \left( a/\pi \right)^{1/2} e^{-at^2} \]  

(12)

or any of its derivatives. The function \( A(t) \) has the advantage of being infinitely
differentiable, i.e., it has an infinite number of continuous derivatives so that any
derivatives appearing on the right-hand side of equation (11) can be included within
the source function itself. Thus we have

$$
\ddot{u}(t) \approx (-1)^n \int \int \cdots \int f(x_1, x_2, \ldots, x_n)
A^{(n)}_1(t - T(x_1, x_2, \ldots, x_n)) \, dS_1 dS_2 \cdots dS_n.
$$

(13)

For a two dimensional problem \( f \) is not completely frequency independent, but its
frequency-dependent part can be easily factored out and included in the source
function term. Since \( A^{(n)}_1 \) is highly oscillatory we are still faced with an oscilla-
tory integrand problem. The method for computing synthetic seismograms using
equation (13) will be referred to as method I. An alternative choice of \( A(t) \) is

$$
A_2(t) = \left( \frac{1}{\Delta t} \right) B(t/\Delta t)
$$

(14)

where \( B(t/\Delta t) \) is the boxcar function given by

$$
B(t/\Delta t) = \begin{cases} 
1/2 & ; |t| < 1 \\
0 & ; |t| > 1
\end{cases}
$$

(15)

and \( \Delta t \) is the sampling interval. Dey-Sarkar and Chapman (1978) used this nor-
malized boxcar window in their time domain WKBJ method for stratified media.
With this smoothing procedure we obtain

$$
\ddot{u}_s(t) \approx (-\partial / \partial t)^n \int \int \cdots \int f(x_1, x_2, \ldots, x_n)
B(t - T(x_1, x_2, \ldots, x_n)) \, dS_1 dS_2 \cdots dS_n
$$

(16)

The computational procedure based on equation (16) will be referred to as method
II.

In the following we discuss the computational techniques using both the time
domain methods mentioned above.

The step common to both time domain methods and the frequency domain
method is to generate tables of travel time, spreading factors, angle of incidence,
and Snell's reflection and transmission coefficients. For media consisting of inho-
mogeneous layers, one needs to trace rays numerically through each layer in order
to compute spreading factors and other such parameters. For homogeneous layers no actual ray tracing is needed since the rays are straight within layers and the spreading factors can be easily computed from the straight line distance between the end points of a ray. No separate table for the travel time needs to be generated in this case since it can be computed easily by dividing the length of the ray by the constant velocity of the homogeneous layer. As an example, we show the different parameters needed for MFPI computation (for the homogeneous layer case) in Fig. 1.2(a). Fig. 1.2(b) shows the list of the tables to be generated. This example is for a particular case of a five-fold MFPI. The extension to MFPIs of higher dimensions is straightforward. Note that for multiple shot/multiple receiver calculations the source and receiver tables are identical and are two dimensional.

Method I: The computational algorithm using method I for a three-fold MFPI is illustrated in Fig. 1.3 by means of a flow chart. In this method we can either choose a Gaussian or any of its derivatives as the source function. In many of the examples here we choose the first derivative of the Gaussian as the source function. The integrals are carried out essentially by table lookup.

Method II: In this method we numerically evaluate the integral of the form of equation (16). Frazer (1987) explains, in detail, the method for doing so. In two dimensions, we choose $x_i$ to be the linear distance along an interface $S_i$. Then $dS_i = dx_i$. Any $(n - 1)$ of the $n$ position variables are treated as ordinary variables of integration and the last variable ($x_n$, say) is treated in a special way. For fixed $x_1, x_2, x_3, ..., x_{n-1}$ we locate those intervals of $x_n$ on which $t - \Delta t < T(x_1, x_2, ..., x_n) < t + \Delta t$; then if $(\Delta x_n)_k$ is such an interval and $(x_n)_k$ is the center of that interval, we have

$$ \int dS_n f(x_1, x_2, ..., x_n) B(t - T(x_1, x_2, ..., x_n)) \approx \Sigma_k f(x_1, x_2, ..., x_{n-1}, (x_n)_k) (\Delta x_n)_k. \quad (17) $$

The remaining integrations over $x_1, x_2, x_3, ..., x_{n-1}$ are handled by some standard quadrature method such as the trapezoidal rule. Since $f$ is a slowly varying function, it can often be assumed to be a constant between two quadrature points while doing this integration.
Figure 1.2 a.) A medium consisting of four layers. Parameters used in MFPI integrand are shown.
$S_i(x_i) =$ linear distance of $x_i$ along $S_i$

from a fixed point on $\sigma_i, i = 1, 2, 3$ for the

present example $A_0, A_1, A_2, A_3$ and $A_4$ are angles

<table>
<thead>
<tr>
<th>Table No.</th>
<th>Content</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$|\vec{x}_o - \vec{x}_1|$ as a function of $S_1(x_1)$</td>
<td>1 for single shot</td>
</tr>
<tr>
<td>2.</td>
<td>$A_0$</td>
<td>2 for multiple shots</td>
</tr>
<tr>
<td>3.</td>
<td>$(\hat{P} \hat{P})_{12}(A_0)$</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>$|\vec{x}_1 - \vec{x}_2|$</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>$A_1$</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>$A_2$ as a function of $S_1(x_1)$ and $S_2(x_2)$</td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>$(\hat{P} \hat{P})_{23}(A_1)$</td>
<td>2</td>
</tr>
<tr>
<td>8.</td>
<td>$(\hat{P} \hat{P})_{21}(A_2)$</td>
<td></td>
</tr>
<tr>
<td>9.</td>
<td>$|\vec{x}_2 - \vec{x}_3|$</td>
<td></td>
</tr>
<tr>
<td>10.</td>
<td>$A_3$</td>
<td></td>
</tr>
<tr>
<td>11.</td>
<td>$A_4$ as a function of $S_1(x_1)$ and $S_2(x_2)$</td>
<td></td>
</tr>
<tr>
<td>12.</td>
<td>$(\hat{P} \hat{P})_{34}(A_4)$</td>
<td>2</td>
</tr>
<tr>
<td>13.</td>
<td>$(\hat{P} \hat{P})_{23}(A_3)$</td>
<td></td>
</tr>
<tr>
<td>14.</td>
<td>$|\vec{x}_1 - \vec{x}_r|$ as a function of $S_1(x_1)$</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 1.2 b.) Tables needed to compute the MFPI.
START

INPUT  Trace length, sampling interval, model etc.

CALL GENTABLE Subroutine GENTABLE generates tables I, II, and III

CALL SIGNAL Subroutine SIGNAL generates the source function

Loop $x_1$
  Exit Loop if $x_1 \cdot GT \cdot x_{1\text{max}}$
  Pick $||x_2 - x_1||$ from Table I
  $S_1 = ||x_2 - x_1||$
  $T_1 = S_1 / AP_1$  \[AP_1 = \text{P-wave velocity for layer 1}\]
  Loop $x_2$
    Exit Loop if $x_2 \cdot GT \cdot x_{2\text{max}}$
    Pick $||x_1 - x_2||$ from Table II
    $S_2 = ||x_1 - x_2||$
    $T_2 = T_1 + S_2 / AP_2$  \[AP_2 = \text{P-wave velocity for layer 2}\]
    $S_3 = S_1 + S_2$
    Loop $x_1$
      Exit Loop if $x_1 \cdot GT \cdot x_{1\text{max}}$
      Pick $||x_1' - x_1||$ from Table II
      $S_4 = ||x_1' - x_1||$
      $T_4 = T_2 + S_4 / AP_1$
      $S_5 = S_3 + S_4$
      Compute $F(x_1, x_2, x_1', x_2')$ using $S_i$ etc.
    Time Loop
      Convolution with Source Function to Generate Synthetics at Different Time Samples
    End Loop
  End Loop
  Exit Loop if $x_2 \cdot GT \cdot x_{2\text{max}}$
End Loop
End Loop
Stop
End

Figure 1.3 Flow chart for the computation of a three-fold MFPI by method I.
Figure 1.4 Flow chart for the computation of a three-fold MFPI by method II.
The basic computational algorithm of method II for a three-fold MFPI is described schematically by means of a flow chart in Fig. 1.4. After evaluating the integral we take the required number of derivatives by transforming into the frequency domain, multiplying by $(i\omega)^n$, and then transforming back to the time domain.

Since the reflection and transmission coefficients are, in general, complex numbers, the MFPI result will be a complex time series of the form

\[ u(t) = u_R(t) + iu_I(t). \]  \hspace{0.5cm} (18)

A real time series is obtained from

\[ u(t) = u_R(t) + F(\text{sign}(\omega)F^{-1}(u_I(t))) \]  \hspace{0.5cm} (19)

where \( F = \int dt \exp(i\omega t) \) is the temporal Fourier transform and the second term on the RHS of equation (19) represents the Hilbert transform of the imaginary part of \( u(t) \).

Some care has to be taken in deciding how closely the quadrature points on the interfaces should be chosen while evaluating the MFPIs either by method I or by method II. Our numerical experiments indicate that if \( DL \) is the spacing between two consecutive quadrature points on an interface and \( V_{\text{max}} \) is the maximum velocity of the medium, then \( DL \) should be so chosen that \( DL/V_{\text{max}} \) is of the order of \( \Delta T \) where \( \Delta T \) is the sampling interval. This means that the difference in the travel time between two consecutive ray paths used in numerical quadrature should not exceed the sampling interval of the time series one wishes to compute. Thus for high frequency computation one needs to use finer sampling of the interfaces. An improper choice of \( DL \) in MFPI calculations gives rise to spurious phases. Fig. 1.5 shows an example of a three-fold MFPI for a model consisting of two flat layers over a half space. A single trace has been computed by method I for a receiver located at 2.2 km. The procedure was repeated for different values of \( DL \). It can be clearly seen that smaller values of \( DL \) give better results. We chose \( DL \) the same for all the interfaces, for convenience, although this may not be necessary.
Figure 1.5 Traces showing the effect of improper sampling.
Optimization of time domain codes: the speed of computation of multifold path integrals in the time domain depends on the following factors:

1. Fold of the integral: for time domain calculations (i.e., methods I and II) the computation time increases geometrically with fold.

2. Number of quadrature points on each surface of integration.

3. Initial time of the time series, length of the time series and the number of time samples in the time series. We may note here that \( \Delta T \approx \frac{DL}{V_{max}} \) needs to be satisfied.

4. Number of generalized rays used for evaluating the integral.

Efficiency of computation can be significantly increased by a careful choice of the parameters based on the above factors. This can be achieved by considering only those ray paths which contribute significantly to the time series. In order to do this we first shoot Snell's rays through the model and compute the travel time curve. The travel time curve is then used to choose an initial time \( T_0 \) and the time series length \( TL \), both of which may be different for different traces at different receiver locations. \( T_0 \) and \( TL \) are so chosen that the computation is restricted to a small time window but they are long enough to include all possible arrivals. Now at each step of our computation, i.e., at the start of each interface loop, we check to see if the travel time exceeds the time series \( T_0 + TL \) in which case we skip the entire computation within that loop. This technique substantially reduces the computation time.

Other Considerations: Complex structures of the interfaces or irregular velocities in the medium may cause the formations of local geometrical shadows on some interfaces. See, for example, Fig. 8 of Frazer (1987). Paths through such shadows should be excluded from the integration (the proper way to include them is to introduce more surfaces of integration). This can be achieved by the use of a 'flag'. Whenever the flag is on, indicating the presence of a shadow zone, the amplitude function \( f \) is set to zero at that point. By this procedure we exclude
inadmissible paths. An example of this for a single-fold KH integral can be found in Sen and Frazer (1985a) in the vertical fault model example.

As mentioned earlier in the text, each interaction of a generalized ray with an interface gives rise to one fold of the integral. This results in a large number of folds of the integral for a multi layered problem. However, it may not always be necessary to compute the complete multifold integral. If an interface is smooth, i.e., it does not have any corners, the surface integral corresponding to that interface can be replaced by its stationary phase value. In addition, by looking at the Snell rays, one can choose an alternate surface(s) of integration which may be more convenient to evaluate than the standard MFPI. One such example is illustrated in the Fig. 6 of Sen and Frazer (1985a).

**Synthetic Seismograms**

In this section we show examples of MFPI synthetic seismograms. All the reflection/refraction seismograms shown in the sequel are for P-P phases (pressure for the acoustic case/vertical displacement for the elastic case) and were computed by using our time domain method I.

**Example 1: Wedge Model:** In this example, shown in Fig. 1.6, we consider an acoustic medium consisting of two layers over a half space. The shallowest interface has the form of a wedge such that at the corner, P, the slope changes discontinuously. The deepest interface is a plane dipping slightly to the right. We used a constant density of unity in the medium. We computed synthetic seismograms for a source located vertically above the corner P on the surface. Figure 1.6(a) shows rays traced from the source, refracted through the shallowest interface and then reflected from the deep interface back to the surface. An interesting feature of this model is that since the slope of the top interface changes discontinuously at point P, the rays after refraction through this interface do not insonify all parts of the reflector. As a result a shadow zone QR forms on the deepest reflector.
Figure 1.6 a.) Ray trace for primary reflection from the deepest interface.
Figure 1.6 b.) Travel time curve obtained from ray trace of (a).

c.) MFPI synthetic seismograms.

d.) Finite difference synthetic seismograms.
Because of this effect, the reflected rays have two distinct branches separated by a shadow. Figure 1.6(b) shows the travel time curve, obtained from the ray trace shown in Fig. 1.6(a), which has two branches well separated by a shadow. Use of geometrical ray theory would give no arrival in the shadow zone, but that is true only in the limit of infinite frequency. Use of the conventional single-fold KH integral would require tracing rays from source and receiver down to the deep interface and integrating over the deep interface, except in the shadow zone QR (Sen and Frazer 1985a). This formulation would give rise to diffractions from points Q and R. Similarly, use of the extended WKBJ method (Frazer and Phinney 1980; Chapman and Drummond 1982) would give rise to spurious branches associated with a single ray (plane wave) from either side of the shadow zone QR. These are the artifacts of the formulation of the above methods. At finite frequencies the shadow region QR is lit by diffractions from the corner P associated with a downward propagating wavefront originating at point P. Similarly, after reflection from the deep interface, when the wave fronts hit the shallow interface, the point P would send out diffractions of which the upward propagating diffracted rays would insonify the shadow zone on the surface. Hence, at finite frequencies we would expect to see diffracted reflections in the shadow zone. Since the MFPI is essentially a multiple Huygen's construction, all of the above mentioned effects are automatically included. In order to synthesize reflections (pressure response) from the deep interface we used a three-fold MFPI. The resulting synthetic seismograms are shown in Fig. 11.6(c). The seismograms show a continuous reflection arrival even in the shadow zone. Careful examination shows a slight decrease of amplitude in the shadow zone. In Fig. 1.6(d) we show synthetic seismograms computed by a second order acoustic finite difference code. The match of the MFPI synthetics with those obtained by the finite difference method is excellent. We may note here that the time of onset of the pulse in the finite difference synthetics corresponds to the reflection arrival, whereas the inflection point on the pulse (zero crossing) in the MFPI synthetics corresponds to the reflection arrivals.

Example 2: The model and ray trace from a source at 2.0 km range are shown in Fig. 1.7(a). The model consists of two acoustic layers over an acoustic half
space in which the shallowest interface forms an inverted wedge. The deep interface has a corner P at which the slope changes discontinuously. The rays drawn from the source, after refraction through the shallow interface, form a triplication and caustics at the deep interface. Due to the discontinuous slope of the deep interface at point P, the rays, after reflection from the deep interface, form a shadow zone on the surface in the range 2.2-2.6 km. In addition, the reflected rays form a triplication and caustics in the range 1.7-2.1 km on the surface. Due to the presence of caustics and a shadow, geometrical ray theory could not have been used to synthesize the reflection. The corner Q on the shallow interface and the corner P on the deep interface both act as point diffractors and the diffractions from these corners insonify the shadow zone. The use of a conventional single-fold KH formulation would fail to include the diffractions from corner Q on the shallow interface and would also require using the method of integrable singularity to repair its breakdown at caustics on the deep interface. We used a three-fold MFPI to synthesize the reflections from the deep interface. The MFPI synthetic seismograms for this model are shown in Fig. 1.7(c). The triplication zone in the range 1.7-2.1 km is very well resolved by these synthetics. The high amplitude in this zone is caused by focusing of the rays. The shadow zone predicted by geometrical ray theory in the range 2.2-2.6 km appears continuous although the amplitudes are slightly smaller in this region. In addition, a diffraction arrival can be marked in the lit region. In Figs. 1.7(d) and 7(e), we show a comparison of the pressure response (for a source with peak frequency 10 Hz) computed by finite difference and MFPI techniques. On the finite difference synthetic seismograms the first arriving phases showing a bowtie pattern correspond to reflections from the shallow interface, while the phase arriving at around 3.5 sec is the reflection from the deep interface. MFPI was used to synthesize only the reflections from the deep interface. The zone of interest in the model lies in the range 1.6 to 2.8 km; there MFPI and finite difference agree well both in travel time and amplitude. At distances far away from the source the reflections from the two interfaces interfere and hence we see a slight mismatch in the amplitudes computed by the two methods.
Figure 1.7 a.) Ray trace for primary reflection from the deepest interface.
Figure 1.7 b.) Travel time curve.

c.) MFPI synthetic seismograms.
Figure 1.7 d.) Finite difference synthetic seismograms.
e.) MFPI synthetic seismograms.
Example 3: The model considered in this example consists of three homogeneous elastic layers over an elastic half space. The interfaces have folded structures dipping slightly to the right. The layers and the half space were assumed to be Poisson solids (shear wave velocity = P-wave velocity $/\sqrt{3}$ and $\lambda = \mu$). The primary and shear wave velocities are shown in the figure. Figure 1.8(a) shows Snell rays for the primary P wave reflection from the deepest interface, drawn from a source located at a range of 2.0 km on the surface. The ray field looks extremely complicated because of the presence of many triplications caused by the highly uneven structures of the strata. Rays that went outside the range of the model have been omitted from the plot. We used a five-fold MFPI to compute reflections from the deepest interface, since neither geometrical ray theory nor a single-fold KH integral could be used because of the presence of many caustics and triplications. The travel time curve obtained from the ray tracing in Fig. 1.8(a) and the MFPI synthetic seismograms (vertical displacements) are shown in Figs. 1.8(b) and 1.8(c), respectively. The travel time curve shows 12 branches, but only a few of them contribute significantly. The effects of focussing and defocussing of the rays can be marked on the seismograms by the variation of amplitude with range. In Fig. 1.8(d) we show the ray trace for the primary reflection from the second interface. A three-fold MFPI was used to synthesize this reflection. The travel time curve obtained from this ray tracing and the series of synthetic seismograms are shown in Figs. 1.8(e) and 1.8(f), respectively.
Figure 1.8 a.) Ray trace of primary reflection from the deepest interface. Note that the rays that went outside the limit of the model have been omitted from the plot.
Figure 1.8 b.) Travel time curve.

c.) Synthetic seismograms computed by a five-fold MFPI.
Figure 1.8 d.) Ray trace of the primary reflection from the second interface.
Figure 1.8 e.) Travel time curve.

f.) Synthetic seismogram computed by a three-fold MFPI.
**Example 4:** In this example we attempted to compute refractions (turning rays) in a medium consisting of a homogeneous acoustic layer over an acoustic half space. The top layer consists of a homogeneous fluid with a velocity of 1.0 km/sec; at the discontinuity the velocity jumps to 1.5 km/sec and then increases linearly with depth at the rate of 1.6/sec. The interface separating the homogenous layer from the half space has a corner at a point P and small basin structure QR. Fig. 1.9(a) shows the ray trace from a surface source located at a range of 1.0 km. An interesting feature of the model is that, due to the discontinuity in the slope of the interface at point P, the Snell rays form a shadow zone on the surface in the range 5.2-6.8 km. We may also note here that the structure QR is not lit at all by the rays. However, at finite frequencies the point P acts as a point scatterer and the shadow zone is lit by diffraction from this corner. None of the existing asymptotic methods could be used to synthesize refractions for this model. We used a two-fold MFPI for this purpose. Figure 1.9(b) shows the MFPI synthetic seismograms for this model. We notice here that the shadow zone is insonified by the diffracted arrivals although the amplitudes of these arrivals are smaller than the refraction in the regions lit by Snell rays. As discussed earlier in the section on computational procedure, the MFPI formulation requires tracing rays from the source at $x_1$ to all points on the interface, then from each point on the interface to every other point on the interface, then from each point on the interface to the receiver at $x_r$. The integral takes the form

$$\int dx_1 \int dx_1' f(x_1, x_1') \exp\{i\omega T(x_1, x_1')\}. \quad (20)$$

When the point $x_1$ coincides with the point $x_1'$, since we are using geometrical optics formulae to compute the function $f$, the spreading factor becomes zero, giving rise to a singularity in the integrand. This singularity is integrable in two dimensions. Similar problems will arise while trying to model reflections from layer pinchouts. In the limit when the point $x_1$ approaches point $x_1'$, the generalized ray path for the refraction shown in Fig. 1.9(a) reduces to the generalized ray path for a primary reflection from the interface. Thus the generalized ray paths for primary reflection are included in the two-fold integral and hence we see an arrival corre-
sponding to the stationary phase points of such a generalized ray. It shows up at the time of arrival of the primary reflection from the interface, but the amplitudes will be incorrect because of using an incorrect integral formulation for them. In Fig. 1.9(b) the phase arriving later than the refraction arrivals correspond to such a phase. At short ranges they interfere with the refracted arrivals but at a long range these two phases are well separated from each other and hence for long range refraction calculation MFPIs can still be used very efficiently.
Figure 1.9 a.) Ray tracing for refraction modeling.
Figure 1.9 b.) Synthetic seismograms computed by using a two fold MFPI.
Discussions and Conclusions

In this paper we have shown the usefulness of the MFPI formulation for the computation of synthetic seismograms in a medium in which elastic properties vary laterally as well as in depth. MFPI is a new asymptotic method, which is as accurate as geometrical ray theory wherever GRT is valid but works even in some regions where GRT breaks down. The novelty of this approach is the multiple Huygen's construction which enables one to include diffractions from any corners present in any of the interfaces in the medium. This feature appears to be an improvement over the single-fold elastic KH integral and other asymptotic methods such as Gaussian beams or extended WKBJ theory. Time domain computation of MFPIs is fairly fast on large computers with vector array processors. However, if one is interested in computing a great number of generalized rays (with many folds of integration) the MFPI method will become expensive. For smoothly varying media the Gaussian beam method or the extended WKBJ method are be faster because these methods require fewer integrations. For a medium in which the layers are not homogeneous, the calculation of synthetics using MFPI may be expansive. However, for such media, the fold of an MFPI can be greatly reduced, in many cases, by replacing the integrals by their stationary phase point values. Use of the geometrical optics Green's function to evaluate the KH integrand causes singularities at layer pinchouts and head wave paths are not included in our MFPI formulation. Both of these limitations can be overcome by the use of multi-fold phase space path integral formulation (Frazer and Sen 1985b) in which case the source and receiver wavefields are decomposed into a series of plane waves and the wavefields are expressed in terms of integrals over ray parameters. The usefulness of phase space KH path integrals is currently being investigated.
APPENDIX A: A Generalized Filon Method (GFM) Formula for Three Fold Integrals

Here we derive a quadrature formula for computing integrals of the form

\[
I = \int_{x_1}^{x_2} \int_{x_2}^{x_3} \ldots \int_{x_n}^{x_n} f(x_1, x_2, \ldots, x_n) 
\]

\[
e^{-i\omega T(x_1, x_2, \ldots, x_n)} \, dx_1 \, dx_2 \, \ldots \, dx_n.
\]

(A-1)

We may note that for large values of \(\omega\) the phase term is highly oscillatory and use of the trapezoidal rule would require a large number of steps, thus making the computation very expensive. Frazer (1977) developed the Generalized Filon Method (GFM) quadrature formula to avoid this problem encountered in a similar single-fold integral. Here we use the GFM quadrature formula iteratively for the general multifold path integrals. For the purpose of illustration we consider a 3-fold integral of the form

\[
I = \int_{x_1}^{x_2} \int_{x_2}^{x_3} \int_{x_3}^{x_3} \, dx_1 \, dx_2 \, dx_3 \, f(x_1, x_2, x_3) \, e^{i\omega T(x_1, x_2, x_3)}.
\]

(A-2)

For the sake of simplicity we assume that \(x_1, x_2, x_3\) are all sampled at equal intervals of \(\Delta x_1, \Delta x_2\) and \(\Delta x_3\) respectively. We first evaluate the integral with respect to \(x_3\) using the trapezoidal analogue of the GFM formula and treat the other variables as constants.

\[
I = \left\{ \begin{array}{l}
\int \int dx_1 dx_2 \sum_{k=2}^{n_1} \Delta x_3 \, f(x_1, x_2, x_3^k) \, e^{i\omega T(x_1, x_2, x_3^k)} \\
\{e^{i\omega T(x_1, x_2, x_3^k)} - (f(x_1, x_2, x_3^k) - f(x_1, x_2, x_3^{k-1}))
\}
\end{array} \right. \]

\[
- e^{i\omega T(x_1, x_2, x_3^{k-1})} \left[ f(x_1, x_2, x_3^{k-1}) - (f(x_1, x_2, x_3^{k-1}) - f(x_1, x_2, x_3^{k-2})) \right];
\]

\[
\text{if } T(x_1, x_2, x_3^k) \neq T(x_1, x_2, x_3^{k-1})
\]

(A-3)

In the foregoing we will just extend (A-3) only for the case when \(\Delta T \neq 0\) between two quadrature points. We next write (A-3) as
\[
I = \frac{\Delta x_3}{\sigma} \int \int dx_1 dx_2 \Sigma_{k=2}^{n_1} \left\{ e^{\sigma T(x_1, x_2, x_3^k)} F_3^k(x_1, x_2, x_3^k, x_3^{k-1}) - e^{\sigma T(x_1, x_2, x_3^{k-1})} F_3^{k-1}(x_1, x_2, x_3^{k-1}, x_3^k) \right\}
\]

where \( F_3^k(x_1, x_2, x_3^k, x_3^{k-1}) = \frac{1}{T(x_3^k) - T(x_3^{k-1})} \left[ f(x_1, x_2, x_3^k) \right] \)

\[
\frac{1}{\sigma (T(x_1, x_2, x_3^k) - T(x_1, x_2, x_3^{k-1}))} \left[ f(x_1, x_2, x_3^{k-1}) \right]
\]

Next we do the integration with respect to \( x_2 \),

\[
I = \frac{\Delta x_3}{\sigma} \int dx_1 \sum_{j=2}^{n_2} \Sigma_{k=2}^{n_1} \frac{\Delta x_2}{\sigma (T(x_1, x_2^j, x_3-k) - T(x_1, x_2^{j-1}, x_3^k))} \left\{ e^{\sigma T(x_1, x_2^j, x_3^k)} \left[ F_3^k(x_1, x_2^j, x_3^k, x_3^{k-1}) \right] \right. \\
- \frac{1}{\sigma (T(x_1, x_2^j, x_3^k) - T(x_1, x_2^{j-1}, x_3^k))} \left[ F_3^{k-1}(x_1, x_2^j, x_3^{k-1}, x_3^k) - F_3^k(x_1, x_2^j, x_3^k, x_3^{k-1}) \right]
\]

\[
\left[ \frac{F_3^k(x_1, x_2^j, x_3^k, x_3^{k-1}) - F_3^{k-1}(x_1, x_2^{j-1}, x_3^{k-1}, x_3^k)}{\sigma (T(x_1, x_2^j, x_3^k) - T(x_1, x_2^{j-1}, x_3^k))} \right] \right\}
\]

Finally we have

\[
I = \frac{\Delta x_3 \Delta x_2}{\sigma^2} \int dx_1 \sum_{j=2}^{n_2} \Sigma_{k=2}^{n_1} e^{\sigma T(x_1, x_2^j, x_3^k)} F_2^{j,k}(x_1, x_2^j, x_3^k, x_2^{j-1}, x_3^{k-1}) \\
- e^{\sigma T(x_1, x_2^{j-1}, x_3^k)} F_2^{j-1,k}(x_1, x_2^{j-1}, x_3^k, x_2^j, x_3^{k-1}) \\
- e^{\sigma T(x_1, x_2^j, x_3^{k-1})} F_2^{j,k-1}(x_1, x_2^j, x_3^{k-1}, x_2^{j-1}, x_3^k) \\
+ e^{\sigma T(x_1, x_2^{j-1}, x_3^{k-1})} F_2^{j-1,k-1}(x_1, x_2^{j-1}, x_3^{k-1}, x_2^j, x_3^k)
\]

\[(A-5)\]
where,

\[ F_{2}^{j,k} \left( x_{1}, x_{2}, x_{3}^{j} ; x_{2}^{j-1}, x_{3}^{k-1} \right) = \frac{1}{\left( T \left( x_{1}, x_{2}, x_{3}^{k} \right) - T \left( x_{1}, x_{2}^{j-1}, x_{3}^{k} \right) \right)} \]

\[ F_{3}^{k} \left( x_{1}, x_{2}, x_{3}^{j} ; x_{3}^{k-1} \right) - \frac{F_{3}^{k} \left( x_{1}, x_{2}^{j-1}, x_{3}^{k-1} \right) - F_{3}^{k} \left( x_{1}, x_{2}^{j-1}, x_{3}^{k-1} \right)}{\sigma \left( T \left( x_{1}, x_{2}, x_{3}^{k} \right) - T \left( x_{1}, x_{2}^{j-1}, x_{3}^{k} \right) \right)} \]

Proceeding in this manner we can write

\[ I = \frac{\Delta x_{1} \Delta x_{2} \Delta x_{3}}{\sigma^{3}} \sum_{i=2}^{n_{1}} \sum_{j=2}^{n_{2}} \sum_{k=2}^{n_{3}} e^{e T(s_{i}, x_{2}^{j})} F_{1}^{i,j,k} \left( x_{1}, x_{2}, x_{3}; x_{1}^{i-1}, x_{2}^{j-1}, x_{3}^{k-1} \right) \]

\[ - e^{e T(s_{i}^{i-1}, x_{2}^{j})} F_{1}^{i,j,k} \left( x_{1}, x_{2}, x_{3}; x_{1}^{i-1}, x_{2}^{j-1}, x_{3}^{k-1} \right) \]

\[ e^{e T(s_{i}^{j-1}, x_{2}^{j})} F_{1}^{i,j,k} \left( x_{1}, x_{2}, x_{3}; x_{1}^{i-1}, x_{2}^{j-1}, x_{3}^{k-1} \right) \]

where

\[ F_{1}^{i,j,k} \left( x_{1}, x_{2}, x_{3}^{j} ; x_{1}^{i-1}, x_{2}^{j-1}, x_{3}^{k-1} \right) = \frac{1}{\left( T \left( x_{1}, x_{2}, x_{3}^{k} \right) - T \left( x_{1}^{i-1}, x_{2}^{j-1}, x_{3}^{k-1} \right) \right)} \]

\[ F_{2}^{j,k} \left( x_{1}, x_{2}, x_{3}^{j} ; x_{2}^{j-1}, x_{3}^{k-1} \right) \]

\[ F_{3}^{k} \left( x_{1}, x_{2}, x_{3}^{j} ; x_{3}^{k-1} \right) - \frac{F_{3}^{k} \left( x_{1}, x_{2}^{j-1}, x_{3}^{k-1} \right) - F_{3}^{k} \left( x_{1}, x_{2}^{j-1}, x_{3}^{k-1} \right)}{\sigma \left( T \left( x_{1}, x_{2}, x_{3}^{k} \right) - T \left( x_{1}, x_{2}^{j-1}, x_{3}^{k} \right) \right)} \]

and so on.

Equation (A-7) can be used for numerical evaluation of (A-2). Extension of this formula to integrals of higher fold is straightforward but they are very long and will not be repeated here. Note that it is not necessary to derive such formulas before coding them. At each step the algorithm is the same and this enables one to create a single computer code which will work on integrals of various folds.
APPENDIX B: Generalized Reflection and Transmission Coefficients

In this appendix we show how to compute the generalized reflection and transmission coefficients that appear in equations (1), (2), and (3) of the text.

The generalized coefficient $T_{12}^{PP}$ is given by

$$T_{12}^{PP} = T_{12}^{PP} \left( \hat{t}(x_1^-) \right) \cdot \frac{\hat{M}_{2}^{P}}{\hat{t}(x_1^-)} \cdot \hat{T}_{12}^{PP} \left( \hat{t}(x_1^-) \right),$$

$$- \hat{n}_1(x_1), \hat{t}(x_1^+), x_1^+ \cdot \hat{t}(x_1^+).$$

(B-1)

where $M$ includes Snell's transmission coefficient etc. defined in equation (8) of the text. Please refer to Fig. (1) above for the meaning of different vectors appearing in (B-1)

$$M_{2}^{P} \left( \hat{T}_{12}^{PP}, -\hat{n}_1, \hat{t}(x_1^+); x_1^+ \right) = - \left[ \lambda_2 \left( \hat{n}_1 \hat{T}_{12}^{PP} + \hat{t}(x_1^+) \hat{n}_1 \right) + 2 \mu_2 \left( \hat{T}_{12}^{PP} \hat{n}_1 + \hat{n}_1 \hat{t}(x_1^+) \right) \right] / (\lambda_2 + 2 \mu_2).$$

(B-2)

Now,

$$T_{12}^{PP} = \left( \hat{P} \hat{P} \right)_{12} \hat{T}_{12}^{PP}$$

(B-3)

where $\left( \hat{P} \hat{P} \right)_{12}$ is the Snell transmission coefficient given by Aki and Richards (1980), which is a function of angle of incidence only. From (B-1), (B-2), and (B-3) we have

$$T_{12}^{PP} = T_{12}^{PP} \cdot \frac{\hat{M}_{2}^{P}}{\hat{t}(x_1^+)} \cdot \hat{t}(x_1^+)$$

$$= - \frac{\left( \hat{P} \hat{P} \right)_{12}}{(\lambda_2 + 2 \mu_2)} \left[ \lambda_2 \left( \hat{T}_{12}^{PP} \cdot \hat{t}(x_1^+) \right) \left\{ \left( \hat{T}_{12}^{PP} \cdot \hat{n}_1 + \hat{n}_1 \cdot \hat{t}(x_1^+) \right) \right\} + 2 \mu_2 \left\{ \left( \hat{T}_{12}^{PP} \cdot \hat{n}_1 \right) + \left( \hat{n}_1 \cdot \hat{t}(x_1^+) \right) \right\} \right]$$

$$= - \frac{\left( \hat{P} \hat{P} \right)_{12}}{(\lambda_2 + 2 \mu_2)} \left\{ \left( \hat{T}_{12}^{PP} \cdot \hat{n}_1 \right) + \left( \hat{n}_1 \cdot \hat{t}(x_1^+) \right) \right\}. \right\}}

(B-4)

Now we know (Frazer and Sen, 1985a, Table 1 A)

$$\sigma = \hat{t}(x_1^-) \cdot (\hat{I} - \hat{n}_1 \hat{n}_1) / \alpha_1$$

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Thus we need to know the angles that the incident and transmitted rays make with the normal at the point on the interface in order to calculate generalized reflection and transmission coefficients.

\[
\begin{align*}
\hat{T}_{12}^{PP} &= \alpha_2 \sigma \hat{\sigma} + \sqrt{1 - \alpha_2^2 \sigma^2} \hat{n}_1 \\
\hat{T}_{12}^{PP} \cdot \hat{n}_1 &= \sqrt{1 - \alpha_2^2 \sigma^2}, \quad \text{Im} (\sqrt{\cdot}) \geq 0 \\
\hat{T}_{12}^{PP} \cdot \hat{i}(x_1^+) &= \alpha_2 \sigma \hat{\sigma} \cdot \hat{i}(x_1^+) + \sqrt{1 - \alpha_2^2 \sigma^2} \hat{n}_1 \cdot \hat{i}(x_1^+).
\end{align*}
\]
REFERENCES


CHAPTER II
ANALYSIS OF MULTIPATH SOUND PROPAGATION IN THE OCEAN NEAR 50°N, 128°W

Abstract

Downslope sound propagation data collected off the west coast of Vancouver Island for a shallow source and a deep receiver are examined. The purpose of the work was, first, to identify different ray paths from source to the receiver and then to explain the variation in measured pressure amplitude with range. The bottom interacting rays, bottom penetrating rays, and channel rays with bottom interaction, were identified by the travel time differences between different bottom bounce paths. Travel times were computed by ray tracing with an algorithm from Hubley in which the time step along the ray is the independent parameter. Different propagation paths could be identified at different bathymetric regions viz., continental shelf, continental slope and deep ocean. A ridge at a range of 79 km. shadowed some of the propagating rays; the diffraction arrivals from the top corner of the ridge were tentatively identified in the data by ray tracing. Data collected in the range 1-60 km were modelled by computing synthetic sonograms using a reflectivity technique modified to remove phase from the vertical wavefunctions. These synthetics were used to constrain the velocity, velocity gradient, sediment thickness and the absorption coefficient in the upper part of the sediment.

Introduction

The study of acoustic wave propagation in a range-dependent ocean environment is an active area of research. Acousticians often study propagation loss versus range at a single frequency and explain variations of propagation loss in terms of changing bathymetry, sound speed profiles and other environmental factors. A departure from this CW(single frequency) approach was the work of Brown (1982), who used
WKBJ synthetic sonograms for a detailed analysis of waveform structure in the time domain. The time domain procedure is the one followed by most seismologists for the analysis of long range seismic reflection and refraction profiles. The first step in such an analysis is to identify different arrivals in the data when plotted in a record section format. Since the sound speed profiles and the water depth are accurately measured in an ocean acoustics experiment, identification of the arrival structure can be done fairly easily by ray tracing, especially for a range-independent ocean environment. Ray tracing has been used successfully by many workers (for example, Worcester 1981, Legters et. al., 1983; Knobles 1983; Northrop & Shockley 1984) to identify multipath arrivals in the ocean.

In this paper we use ray tracing based on an algorithm given by Hubley (1971) to identify different arrivals in the data collected off the west coast of Vancouver Island. Measured propagation loss curves for the same data were given by Dosso & Chapman (1986), who modeled these data using the parabolic equation (P.E.) method (Tappert 1977; Thompson & Chapman 1983). Results obtained here by ray tracing explain the observed propagation loss. The ocean floor is highly rugged beyond a range of 70 km and there many different ray paths can be identified for propagation from source to the receiver. For the region in which bathymetry is almost flat we use reflectivity synthetic sonograms to infer the properties of the bottom.

Experiment

Pressure measurements were obtained along two shot run tracks shown in Fig. 2.1 off the west coast of Vancouver Island, British Columbia. Dosso and Chapman (1986) described the experiment in detail and modeled the propagation loss for both shot runs using a P.E. method. Here, we only summarize the experiment. The shot tracks were so chosen that they traverse the continental slope in regions where the slope of the ocean bottom was relatively regular and monotonic. In the sequel we analyze data only for shot run 1. In shot run 1, the receiving system was in deep water and the source ship moved progressively into shallower water.
Figure 2.1 Map showing location of shot runs 1 & 2.
Figure 2.2 Bathymetry & measured sound speed profile for shot run 1.
This is a downslope propagation shot run since the waves propagate in the downslope direction towards the receiver. During the experiment, bathymetry along the track was continuously measured using a 12 KHz echo sounding sonar on the source ship. Expendable bathythermographs were used to measure the temperature profile to a depth of 500 m and the sound speed profile was extended to the ocean floor with the aid of historical data. The bathymetry and sound speed profiles for shot run 1 are shown in Fig. 2.2. It can be seen that there is no appreciable variation in sound speed profile with range. The depth of the sound channel axis is about 400 m. The source depth was 22 m, i.e., within the steep negative sound speed gradient, which ensured strong interaction with the ocean bottom. The receiver was placed at a depth of 388 m, very close to the sound channel axis since that is the optimum depth for detection in shallow water. Two ships, CFAV ENDEAVOUR, the receiving ship and CSS PARIZEAU, the source ship, were used in the experiment. The hydrophone signals were sampled at a rate of 1500 samples/sec and the digital signals were transmitted by a radio frequency telemetry link to the receiving ship for recording on magnetic tape.

Data analysis

In this paper we analyze the hydrophone data for the downslope propagation shot run 1 (Fig. 2.1). The first step in such an analysis is to identify the arrivals of different phases. This was done by ray tracing from source to the receiver, i.e., by locating different sets of eigenrays from source to the receiver. For convenience in ray tracing we used reciprocity and interchanged the source and receiver positions. Acousticians have used different approaches for ray tracing in a range dependent ocean environment of which Foreman’s approach (1983) has been one of the most popular. We used a simple and elegant approach suggested by Hubley (1971). In many ray tracing schemes in a medium in which velocity is a continuous function of depth, the depth range is divided into small segments in each one of which the velocity function is assumed to be linear. Snell’s law is applied at each step of ray tracing using the local velocity function. For a single step in ray tracing, the
velocity gradient is assumed constant and the new gradient is used at the beginning of the new step. For example, each step in Gebrane's (1976) algorithm represents a specified change $d\theta$ in the grazing angle $\theta$ of the ray. However, Gebrane's formulation breaks down in two cases: (1) when the gradient of sound speed vanishes and (2) when the ray is vertical. Each step in Hubley's (1971) algorithm represents a specified increment $dt$ in the travel time of the ray and the algorithm is stable for all angles and velocity gradients.

Hubley's algorithm

The scalar Helmholtz equation for wave propagation is

$$\nabla^2 \psi + \omega^2 u^2 \psi = 0 \quad (1)$$

in which a time dependence of $\exp(i\omega t)$ is assumed. $\omega$ is angular frequency and $u = 1/c$ where $c$ is the sound speed. If we write

$$\psi = Ae^{i\omega \phi} \quad (2)$$

where $\phi$ is the travel time, then it is straightforward to show that

$$\nabla^2 \psi + \omega^2 u^2 \psi = (\nabla^2 A) e^{i\omega \phi} + i\omega (2 \nabla \phi \cdot \nabla A + A \nabla^2 \phi) e^{i\omega \phi} - \omega^2 ((\nabla \phi)^2 - u^2) A e^{i\omega \phi} \quad (3)$$

In geometrical ray theory (GRT) we require a solution accurate at large values of frequency $\omega$. Thus, we require

$$\nabla^2 \phi = 0 \quad (4)$$

and

$$2 \nabla \phi \cdot \nabla A + A \nabla^2 \phi = 0. \quad (5)$$

Equations (4) and (5) are the eikonal and transport equations, respectively. The eikonal equation is what we solve to get ray paths. Let $\mathbf{x}(s)$ be a ray parameterized by arc length $s$ along the ray. By definition, rays are normal to wavefronts and the wavefronts are surfaces of constant $\phi$. As $u$ is the slowness we have

$$\frac{\partial \mathbf{x}}{\partial s} = (u(\mathbf{x}))^{-1} \nabla \phi \quad (6)$$
Differentiating this equation with respect to \( s \) and using the chain rule yields

\[
\partial_s (u \partial_s \psi) = \left( \frac{1}{u} \nabla \phi \cdot \nabla \right) \nabla \phi. \tag{7}
\]

It can be shown that for any function \( \phi \)

\[
\nabla \phi \cdot \nabla \phi = \frac{1}{2} \nabla (\nabla \phi \cdot \nabla \phi). \tag{8}
\]

Thus substituting (8) into (7) and using (4) we get

\[
\partial / \partial s(u \partial_s \psi / \partial s) = \left( \frac{1}{2u} \right) \nabla \phi \cdot \nabla \phi = \nabla u. \tag{9}
\]

Now using the chain rule on the leftmost side of (9) gives

\[
(\partial \psi / \partial s \cdot \nabla u) \partial \psi / \partial s + u \partial_s(\partial \psi / \partial s) = \nabla u. \tag{10}
\]

At this stage we introduce the development due to Hubley. In order to get simple equations in the cartesian axis system \( XZ \) we let

\[
x(s) = [x(s), z(s)] \tag{11}
\]

and introduce the grazing angle \( \theta \) (Fig. 2.3) so that \( \partial \psi / \partial s \) can be written as

\[
\partial \psi / \partial s = \dot{x} \partial x / \partial s + \dot{z} \partial z / \partial s = [\cos \theta, \sin \theta]^T. \tag{12}
\]

Next we substitute for \( \partial \psi / \partial s \) in (10) and use (12) to get

\[
(\cos \theta \partial_s u + \sin \theta \partial_z u)[\cos \theta, \sin \theta]^T + u[\partial_s cos \theta, \partial_z sin \theta]^T = [\partial_x u, \partial_z u]^T. \tag{13}
\]

After some algebra each component in (13) is seen to satisfy the relation

\[- \sin \theta \partial_s u + \cos \theta \partial_z u = u \partial_s \theta. \tag{14}\]

We know that \( ds = cd t = dt / u \) and so

\[
u \partial_s \theta = u^2 \partial_t \theta. \tag{15}\]
Figure 2.3 Arc of a ray: parameters used in ray tracing using Hubley's algorithm.
Substitution of (15) into right hand side of (14) now gives

\[ d\theta = \frac{1}{u^2} (-\sin \theta \partial_z u + \cos \theta \partial_x u) dt. \]  \hspace{1cm} (16)

Finally noting that

\[ 1/u^2 du = c^2 d(1/c) = -dc, \]  \hspace{1cm} (17)

We rewrite (16) in terms of sound speed

\[ d\theta = (\sin \theta \partial_x c - \cos \theta \partial_z c) dt \]  \hspace{1cm} (18)

As usual, we also have

\[ dx = c \cos \theta \ dt \]  \hspace{1cm} (19)

and

\[ dz = c \sin \theta \ dt. \]  \hspace{1cm} (20)

Equations (18)-(20) are very stable for stepping. Unlike many other methods, they do not become unstable when the gradient of velocity vanishes. The time increment \( dt \) is the independent variable for stepping. Our procedure for ray tracing using this algorithm is simple as follows:

1. Start at \((x_i, z_i)\) with grazing angle \( \theta_i \),
2. Compute \( \partial_x c \) and \( \partial_z c \), the horizontal and vertical derivatives, respectively, of the sound speed at point \((x_i, z_i)\),
3. For a given \( \Delta t \) calculate the new position \((x_{i+1}, z_{i+1})\) of the ray (Fig. 2.3), and the new grazing angle, \( \theta_{i+1} \), by the following relations:

\[ \theta_{i+1} = \theta_i + (\sin \theta_i \partial_x c - \cos \theta_i \partial_z c)\Delta t \]  \hspace{1cm} (21)

\[ x_{i+1} = x_i + c \cos \theta_i \Delta t \]  \hspace{1cm} (22)

\[ z_{i+1} = z_i + c \sin \theta_i \Delta t \]  \hspace{1cm} (23)

\[ t_{i+1} = t_i + \Delta t, \]  \hspace{1cm} (24)

4. Gone far enough in range or depth? If so, stop.
5. Update \( \sin \theta, \cos \theta, \partial_x c, \partial_z c \) and go to step (3).
As step (3) contains a predictor step but no corrector step, the step size in travel time, $\Delta t$, is critical for ray tracing. If $\Delta t$ is chosen too big then the sound speed will not be sampled well enough, whereas a very small $\Delta t$ will result in increments $d\theta$, $dx$ and $dz$ so small as to result in a loss of precision when $\theta, x$ and $z$ are updated. However $\Delta t$ need not be a constant along the entire ray path. Usually the sound speed gradient varies rapidly in the shallow part of the ocean while in deep water the sound speed gradient is nearly constant. Thus a small value of $\Delta t$ in the shallow water and a large value in deep water may be a good choice in order to get fairly accurate ray paths. For the ray tracing in this study $\Delta t$ was 0.04 sec for depths greater than 400 m and 0.001 sec for depths less than 400 m. The sound speed profile may be smoothed to avoid discontinuities in the higher order depth derivatives in the velocity function.

The ocean bottom can be defined in several ways. One approach is to represent the ocean bottom as a piece-wise straight line. Gebrande(1976) used normalized inverse tangent functions to define an arbitrarily shaped curved interface. In Gebrande's parameterization a curve in the XZ plane is defined by the equation

$$Z = Z_0 + \sum_{k=1}^{n} \Delta Z_k (0.5 + \pi^{-1}tan^{-1}(X - X_k)/b_k)$$

(25)

where $Z_0$ is the depth of the interface at $X = -\infty$. Each term within the summation represents a flexure with the centre at x-coordinate $X_k$ and throw $\Delta Z_k$. The width of the flexure is $2b_k$. This algorithm is very easy to code and many realistic bottom structures can be modelled by suitably combining flexures of different parameters. An alternative to this approach is that of Foreman(1983), who used a T-spline fit to the bathymetry.

**Identification of eigenrays**

The pressure-time series recorded by a hydrophone at a particular range requires locating all the eigenrays connecting the source to the receiver. One way to locate an eigenray from the source to the receiver is by two-point ray tracing. This, however, is a very time consuming procedure. Alternatively, one can shoot a fan
of rays from the source to a horizontal plane at the receiver depth. The fan is so chosen that the rays bracket the desired receiver range. The eigenray at the desired receiver position can be calculated by linear interpolation between the rays which bracket the receiver range. If the rays cross over or go through caustic(s), there will be multiple ray-paths and hence interpolation has to be done on each branch separately. The travel time, take-off angle, emergence angle etc. can be calculated by interpolation on the parameters between the rays that bracket the receiver range on each branch of the ray-field. These parameters can be used to calculate ray theoretical amplitudes, if desired.

**Phase identification for shot run 1**

Shot run 1 (Fig. 2.1) is a downslope propagation run in which the receiver was in deep water while the source moved progressively into shallower water. The pressure-time series thus recorded were sampled at the rate of 1500 samples/s. For frequency domain calculations such as for studying propagation loss as a function of frequency and range, no absolute timing information is needed but for waveform modeling it is necessary to know the time of arrival of different phases appearing in the sonograms. Since the primary goal of the original experiment was to study the propagation loss (Dosso and Chapman 1986), a careful record of the absolute time information was not kept. At each shot location 10240 samples were recorded at the rate of 1500 samples/sec (nearly 6.83 s). All the useful data, i.e. arrivals with significant pressure amplitude, were contained in this time window. Thus we needed to obtain the starting time of each trace and then identify different groups of arrivals in the trace. Both were done simultaneously by ray tracing. Since the sound speed did not vary significantly with range, we chose a single sound speed profile (one from deep water; the second profile from the left in Fig. 2.2) to represent the sound speed structure at all ranges. All of the significant arrivals could be identified by ray tracing using this profile. For the purpose of ray tracing, reciprocity was used, i.e., the source and receiver positions were interchanged. From the receiver location (source location for the ray trace) a cone of rays was shot up to a horizontal plane at a depth of 22 m.
Eigenrays at a particular range were obtained by interpolation between the rays that bracket that range unless the receiver was in a region of geometrical shadow. This procedure was repeated for different sets of generalized rays. Thus a travel time curve (i.e., travel time versus range) was obtained for each generalized ray. In the discussion below the labels source and receiver are used in a manner consistent with the ray tracing and not with the physical experiment.

The simplest path of propagation is the direct path from the source to the receiver. At near ranges, the direct arrivals appear with high amplitude and die off rapidly with increasing range. Up to a range of 10 km, the first arriving phase in the data is the direct arrival. Ray tracing for the direct path (not shown) shows that beyond a range of 10 km, the rays turn even before reaching the receiver. Since at near ranges a large number of bubble pulses arrive following the main shock wave (this is a property of SUS charges in water, discussed later in the text) and because of the small travel time differences between the direct and the surface reflected rays, the direct wave arrivals are highly oscillatory and of long duration.

The rays that immediately follow the direct arrivals are the ones that have undergone only one reflection at the water-sediment interface. This family of single bottom interacting rays always shows a characteristic four leg propagation as sketched in Fig. 2.4. These four paths together will be referred to as first bottom bounce rays. As can be seen in Figs. 2.5(a) and (b), the travel time difference between the direct and the first bottom bounce rays is large at near ranges but decreases with increasing range. The amplitude of the first bottom bounce rays is small at short ranges, increasing gradually with increasing range up to about 20 km, and then decreasing gradually with increasing range until it disappears at a range of approximately 50 km. In order to explain the disappearance of this group of arrivals at 50 km, we examine the ray trace diagram in Fig. 2.6(a). It can be seen that the maximum range to which the first bottom interacting rays appear is about 43 km. A ray with a grazing angle slightly less than that of this limiting ray turns even before reaching the water-sediment interface and reaches the receiver at a range of about 48 km.
Figure 2.4 Characteristic four path propagation of the single bottom interacting ray.
Figure 2.5 a.) Display of broadband data in waterfall format.

b.) Travel time curves obtained by ray tracing.
Figure 2.6 a.) Ray paths for the first bottom bounce rays.

b.) Ray paths for the second bottom bounce rays.

c.) Ray paths for the third bottom bounce rays.
Rays with grazing angles smaller than that of this ray appear at gradually smaller distances and thus constitute a backward travelling branch of the ray field. We include these rays in the family of single bottom interacting rays. The rays tend to converge near a range of 40 km where a decrease in propagation loss has been observed (Dosso & Chapman 1986). The ray-path diagram in Fig.2.6(a) clearly explains the disappearance of this group of arrivals beyond a range of about 50 km. The characteristic four arrivals with their associated bubble pulses make the first bottom bounce group of arrivals resemble a train of wavelets stretched out in time.

The rays following the first bottom bounce rays are the ones that reflect twice from the water/sediment interface and are referred to as second bottom bounce rays. These rays also show a characteristic four-path propagation similar to that of the first bottom bounce rays (Fig.2.4). It has been noted earlier that the first bottom bounce rays cut off near a range of about 50 km and hence beyond this range the first arriving phase in the sonograms is the second bottom bounce. Fig. 2.6(b) shows the ray trace for the second bottom bounce. It can be seen that the longest range at which the second bottom bounce is observed is about 82 km. This is the limiting ray for the second bottom bounce family of rays. Rays with lower grazing angles are obstructed by the ridge at a range of 79 km. Beyond this range is the shadow zone for this family of rays in which we expect to see diffraction arrivals originating at the top of the ridge.

The phases immediately following the second bottom interacting arrivals are the third bottom bounce arrivals which have been reflected three times at the water/sediment interface. The characteristics of this group of arrivals are very similar to those of the second bottom interacting group of arrivals. Fig.(2.6c) shows the ray-paths for this group of arrivals. The interesting feature of this group is that it can be observed up to a range of 78 km where it is cut off by the ridge. It reappears on the other side of the ridge at 90 km and can be seen from there out to a maximum range of 100 km. The shadow region contains appreciable diffracted energy. Proceeding in this manner, we were able to identify seven bottom bounce paths to a maximum range of 104 km. In most of the sonograms, the regular pattern of bottom bounce/channel rays-bottom reflected paths can be observed. However, as
we move into shallower water the regular pattern of bottom bounce paths breaks down. Especially in the region of 78-86 km range, the data lose their regular pattern. This is expected due to multiple scattering and diffraction from the ridge at a range of 79 km.

We next attempted to identify the diffraction arrivals by computing diffraction ray paths from the source to the receiver via the corner, P, of the ridge. This involved two steps: (i) for a defined generalized ray path, tracing ray from the source to the corner P; and (ii) tracing rays from the corner P to the plane containing the receiver. Recall that, as the first bottom bounce rays cut off at 50 km, the first ray family to excite the diffraction from corner P is the second bottom bounce. Therefore, we first traced rays for a specified family of rays (for example, second bottom bounce family) from the source to a plane at the depth of the corner P and then interpolated these to obtain the travel time for the ray from the source to the corner P (Fig. 2.7(b,c)). Next we shot a fan of rays from the corner P up to a depth of 22 m (Fig. 2.7(a)) and obtained a time distance chart. The travel time from the source to the corner P was added to this to obtain a diffraction travel time curve. This procedure was repeated for each group of bottom bounce rays. Two of the diffraction travel time curves thus obtained are marked DF in the travel time plot (Fig. 2.5(2)).

The regular pattern of bottom bounce rays is completely lost in the data beyond a range of about 106 km. To illustrate the wave propagation in this continental shelf region, Fig. 2.8 shows some rays from a source at a depth of 22 m and at a range of 110 km. It can be seen that at each bottom reflection, the grazing angle of the ray is reduced by twice the slope angle and that after a few successive reflections, the ray angle is sufficiently reduced to become continuously refracted within the water column. Thus between the source and the receiver, we have numerous ray paths continuously refracted in the water. As the propagation in this region is mostly by channel ray paths with minimal bottom interaction, the low propagation loss seen in the data and found by Dosso & Chapman (1986) is consistent with the ray picture.
Figure 2.7 a.) Second step in diffraction ray tracing.
b.) First step in diffraction ray tracing 1.
c.) First step in diffraction ray tracing 2.
Figure 2.8 Ray diagram a source at 110 km. range showing channel rays.
Figure 2.9 a.) Dosso & Chapman model.
  b.) Turning rays through sediments.
So far we have discussed only the water-borne ray paths. In addition to these, there is energy that enters the sediment layer, turns through it and reenters the water. Dosso and Chapman (1986) represented the ocean by a three layer model consisting of the water layer, a 150 m thick sediment layer and an absorbing basement (Fig. 2.9(a)). The sound speed structure of the ocean is shown in Fig. 2.2. Based on Chapman's (1984) result obtained by a deconvolution technique, the layer of unconsolidated sediments with an initial speed \( c_0 \) of 1540 m/sec. and a linear sound speed gradient of 1.6 per sec. were used. The \( c_0 \) and the gradient values suggest a clayey-silt composition, which is typical for a continental slope area where the sediments are mostly turbidites from terrestrial erosion (Hamilton, 1980). The density and compressional wave attenuation for this type of sediment are assumed to be 1.5 gm/cm\(^2\) and 0.1 db/ wavelength respectively. Whenever a ray bounces off the ocean bottom (in the pre-critical zone) some part of the energy is transmitted into the sediment. Based on the angle of reflection, the transmitted ray will either be absorbed in the basement or it may turn through the sediment and return to the receiver. Fig.2.9(b) shows the refracted rays through the sediment associated with the first bottom bounce family of rays. Beyond 14 km is the post-critical zone where no energy is transmitted into the sediment. These rays will also have the characteristic four-path propagation noted earlier in the context of bottom bounce paths. An interesting feature of this family of rays is that they develop a caustic surface during their turning through the sediments and therefore these phases, if observed, will show a phase shift of \( \pi/2 \). We were, however, unable to resolve this phase in the broad band data. The shallow refracted path through sea floor sediments plays a significant role in transmission of acoustic energy at low frequencies (Helmberger et.al. 1979).

The travel time curves obtained by ray tracing for different family of rays, as discussed earlier in the text (shown in Fig. 2.5(a)) were used to identify different phases in the data. We then applied static time shifts to each one of the traces and obtained the water fall plot shown in Fig. 2.5(b). Our ray trace results for this data set can be summarized as follows:
(i) In the deep ocean region, the propagation is via direct path, bottom bounce paths of different orders, and by refracted ray paths through sediments.

(ii) In the continental slope region, sound propagates by bottom bounce paths but there is a region of geometrical shadow past the ridge at 79 km. The geometrical shadow region is lit by diffractions from the corner P.

(iii) In the continental shelf region, the sound wave propagation is mostly by ray paths which are refracted through the water layer and by multiply reflected paths in the shallow parts of the ocean.

**Waveform modelling**

A realistic prediction of the received signal depends on two factors:

(i) Correct modelling of the source function; and

(ii) An adequate Green's function which contains all the properties of the ocean environment, namely the structure of the water column and the geoacoustic profile.

**Synthesis of source waveform**

The sources used in the experiment were 0.82 kg SUS charges (TNT) detonated at a depth of about 22 m. SUS charges have been used extensively in ocean acoustics experiments and their general characteristics and waveforms have been studied in considerable detail (e.g., Cole, 1948). The explosion generates a small sphere of gas at high temperature and pressure, and a spherically symmetric shock wave is radiated in the water. The first pulse is the shock wave pulse, which is followed by a series of bubble pulses with gradually decreasing amplitude. The shock wave dominates the radiation at frequencies above 200 Hz, whereas at lower frequencies the first bubble pulse dominates. The shock wave pressure history is represented by
an instantaneous pressure rise followed by an exponential decay, but the decrease at times greater than about two time constants is slower. The acoustic signal thus generated by TNT depends mainly on the depth of the explosion and mass of the explosive materials. Chapman (1985) analyzed the source waveform parameters as functions of source depth, range and charge weight used in explosion. Since direct measurement of source waveform parameters for shallow charges (such as the one used in this experiment) was not possible due to surface reflection, many parameters could only be obtained by extrapolation of parameter curves obtained from deeper charges. We first attempted to generate a source function that fit the measurements of Chapman (1985) by using different orders of polynomials. Unfortunately, it was not possible to fit all the parameters. Therefore, we decided to use an analytical expression given by Wakely Jr. (1978) to define the source function. The parameters used for the source function are given in the appendix.

A modified reflectivity algorithm

As noted earlier, the bathymetry in the range interval 0-60 km is fairly flat and sound speed profiles do not show appreciable variation with range. Therefore in the range 0-60 km, the medium can be assumed to be stratified; the sound speed profile in the ocean can be approximated by thin homogeneous layers with very small discontinuous jumps in velocity across interfaces.

The reflectivity method has long been used by seismologists for modeling both land and marine reflection and refraction data. The method, originally proposed by Fuchs and Muller (1971), was extensively modified by Kennett (1983). The calculations are done essentially in two steps:

(i) First, a reflectivity function \( R(\omega, p) \) is calculated in the ray-parameter (wave number)-frequency domain. This is done by a layer by layer iteration starting from the free surface down to the deepest layer over the half space. In each step of the iteration all orders of multiple bounce paths in the layer are included. At the final step, one obtains a reflectivity function that includes all possible ray paths from the source to the receiver.
(ii) The second step of the computation involves numerical evaluation of a double integral of the form

\[ u(x, t) = \frac{1}{2\pi} \int d\omega \exp(-i\omega t) \int dp f(\omega, p) \exp(\sigma g(p)) \]  

(26)

where \( \sigma = i \omega x \) and \( g(p) = p \).

The integration over frequency \( \omega \) is usually carried out by a fast Fourier transform (FFT). A complex frequency with constant imaginary part is used in the integral to attenuate the wrap-around caused by the use of an FFT. The correct amplitude of the computed time series is later restored by multiplying by \( \exp(Im(\omega)t) \). The use of complex frequency is explained in detail in Mallick and Frazer(1987).

The integrand of the integral over the ray parameter \( p \) is highly oscillatory and the use of trapezoidal rule would require a very small step size in \( p \). Use of a generalized Filon method (GFM) (Frazer, 1977; Frazer and Gettrust, 1984) allows one to use a much larger step size in \( p \). The sampling interval \( \Delta p \) depends both on frequency and the maximum range needed for calculation i.e., long range and high frequency calculations require a very small step size in \( p \) to avoid spatial aliasing.

The GFM scheme takes advantage of the fact that although the function \( f(\omega, p) \exp(\sigma g(p)) \) is highly oscillatory, each of the functions \( f(\omega, p) \) and \( g(p) \) is slowly varying. The trapezoidal analogue of the first order GFM formula fits a straight line between two quadrature points for the functions \( f(\omega, p) \) and \( g(p) \) separately thus allowing a much larger step size than that required by the conventional trapezoidal rule. Here, we note that the function \( f(\omega, p) \) contains the reflectivity matrix and includes all orders of multiples in each layer. This function includes terms of the form \( \exp(i\omega \sum_j q_j d_j) \) where \( q_j \) and \( d_j \) are the vertical slowness and thickness, respectively, in layer \( j \). The sum is taken over the entire ray path from the source to the receiver for different generalized rays. This term gives rise to oscillations in \( f(\omega, p) \) and thus \( \Delta p \) must be so chosen that the oscillations of \( f(\omega, p) \) are properly tracked along the path of integration. This \( \Delta p \) is, however still much bigger than that required by the conventional trapezoidal rule because only the
oscillations of $f(\omega, p)$ need be tracked; $f(\omega, p)$ oscillates much more slowly than $f(\omega, p) \exp(i\omega x)$ unless the product $\omega x$ is very small.

In the present problem, we are interested in modeling only the first bottom bounce phase. Our model consists of a thick water layer (2.5 km) over a thin sediment layer. Most of the propagation paths are bottom reflected and water borne paths, with some refracted/reflective phases from the sediment. The sound speed profile in the water varies very smoothly and continuously with depth.

Usually the reflectivity function (Kennett, 1983) contains all possible ray paths (including all orders of multiples) between the source and the receiver. In order to calculate only the first bottom bounce phase, we first modified the reflectivity function to include only paths 1-4 (Fig. 2.4). The reflectivity matrices for these four paths as shown in Fig. 2.10, are $R_D^{NR}R_D^{RS}$, $R_D^{NR}T_D^{RS}R_U^{FS}$, $R_U^{FR}R_D^{NR}T_D^{RS}$ and $R_U^{FR}R_D^{NR}T_D^{RS}R_U^{FS}$ respectively (Kennett, 1983).

Where,

$R_U^{FS}$ = Upward looking reflection coefficient matrix between the source and the free surface.

$T_D^{RS}$ = Downward looking transmission coefficient matrix between the source and receiver levels.

$R_D^{NR}$ = Downward looking reflection coefficient matrix between the receiver level and the bottom of the stack of layers.

$R_U^{FR}$ = Upward looking reflection coefficient matrix between the receiver level and the free surface.

These quantities are obtained by layer by layer iteration from the free surface down to the deepest interface.

A unique feature of our model is that the velocity varies very slowly with depth in the water column. Thus the plane wave reflection coefficients at the interfaces of each one of these thin layers are small except at low grazing angles. This means the integrand $f(\omega, p)$ behaves like the series

$$f(\omega, p) = \sum_k a_k(\omega, p)\exp(i\omega \phi_k)$$  \hspace{1cm} (27)
in which no individual $\phi_k$ differs greatly from $\sum_j q_j d_j$. More precisely, each term in the series satisfies

$$|\frac{\partial}{\partial p} \text{Re}(\phi_k - \sum_j q_j d_j)| \ll |\frac{\partial}{\partial p} \text{Re}(\phi_k)|.$$  

(28)

Under these circumstances it is efficient to remove some of the phase from $f(\omega, p)$ by rewriting the $p$-integral in equation (26) as follows:

$$u(\omega, x) = \int dp f_1(\omega, p) \exp(\sigma g_1(p))$$  

(29)

where,

$$f_1(\omega, p) = f(\omega, p) \exp(-i\omega \sum (q_j d_j))$$  

(30)

$$g_1(p) = (p + \sum (q_j d_j)/x)$$  

(31)

and

$$\sigma = i\omega x.$$  

(32)

In Equation (29), one part of the integrand is divided by $\exp(i\omega \sum(q_j d_j))$ while the other part is multiplied by the same factor. Thus the integrands in (26) and (29) are exactly the same; Equation (29) is simply a computationally more efficient way of writing equation (26).

Heterodyning an approximate phase factor out of $f(\omega, p)$ gives the function $f_1(\omega, p)$ which oscillates more slowly with $p$ allowing us to use a still bigger size in $\Delta p$ in the GFM integration. The key point is to use the proper phase factor for a chosen ray path so that the oscillations in $f(\omega, p)$ can be reduced. In our experiment, the source was close to the surface while the receiver was at a depth of 380 m, close to the sound channel axis. Hence the reflectivity integrands for paths 1 and 2 were summed before integration, while the integrands for paths 3 and 4 were integrated separately since the phase factors for these two groups of paths are significantly different.
Figure 2.10 Reflectivity matrices for paths 1, 2, 3 and 4.
Figure 2.11 Real & Imaginary parts of the integrand.

a.) \( f(\omega, p) \): original reflectivity.

b.) \( f_1(\omega, p) \): modified reflectivity.
Figure 2.12 Impulse response.

a.) Original reflectivity.

b.) Modified reflectivity.
As the vertical portion of the direct path between source and receiver is very small, there is no need to heterodyne phase out of the reflectivity function for the direct path.

To illustrate this technique, we show calculations for a model of a water layer over a half space in the frequency range 0-75 Hz. The water layer has the sound speed profile shown in Fig 2.13 except that the sub-bottom is homogeneous. Calculations are shown for paths 1 & 2. Figure 2.11(a,b) shows the real and imaginary parts of $f(\omega, p)$ at a frequency of 75 Hz as a function of the ray parameter $p$. The top part of the figure shows the usual form of $f(\omega, p)$ while the bottom part shows $f_1(\omega, p)$, the function obtained from $f(\omega, p)$ by heterodyning out part of the phase. It can be clearly seen that the approximate heterodyning out of the phase makes the function $f_1(\omega, p)$ very smoothly varying compared to $f(\omega, p)$. However, there still remain some rapid oscillations in $f_1(\omega, p)$, for $p$ values greater than 0.4 sec./km due probably to rapid changes in reflection coefficient with $p$. Accordingly we used a large $\Delta p$ in the range $0 \leq p \leq 0.4$ and used a small $\Delta p$ for $p$ values greater than 0.4. Fig. 2.12(b) shows the impulse response computed by this technique. In this example, for $0 \leq p \leq 0.4$, we used a $\Delta p$ five times larger than that used for $p$ values greater than 0.4. Figure 2.12(a,b) is a comparison of this new method with the conventional one and it can be noted that when we use conventional form of $f(\omega, p)$, the synthetics show spatial aliasing at much shorter range than do those calculated by the new algorithm.
Figure 2.13 a.) Model 1: Dosso & Chapman model.
b.) Model 2: I.T & A model.
c.) Model 3: New model.
Figure 2.13 d.) Synthetics for model 1.

e.) Synthetics for model 2.

f.) Synthetics for model 3.

g.) Data.
Results from waveform modeling

In this section, we explain the results from the waveform modeling of the first bottom bounce group of arrivals in the hydrophone data for the downslope propagation shot run 1 in the frequency range 0-100 Hz. The data were first filtered through a 5-100 Hz bandpass filter (6 poles) and were then passed through a 0-200 Hz Hanning window. As noted earlier, the first bottom bounce group of arrivals appear only in the range 0-52 km where the bathymetry is quite flat. The sound speed in the water was measured accurately but the sound speed, density, attenuation and thickness of the sediment were not well known. We began our waveform modeling by computing synthetic sonograms for some existing models.

MODEL 1: Dosso and Chapman (1986) model: Dosso and Chapman analyzed the same data set in the frequency domain. In their P.E results, they used the model shown in Fig 2.13(a). Their model consists of a water layer overlying a 150 m thick sediment, the initial sound speed and sound speed gradient in the sediment being 1.54 km/s and 1.6/s respectively, overlying an absorbing basement. As the P.E. method is acoustic, no information on the shear wave speed of the sediment was obtained. Figure 2.13(d) shows the reflectivity synthetics that we computed for this model. Although the overall match of the amplitude of the synthetics with those of the data is good, there are some obvious mismatches in the waveforms. The data show an arrival marked SS' closely following the first bottom reflection in the range 0-20 km with a moveout which does not change appreciably with range. This arrival makes the first reflection appear like a double doublet. This character is absent from the synthetics. However, we do see an arrival (marked SS') in the synthetics which gets cut off at a range of 14 km that has a moveout different from that of the SS' branch on the data. This is an arrival corresponding to the turning rays through the sediment (Fig. 2.9(b)). The sound speed contrast at the water/sediment interface is such that the rays become post-critical near a range of 14 km and beyond that range no turning rays could be observed.
MODEL 2: I.T & A model: Inverse theory & Applications Inc. (1985) carried out an inversion of a similar data set (one with a deep source) for a line about 15 min to the south of line 1. They used a Garmany $p - \tau$ inversion (Garmany 1979) which resulted in a model shown in Fig. 2.13(b). In their model the sound speed in the sediment rises sharply to 1600 m/s over the first 15 m and, with a gradient of about 2.3/s, rises to 1.76 km/s at a depth of about 80 m. At 150 m into the sediment, the sound speed appears to be 1.88 km/s. The synthetics computed for this model are shown in Fig. 2.13(e). It can be clearly seen that in the range 0-20 km, the match of the amplitude and waveform of the synthetics and the data is poor. The branch $SS'$ in the synthetics corresponds to turning rays through the top 80 m (high gradient zone) of the sediment layer.

MODEL 3: Since none of the existing models gave a very good match of either the amplitude or waveform of the synthetics with those of the real data, we computed a series of synthetics for a variety of models by changing the sediment parameters. The best model obtained by this trial and error technique is shown in Fig. 2.13(c). Our model shows a 40 m thick sediment layer underlying the water column. The compressional wave speed at the water/ sediment interface is 1.52 km/s; then it rises very slowly with a gradient of 0.05/s. At the bottom of the layer the compressional wave velocity jumps to 1.55 km/s. It is known that almost all marine sediments possess enough rigidity to transmit shear waves which are important in underwater sound propagation because compressional waves can be partially converted to shear waves and the energy is rapidly attenuated. We estimate a shear wave speed of $50\pm10$m/s in the layer which is smaller than that predicted from the compressional wave speed by the regression equation of Hamilton (1980). The $Q_p$ and $Q_s$ are respectively $50\pm10$ and $30\pm10$. It may be noted here that we use a frequency dependent power law for $Q$ (Strick 1967, 1970; Mallick & Frazer 1987) and a constant density of 1.5 gm/cc in the sediment. The modified Strick’s power law formula to calculate the frequency dependent complex velocity is given as
\[
\frac{1}{c(\omega)} = \frac{1}{c_\infty} + \frac{k_0}{\sin(\sigma \pi/2)(\epsilon - i\omega)^\sigma} \frac{1}{\sin((\omega + ic)(\pi/2))} \]

(33)

\[
-\pi/2 < \arg(\omega + ic) < 3\pi/2 \]

(34)

where

\(c(\omega)\) = complex velocity as a function of frequency \(\omega\).

\(c_\infty\) = velocity at infinite frequency.

\(\sigma\) = first attenuation parameter \((0 < \sigma < 1)\) equal to about 0.1 for most earth materials.

\(k_0\) = second attenuation parameter.

If \(Q\) is specified at the reference frequency \(\omega_0\), then \(k_0 \approx \omega_0^2/(2c_\infty Q(\omega_0))\)

\(\epsilon\) = third attenuation parameter. Set \(\epsilon=0.001\).

The variation of \(Q_p\) and \(Q_s\) with frequency in the sediment is shown in Fig. 2.15. The \(\sigma\) and reference frequency are 0.1 and 1 Hz respectively. The \(Q_p, Q_s\), compressional and shear wave speed beneath the layer are 300, 100, 1.55km/s and 200m/s respectively. The synthetics for this model (Fig. 2.13 f) show a fairly good agreement in both amplitude and waveform with those in the data (Fig.2.13 g). The branch \(SS'\) appears with the correct moveout. The branch \(SS'\) in the synthetics corresponds to the reflections from the bottom of the layer. Fig. 2.14 is an expanded plot of the data and the synthetics. The branch \(SS'\) in the data shows an excellent match with that in the synthetics for this model up to a range of about 16 km. The branch \(SS'\) can be marked on the data to a range of about 26 km while in the synthetics for model 3, this branch merges with the first bounce phase beyond a range of 16 km. We believe that this can be explained by gradual thickening of the sediment layer beyond a range of 20 km. Thickening of the sediment layer toward the coast is to be expected because the age of the crust increases towards the coast. In order to obtain a more accurate geoaoustic model of the seafloor some information on sediments from the core samples, are needed. As no cores are available at the site of the experiment, the model can not at present be any better constrained.
Figure 2.14 Expanded plot of data and synthetics.
Figure 2.15 Variation of $Q_p$ and $Q_s$ in the sediment.
Conclusions

A downslope propagation shot run line collected off the west coast of Vancouver Island has been analyzed in the time domain first by ray tracing using Hubley's algorithm and then by waveform modeling using a reflectivity technique. A modified reflectivity technique has been developed in which approximate phase factors for particular ray paths are used to heterodyne part of the phase out of the reflectivity function. This enables one to use a large step size in quadrature. The data in the frequency range 0-100 Hz show a distinct arrival, closely following the first bottom bounce, which appears to be caused by a thin sediment layer 40 m beneath the water column.
**APPENDIX: Source Function**

The SUS charge pressure waveform can be described by the following analytical expression (Wakeley Jr. 1978):

\[
s(t) = P_s \exp(-t/r_s) + \sum_{i=1}^{n}[P_i(\exp((t - T_i)/\tau_i^r) + \exp(-(t - T_i)/\tau_i^d))]
\]

\[
+ P_{-i}\sin\pi((t - T_{i-1})/(T_i - T_{i-1}))(H(t - T_{i-1}) - H(t - T_i))
\]

where

- \(P_s\) = shock wave peak pressure
- \(P_i\) = bubble pulse peak pressure of the i-th bubble
- \(P_{-i}\) = negative bubble pulse peak pressure of the i-th bubble
- \(T_i\) = period of the i-th bubble pulse
- \(T_0 = 0\)
- \(\tau_i^r\) = rise time constant of the i-th bubble pulse
- \(\tau_i^d\) = decay time constant of the i-th bubble pulse
- \(H(t)\) is the heavyside step function
  - \(H(t) = 0\) if \(t \leq 0\) and \(1\) if \(t > 0\)

The parameters used in the above expression were obtained mostly from Chapman's results for a 22 m source depth. Although this expression does not use all the parameters measured by Chapman (1985), it gives a fairly realistic source function. The following expressions were used for calculating the parameters used in the above equation.

Let \(w\) be the charge weight in kg, \(R\) the range in meters and \(Z_0\) the hydrostatic depth in meters (hydrostatic depth = shot depth + 10.1 in meters) then

- \(P_s = 5.04(10^{13})(w^{1/3}/R^{1.13})\)
- \(P_1 = 1.49(10^{12})(w^{1/3}/R)Z_0^{0.28}\)
- \(P_2 = 0.49P_1\)
- \(P_3 = 0.14P_1\)
- \(P_{-1} = 5.0(10^{10})(w^{1/3}/R)Z_0^{0.6}\)
\[ P_{-2} = 0.62P_{-1} \]
\[ P_{-3} = 0.6P_{-2} \]
\[ P_{-4} = 0.84P_{-3} \]
\[ T_1 = 2.11w^{1/3}(w^{1/3}/R)^{-0.14} \]
\[ T_2 = 0.80T_1 \]
\[ T_3 = 0.65T_1 \]
\[ \tau_s = 8.12(10^{-5})w^{1/3}(w^{1/3}/R)^{-0.14} \]
\[ \tau_r = 1.36(10^{-2})w^{1/3}Z_o^{-0.6} \]
\[ \tau_d = 0.87(10^{-2})w^{1/3}Z_o^{-0.6} \]

A plot of the broad band (0-750 Hz.) source function is shown in Fig.(2.16).
Figure 2.16 Broad band source function.
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APPENDIX A

Kirchhoff–Helmholtz reflection seismograms in a laterally inhomogeneous multi-layered elastic medium—I. Theory

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Summary. In a medium consisting of elastic layers with irregular interfaces, Kirchhoff–Helmholtz (KH) theory can be extended to synthesize the motion due to various generalized rays. An exact elastic form of the KH integral is first derived, then various asymptotic approximations are used to convert this integral into one which can be rapidly evaluated to give the motion of a single generalized ray. The approximations used are those of geometrical optics, for propagation across layers, and the Kirchhoff or tangent-plane approximation for propagation across boundaries. It is shown how the KH method leads naturally to a generalization of our usual notion of elastic reflection and transmission coefficients. The new coefficients are functions of both angle of incidence and angle of reflection or transmission and they are derived so as to obtain coordinate-free formulae that show clearly their relation to the conventional Snell’s law coefficients. The elastic KH method is applied first to the problem of a single interface, where its performance is compared to that of the Gaussian beam and Maslov methods. (For synthesizing reflections from irregular interfaces the KH method is superior because it includes signals diffracted from corners. However, when the interface is very smooth on the scale of a wavelength the Maslov and Gaussian beam methods are superior because they do not break down when there is a caustic on the reflector.) KH theory is then applied to a multilayered elastic medium and it is shown how the effects of frequency-dependent attenuation and dispersion can be incorporated into the theory by taking advantage of the approximately logarithmic variation of slowness with frequency in most earth materials. The limitations of the KH theory are discussed and some recent attempts to overcome these difficulties are reviewed. A new method for overcoming the problem of a caustic on the reflector becomes apparent when the KH integral is regarded as a member of a larger family of equivalent 1-fold integrals all of which are derivable from the same multifold path integral. Refracted or diving rays can be treated within the same formalism with equal benefit. For velocity models that are independent of one spatial direction (strike) a method is given for approximately converting 2-D results into 3-D results.
1 Introduction

The Kirchhoff-Helmholtz (KH) integral (Helmholtz 1860; Kirchhoff 1883) has been applied to many problems in wave propagation. Mow & Pao (1971) review its application in the diffraction of elastic waves by cylinders and spheres. Burridge (1962, 1963) used it to calculate reflections in liquid and solid spheres and recently Haddon & Buchen (1981) adapted Burridge's method to the synthesis of PKP. Scott & Helmberger (1983) used the KH integral to model body wave reflections from mountain topography and spall from nuclear blasts. Application of the method to the synthesis of finite frequency body wave synthetic seismograms in media with laterally inhomogeneous but continuous velocity was made by Sinton & Frazer (1981), Haddon (1982), Zherniak (1983), and Frazer & Sinton (1984).

In exploration seismology Hilterman (1970, 1975, 1982), Trorey (1970, 1977) and Berryhill (1977) have used the KH technique to model small offset reflection data for reflecting surfaces embedded in a homogeneous acoustic half-space. Extensions of these methods to the case of a laterally varying velocity were given by Hilterman & Larsen (1975), Berryhill (1979), Carter & Frazer (1983), and Deregowski & Brown (1983).

The elastodynamic form of the KH integral (Love 1904, 1944; de Hoop 1958; Wheeler & Sternberg 1968) has been applied to the calculation of reflected wavefields less often than the acoustic form. Until recently most reflection data were gathered with small lateral separation of sources and receivers so that very little conversion of compressional energy to shear energy occurred and losses due to mode conversion could be neglected. Also sources and receivers were designed to enhance compressional energy at the expense of shear wave energy. In this paper we are interested in wide angle reflections as well as near vertical reflections so that (even when shear wave arrivals are not computed) the loss in amplitude and change in phase of compressional arrivals due to shear conversion must be accounted for.

A derivation of the time domain form of the elastodynamic Kirchhoff integral can be found in Aki & Richards (1980). For completeness we include here a brief derivation of the frequency domain form of the integral that will be needed in the sequel. As shown in Fig. 1 let $V$ be an open volume in a 2- or 3-D elastic medium and let $\partial V$ be the boundary of $V$ with outward pointing unit normal $\hat{n}$. Let $f_1$ be some distribution of body force density which vanishes on $V$ and $\partial V$, and let $u_1$ and $\tau_1$ be the displacement and stress associated with $f_1$ so that these quantities satisfy the frequency domain momentum equation $-\rho \omega^2 u_1 = \nabla \cdot \tau_1 + f_1$. Also, let $f_2$ be some distribution of body force which vanishes both outside $V$ and on $\partial V$ and let $u_2$ and $\tau_2$ be the displacement and

\[ \text{Figure 1. A volume } V \text{ in } (E^2 \text{ or } E^3) \text{ with boundary } \partial V. \text{ The vector } \hat{n} \text{ is the outward-pointing unit normal to } \partial V. \text{ We wish to calculate the displacement at } x_1 \text{ due to a force } f_1 \text{ at } x_1. \]
Kirchhoff–Helmholtz reflection seismograms

stress associated with \( f_2 \), so that \(- \rho \omega^2 u_2 = \nabla \cdot \tau_2 + f_2\). Using the divergence theorem we write

\[
\int_{\partial V} \hat{n} \cdot (\tau_1 \cdot u_2 - \tau_2 \cdot u_1) \, dA = \int_V \nabla \cdot (\tau_1 \cdot u_2 - \tau_2 \cdot u_1) \, dV. \tag{1}
\]

However, for any vector \( u \) and second-order tensor \( \tau \), \( \nabla \cdot (\tau \cdot u) = (\nabla \cdot \tau) \cdot u + \tau : \nabla u \). (To interpret this last equation let \( \{x^i\}_{i=1}^3 \) be any system of coordinates for \( E^3 \). Then \( \nabla = \nabla x^i \partial / \partial x^i \) and \( \tau \cdot \nabla u = \nabla x^i \cdot \tau \cdot \partial_j u \) — e.g. Backus 1967.) Thus the integral on the right side of (1) becomes

\[
\int_V \left[ (\nabla \cdot \tau_1) \cdot u_2 - (\nabla \cdot \tau_2) \cdot u_1 + \tau_1 : \nabla u_2 - \tau_2 : \nabla u_1 \right] \, dV. \tag{2}
\]

But \( \tau_1 \) and \( u_1 \) are related by the fourth-order elastic tensor \( c \) and because of the symmetries of \( c \) we have \( \tau_1 \cdot u_2 = (c \cdot \nabla u_1) \cdot u_2 = \nabla u_1 \cdot c \cdot \nabla u_2 = c \cdot \nabla u_1 = \tau_2 \cdot \nabla u_1 \). Thus the last two terms in the integral (2) cancel each other. In the remainder of (2) we replace \( \nabla \cdot \tau_1 \) by \(- \rho \omega^2 u_1 - f_1 \) and \( \nabla \cdot \tau_2 \) by \(- \rho \omega^2 u_2 - f_2 \), respectively. Since \( f_1 \) vanishes on \( V \) and \( \partial V \) the integral (2) reduces to \( \int_V f_2 \cdot u_1 \, dV \). To summarize, we have shown that if \( f_1 \) vanishes on \( V \) and \( \partial V \), and if \( f_2 \) vanishes outside \( V \) and on \( \partial V \), then

\[
\int_V f_2 \cdot u_1 \, dV = \int_{\partial V} \hat{n} \cdot (\tau_1 \cdot u_2 - \tau_2 \cdot u_1) \, dA. \tag{3}
\]

When this relation is used for calculations the fictitious force \( f_2 \) is chosen in accordance with the nature of the actual receiver located at \( x_2 \). If there is a pressure sensor at \( x_2 \) then choosing \( f_2 = \nabla \delta (x - x_2) \) and using the fact that \( x_2 \) is an interior point of \( V \) we may write

\[
\int_V f_2 \cdot u_1 \, dV = \int_V u_1 \cdot \nabla \delta (x - x_2) \, dV
\]

\[
= \int_V \nabla \cdot \{ \delta (x - x_2) \, u_1 \} \, dV - \int_V \delta (x - x_2) \, \nabla \cdot u_1 \, dV
\]

\[
= - \nabla \cdot u_1(x_2).
\]

Then since pressure \( P \) is given by \(- k \nabla \cdot u \) where \( k \) is bulk modulus, the pressure field at \( x_2 \) due to \( f_1 \) is

\[
P_1(x_2) = k(x_2) \int_{\partial V} \hat{n} \cdot (\tau_1 \cdot u_2 - \tau_2 \cdot u_1) \, dA. \tag{4}
\]

If the detector at \( x_2 \) measures motion in the direction \( \hat{a}_2 \) then choosing \( f_2 = \hat{a}_2 \delta (x - x_2) \) yields

\[
\int_V f_2 \cdot u_1 \, dV = \int_V \delta (x - x_2) \, \hat{a}_2 \cdot u_1 \, dV = \hat{a}_2 \cdot u_1(x_2)
\]

and so

\[
a_2 \cdot u_1(x_2) = \int_{\partial V} \hat{n} \cdot (\tau_1 \cdot u_2 - \tau_2 \cdot u_1) \, dA. \tag{5}
\]
Figure 2. Application of the KH equations (4) and (5) to the calculation of waves reflected from a material discontinuity $\Sigma$. As $u_1$ is a reflected field it appears to originate from sources outside $V$.

In the sequel we will also use these formulae to calculate energy reflected from a material discontinuity which coincides with a part of the surface of integration as shown in Fig. 2. Here the volume $V$ contains both $x_1$ and $x_2$; however, at each point on the scattering surface $\Sigma$ the quantity $u_1$ in (4) and (5) is the reflected $P$- (or $S$-) wavefield, which appears to emanate from points outside of $V$. Thus, although the physical source point is inside $V$, the virtual source region is not, and so none of the assumptions used in the derivation above are violated. In Fig. 2, the reflected field $u_1$ does not vanish on $\partial V \setminus \Sigma$. However, since $\partial V \setminus \Sigma$ is very distant from $x_2$, the contribution to $u_1(x_2)$ of the integral over $\partial V \setminus \Sigma$ arrives much later in time than the contribution from the integral over $\Sigma$. Thus the former may be neglected.

If the scattered field $u_1$ and the incident field $u_2$ are exactly known on $\partial V$ then equations (4) and (5) hold exactly. However, the usefulness of these equations inheres in the fact that often $u_1$ and $u_2$ are easy to calculate on $L$ whereas $u_1(x_2)$ may be difficult to calculate directly because of the presence of caustics or multiple arrivals. In such circumstances equation (5) can be an efficient method of computing $u_1(x_2)$.

### 2 Plane wave theory

In Section 4 we will use equation (5) to derive formulae for the motion due to various kinds of generalized rays in 2- and 3-D models. These formulae will have the property that they give the geometrical optics solution for $u_1(x_2)$ whenever geometrical optics are valid but that, in addition, they also give a correct solution when $x_2$ is located on a caustic of the $u_1$ wavefield. To derive these formulae we assume that the length scale of the variation in $\Sigma$ is much greater than the wavelength of the signal. Thus near $\Sigma$ we may treat both $u_1$ and $u_2$ as if they were plane waves. To obtain the stress fields associated with these waves we use the constitutive relation $\tau = \lambda \nabla \cdot u + \mu [\nabla u + (\nabla u)^T]$ in which $\lambda$ and $\mu$ are Lamé parameters and $l$ is the identity tensor in $E^3$. For a $P$-wave with displacement $u^p$, local direction of propagation $t^p$, amplitude $A$ and local $P$-wave speed $\alpha$ we have

$$ u^p = A t^p \exp \{i \omega (t^p \cdot r)/\alpha \}, $$

$$ \nabla u^p = \frac{i \omega}{\alpha} t^p u^p, $$

$$ \nabla \cdot u^p = \frac{i \omega}{\alpha} u^p \cdot t^p $$

and

$$ \tau^p = \frac{i \omega}{\alpha} \{ \lambda (u^p \cdot t^p) t^p + 2 \mu u^p t^p \}. $$

(6)
Kirchhoff–Helmholtz reflection seismograms – I

For an $S$-wave with displacement $u^S$, local direction of propagation $t^S$, amplitude $A$ and local $S$-wave speed $\beta$ we have $u^S = A \exp \{i \omega (t^S \cdot r) / \beta \}$ where $t^S \cdot A = 0$, $\nabla u^S = (i \omega / \beta) t^S u^S$, $\nabla \cdot u^S = 0$ and

$$\tau^S = \frac{i \omega}{\beta} (t^S u^S + u^S t^S).$$  \hfill (7)

Then for two $P$-waves, $u^P_A$ and $u^P_B$, the integral of (5), henceforth referred to as the $P$ interaction, is

$$\hat{n} \cdot (\tau^P_A \cdot u^P_B - \tau^P_B \cdot u^P_A) = \frac{i \omega}{\alpha} u^P_A \cdot \{ \lambda (t^P_A \hat{n} - \hat{n} t^P_A) + 2 \mu (\hat{n} t^P_A - i \hat{n} \hat{n}) \} \cdot u^P_B$$  \hfill (8a)

and for two $S$-waves, $u^S_A$ and $u^S_B$, the $S$ interaction is

$$\hat{n} \cdot (\tau^S_A \cdot u^S_B - \tau^S_B \cdot u^S_A) = \frac{i \omega}{\beta} \mu u^S_A \cdot \{ \hat{n} \cdot (t^S - i \hat{n}) I + (\hat{n} t^S - i \hat{n} \hat{n}) \} \cdot u^S_B.$$  \hfill (8b)

![Figure 3. Two elastic half-spaces welded together along an irregular boundary.](image)

In our use of equation (8) $u_A$ will generally be a reflected wavefield of the kind shown in Fig. 3. We will use $u^P_1$ and $u^S_1$ to denote the $P$- and $S$-waves radiated by the source at $x_1$. The $P$- and $S$-wave reflections of $u^P_1$ from $\Sigma$ will be denoted $u^{PP}_1$ and $u^{PS}_1$, respectively and the $P$- and $S$-wave reflections of $u^S_1$ will be denoted $u^{SP}_1$ and $u^{SS}_1$, respectively. The reflected fields are obtained from the incident field by use of the invariant plane wave reflection formulae given in Appendix 1. For example let $u^{PP}_1 = A^{PP}_1 t^{PP}_1$ be the $P$-wave reflected by $\Sigma$ from $u^P_1 = A^{PP}_1 t^P_1$. Then $u^{PP}_1 = u^P_1 \cdot t^{PP}_1 t^{PP}_1$ in which $t^{PP}_1$ is one of the plane wave coefficients of Aki & Richards (1980) and $t^{PP}_1 = t^P_1 \cdot (1 - 2 \hat{n} \hat{n})$ where $\hat{n}$ is the local normal to $\Sigma$. We can use this relation to write the $P$ interaction for $u^{PP}_1$ and $u^P_2$ in terms of $u^P_1$ and $u^P_2$. Thus we obtain the $PP$ interaction

$$\hat{n} \cdot (\tau^{PP}_1 \cdot u^P_2 - \tau^P_2 \cdot u^{PP}_1) = u^{PP}_1 \cdot R^{PP}_{12} \cdot u^P_2;$$  \hfill (9a)

$$R^{PP}_{12} = t^{PP}_1 t^{PP}_2 \frac{i \omega}{\alpha} ^{PP}_1 \cdot \{ \lambda (t^{PP}_1 \hat{n} - \hat{n} t^{PP}_1) + 2 \mu (\hat{n} t^{PP}_1 - i \hat{n} \hat{n}) \}$$  \hfill (9b)

in which

$$t^{PP}_1 = t^P_1 \cdot (1 - 2 \hat{n} \hat{n}).$$  \hfill (9c)

Similarly for $u^{PS}_1$ and $u^S_2$ we have the $PS$ interaction

$$\hat{n} \cdot (\tau^{PS}_1 \cdot u^S_2 - \tau^S_2 \cdot u^{PS}_1) = u^P_1 \cdot R^{PS}_{12} \cdot u^S_2;$$  \hfill (10a)

$$R^{PS}_{12} = t^{PP}_1 t^{PP}_2 \frac{i \omega}{\beta} \mu \psi^{PS}_1 \cdot \{ \hat{n} \cdot (t^{PS}_1 - i \hat{n}) I + (\hat{n} t^{PS}_1 - i \hat{n} \hat{n}) \}$$  \hfill (10b)
in which
\[ \psi_{PS}^{12} = t_{1ps} \cdot (\sigma n - \hat{n} \sigma) \]
\[ \sigma = t_{1ps} \cdot (1 - \hat{n} \hat{n})/\alpha; \quad \sigma = || \sigma ||; \quad \hat{\sigma} = \sigma/\sigma. \] (10c, d)

For \( u_{1SP}^P \) and \( u_{1SP}^S \) we obtain the SP interaction
\[ \hat{n} \cdot (\tau_{1SP}^P \cdot u_{2SP}^P - \tau_{1SP}^S \cdot u_{1SP}^P) = u_{1SP}^S \cdot R_{12}^{SP} \cdot u_{2SP}^P \] (11a)
\[ R_{12}^{SP} = \psi_{1}^{SP} \hat{P} \left( \frac{i \omega}{\alpha} \right) t_{1SP}^P \cdot \frac{(\lambda(t_{1SP}^P \hat{n} - \hat{n} t_{2SP}^P) + 2 \mu(\hat{n} t_{1SP}^P - t_{2SP}^P))}{\alpha} \] (11b)
in which
\[ \psi_{1}^{S} = t_{1}^{S} \cdot (\hat{n} \hat{\sigma} - \sigma \hat{n}) \]
\[ t_{1}^{SP} = \sigma \sigma \hat{\sigma} - \sqrt{1 - \sigma^2} \hat{\sigma} \hat{n}, \quad \text{Im}(\sqrt{\sigma}) > 0; \] (11c, d)
\[ \sigma = t_{1}^{S} \cdot (1 - \hat{n} \hat{n})/\beta; \quad \sigma = || \sigma ||; \quad \hat{\sigma} = \sigma/\sigma. \] (11e, f, g)

And for \( u_{1SS}^P \) and \( u_{2SS}^S \) we obtain the SS interaction
\[ \hat{n} \cdot (\tau_{1SS}^P \cdot u_{2SS}^S - \tau_{1SS}^S \cdot u_{1SS}^S) = u_{1SS}^S \cdot R_{12}^{SS} \cdot u_{2SS}^S \] (12a)
\[ R_{12}^{SS} = \psi_{1}^{SS} \hat{P} \left( \frac{i \omega}{\beta} \right) \mu \left( \hat{n} \cdot (t_{1SS}^{S} - t_{1SS}^S) + (\hat{n} t_{1SS}^{S} - t_{1SS}^S) \right) \] (12b)
in which
\[ \psi_{1}^{S} = t_{1}^{S} \cdot (\hat{n} \hat{\sigma} - \sigma \hat{n}); \quad t_{1}^{SS} = t_{1}^{S} \cdot (1 - 2 \hat{n} \hat{n}); \quad \psi_{1}^{SS} = \psi_{1}^{S} \cdot (1 - 2 \hat{n} \hat{n}) \] (12c, d, e)
\[ \sigma = t_{1}^{S} \cdot (1 - \hat{n} \hat{n})/\beta; \quad \sigma = || \sigma ||; \quad \hat{\sigma} = \sigma/\sigma. \] (12f, g)
\[ \hat{n} = (t_{1}^{S} \times \hat{n})/||t_{1}^{S} \times \hat{n}||. \] (12h)

The quantities \( R_{12}^{PP}, R_{12}^{PS}, R_{12}^{SP}, R_{12}^{SS} \) will be referred to as interaction coefficients, or elastic Kirchhoff–Helmholtz reflection coefficients. Note that with these definitions \( R_{12}^{PP} \) is a function of \( t_{1}^{P}, t_{2}^{P} \) and \( \hat{n} \); \( R_{12}^{PS} \) is a function of \( t_{1}^{P}, t_{2}^{S} \) and \( \hat{n} \); \( R_{12}^{SP} \) is a function of \( t_{1}^{S}, t_{2}^{P} \) and \( \hat{n} \), and \( R_{12}^{SS} \) is a function of \( t_{1}^{S}, t_{2}^{S} \) and \( \hat{n} \). These definitions are completely coordinate-free so that to apply them in any particular coordinate system one needs only to have a formula for the dot product in that system. However, it should be emphasized that these interaction coefficients are asymptotic, being correct only at high frequencies where the radii of curvature of \( \Sigma \) are much larger than a wavelength of the signal. They give only the effect of a single interaction of the incident field with \( \Sigma \). Phenomena such as head waves, which involve multiple interactions with the surface, are not included.

3 Results from geometrical optics

In geometrical optics (Babich & Alekseev 1958; Karal & Keller 1959) one obtains the radiation field of a point source by tracing rays, to obtain travel times; and by assuming
that energy flux is conserved along ray tubes, to obtain amplitudes. We omit the derivations of the geometrical optics formulae as these can be found in many books and papers (e.g., Červený, Molotkov & Pšeničk 1977; Aki & Richards 1980; Frazer & Phinney 1980).

Consider first the problem of an elastic medium with a point source located at \( x_1 \). We assume that \( \alpha, \beta \) and \( \rho \) are smooth functions of spatial position and that the wavelength of the signal is small compared to the scale length of the variations in these parameters. Then the \( P \)-wave motion \( \mathbf{u}_P^1 \) and \( S \)-wave motion \( \mathbf{u}_S^1 \) propagate independently and we have for the motions in a 2- or 3-D medium

\[
\mathbf{u}_P^1(x) = \frac{i_P \Phi_P^{1\rho} \exp(i \omega T_P^1)}{\sqrt{\rho \alpha \alpha_1 \beta_1}} B_P^1
\]

and

\[
\mathbf{u}_S^1(x) = \frac{(b_1 \Phi_S^{1\rho} + c_1 \Phi_S^{1\beta}) \exp(i \omega T_S^1)}{\sqrt{\rho \beta \beta_1}} B_S^1
\]

In equation (13) \( T_P^1 = T_P^{1\rho}(x, x_1) \) is the tangent at \( x \) to the \( P \)-ray from \( x_1 \) to \( x \), and \( T_S^1 = T_S^{1\beta}(x, x_1) \) is the \( P \)-wave travel time from \( x_1 \) to \( x \). In the denominator the densities are \( \rho = \rho(x) \) and \( \rho_1 = \rho(x_1) \) and the \( P \)-wave velocities are \( \alpha = \alpha(x) \) and \( \alpha_1 = \alpha(x_1) \). The source term \( \Phi_P^{1\rho} = \Phi_P^{1\rho}(x, x_1) \) and spreading factor \( B_P^1 = B_P^{1\rho}(x, x_1) \) are discussed in detail below. The unit vectors \( \mathbf{b}_1 = \mathbf{b}(x, x_1) \) and \( \mathbf{c}_1 = \mathbf{c}(x, x_1) \) shown in Fig. 4(c) are normal to each other and normal to the unit vector \( \mathbf{t}_1^P = \mathbf{t}^P(x, x_1) \) which is tangent at \( x \) to the \( S \)-ray from \( x_1 \) to \( x \). The vectors \( \{\mathbf{t}_1^P, \mathbf{b}_1, \mathbf{c}_1\} \) are a frame field on the \( S \)-ray. Thus for rays in a 3-D medium \( \mathbf{b}_1 \) and \( \mathbf{c}_1 \) rotate about \( \mathbf{t}_1^P \) at a rate equal to the torsion of the ray. In the numerator of (14) \( F_P^1 = F_P^{1\rho}(x, x_1) \) and \( F_S^1 = F_S^{1\beta}(x, x_1) \) are source terms, and in the denominator \( B_S^1 = B_S^{1\rho}(x, x_1) \) is a spreading factor.

In a 3-D medium the spreading factors are

\[
B_P^1 = 4\pi \alpha_1 \sqrt{dA/d\Omega} \quad \text{and} \quad B_S^1 = 4\pi \beta_1 \sqrt{dA/d\Omega}
\]

where, as shown in Fig. 4(a), \( dA \) is the cross-sectional area at \( x \) of the ray tube which subtends solid angle \( d\Omega \) at the source. Since the ray tube for the \( P \)-wave will in general be different from that of the \( S \)-wave, \( dA/d\Omega \) refers to the \( P \)-wave in (15a) and to the \( S \)-wave in (15b). In a homogeneous medium \( \sqrt{dA/d\Omega} = || x - x_1 || \) for both \( P \) and \( S \). In a 2-D medium the spreading factors are

\[
B_P^1 = \sqrt{8\pi \omega \alpha_1 d/d\theta} \exp(-i\pi/4) \quad \text{and} \quad B_S^1 = \sqrt{8\pi \omega \beta_1 d/d\theta} \exp(-i\pi/4)
\]

where as shown in Fig. 4(b) \( d\theta \) is the cross-sectional width of the ray tube which subtends an angle \( d\theta \) at the source. Again \( d\theta/d\theta \) is in general different for \( P \) and \( S \)-waves. However, in a homogeneous medium \( d\theta/d\theta = || x - x_1 || \) for both \( P \) and \( S \). The quantities \( dA/d\Omega \) and \( d\theta/d\theta \) can be calculated either by integration of the ray equations (e.g., Červený & Hron

![Figure 4](image-url)

Figure 4. (a) Ray tube in three dimensions; (b) ray tube in two dimensions; (c) frame field on a ray.
or by numerical differencing (e.g. Sinton & Frazer 1982). An approximate method for converting 2-D solutions into 3-D solutions is given in Appendix B.

Equations (15a, b) and (16a, b) assume that the ray between $x_1$ and $x$ has encountered no caustics. If in fact the ray has encountered one or more caustics then $B$ must be multiplied by $\exp\left(-i\text{sgn}(\omega)\sigma(x, x_1)\pi/2\right)$ where $\sigma(x, x_1)$ is the KMAH index (Ziolkowski & Deschamps 1980). Here we need only remember that for 2-D problems the KMAH index of a ray is initially zero and increases by 1 every time the ray encounters a caustic.

In equations (13) and (14) the form of the terms $F_a^P$, $F_b^P$, and $F_c^P$ depends on the nature of the source. For a point force the equivalent body force density (Burridge & Knopoff 1964) is $f_1 = a(\omega)\delta(x - x_1)$ and we have

$$F_a^P = a \cdot t_1^P(x_1); \quad F_b^P = a \cdot b_1(x_1); \quad F_c^P = a \cdot c_1(x_1).$$

(17a, b, c)

In these relations $t_1^P(x_1)$ is the tangent at the source $x_1$ to the $P$-ray from $x_1$ to $x$. Similarly $b_1(x_1)$ and $c_1(x_1)$ are the normals to the $S$-ray from $x_1$ to $x$, evaluated at $x_1$. For a point double couple the equivalent body force density is $f_1 = M \cdot \nabla \delta(x - x_1)$ where $M = M(\omega)$ is the second-order symmetric moment tensor, and we have

$$F_a^P = \frac{i\omega}{\alpha_1} t_1^P(x_1) \cdot M \cdot t_1^P(x_1)$$

(18a)

$$F_b^P = \frac{i\omega}{\beta_1} b_1(x_1) \cdot M \cdot t_1^P(x_1)$$

(18b)

$$F_c^P = \frac{i\omega}{\gamma_1} c_1(x_1) \cdot M \cdot t_1^P(x_1).$$

(18c)

For a point explosion $f_1 = P(\omega) \nabla \delta(x - x_1)$ and

$$F_a^P = \frac{i\omega}{\alpha_1} P(\omega); \quad F_b^P = 0; \quad F_c^P = 0.$$  

(19a, b, c)

These last relations can be obtained from equations (18) by setting $M(\omega) = P(\omega)I$, where $I$ is the identity tensor. Finally we note that for any travel-time function $T(x)$ we have

$T(x) = \overline{t}(x)/v(x)$ where $v$ is the local propagation speed and $\overline{t}$ is the unit normal to the wavefront. Since pressure is $-\nabla \cdot \mathbf{u}$ the pressure field associated with equation (14) is zero and the pressure field associated with (13) is $(-i\omega/\alpha) F_a^P \exp(i\omega T_1^P)((\sqrt{\rho \mu_0} \alpha_1 B_1^P)$.

A scheme to obtain $d/d\theta$ by numerical differencing is shown in Fig. 5. Let $\Sigma$ be a line through the field point $x$ and $s$ denote distance along $\Sigma$. By shooting rays from $x_1$ to $\Sigma$ we

![Figure 5](image-url)

Figure 5. A differencing scheme for $dl/d\theta$. If $s$ denotes distance along $\Sigma$ then $dl/d\theta = \mathbf{n} \cdot \mathbf{t} ds/d\theta$. For the 3-D case see text.
Kirchhoff–Helmholtz reflection seismograms

obtain a function \( s(\theta) \). If \( \hat{n} \) is the normal to \( \Sigma \) at \( x \) and \( \hat{t} \) is the unit tangent to the ray then
\[
dl/d\theta = \hat{n} \cdot \hat{t} ds/d\theta.
\] (20)

In the 3-D case where \( \Sigma \) is a surface let \( s_1, s_2 \), be coordinates on \( \Sigma \) and \( f(s_1, s_2) \) the function such that \( dA = f(s_1, s_2) ds_1 ds_2 \) is an element of area on \( \Sigma \). By shooting rays we get \( s_1 = s_1(\theta, \phi), s_2 = s_2(\theta, \phi) \) where \( \theta \) and \( \phi \) are any spherical polar coordinates on the focal sphere \( \Omega \). Then
\[
dA/d\Omega = \hat{n} \cdot \hat{t} \left( (\partial s_1/\partial \theta)(\partial s_2/\partial \phi) - (\partial s_1/\partial \phi)(\partial s_2/\partial \theta) \right) f \{ s_1(\theta, \phi), s_2(\theta, \phi) \}/\sin \theta.
\] (21)

4 Reflected waves

4.1 A single interface

We are now in a position to write fairly simple expressions for the reflected wavefields from a surface \( \Sigma \) as shown in Fig. 3. In general, two inhomogeneous elastic media are in welded contact along \( \Sigma \). We substitute (9a) into (5) to obtain \( u_{PP} \), the field of the \( PP \) reflection, and then use (13) to get \( u_{PP}^1 \) and \( u_{PP}^2 \) under the integral.

\[
a_2 \cdot u_{PP}^1(x_2) = \int_{\Sigma(x)} u_{PP}^1(x) \cdot R_{12}^{PP} \cdot u_{PP}^2(x) d\Sigma(x)
\] (22a)

\[
= \frac{1}{\sqrt{\rho_1 \alpha_1 \rho_2 \alpha_2}} \int_{\Sigma(x)} \frac{F_1 F_2^{PP}}{\rho_1 \alpha_1} \hat{t}_1^* \cdot R_{12}^{PP} \cdot \hat{t}_2^* \frac{\exp[i\omega(T_1^P + T_2^P)]}{B_1^P B_2^P} d\Sigma(x).
\] (22b)

In this expression \( F_1^P \) is one of the forms (17a), (18a) or (19a), depending on our choice of source, but \( F_2^{PP} = a_2 \cdot \hat{t}_2^*(x_2) \) where \( \hat{t}_2^* \) is the unit tangent to the ray from \( x_2 \) to \( x \), evaluated at \( x_2 \) (our notation is as in Fig. 4c but with \( x_2 \) instead of \( x_1 \)). Since both sides of (22) are linear in \( a_2 \) we may suppress the factor \( a_2 \), and write just

\[
u_{PP}^1(x_2) = \frac{1}{\sqrt{\rho_1 \alpha_1 \rho_2 \alpha_2}} \int_{\Sigma(x)} \hat{t}_2^*(x_2) \frac{F_1}{\rho_1 \alpha_1} \hat{t}_1^* \cdot R_{12}^{PP} \cdot \hat{t}_2^* \frac{\exp[i\omega(T_1^P + T_2^P)]}{B_1^P B_2^P} d\Sigma(x).
\] (23)

Let us review the meaning of each of the symbols in (23). Outside the integral sign we have \( \rho_1 = \rho(x_1), \rho_2 = \rho(x_2), \alpha_1 = \alpha(x_1), \alpha_2 = \alpha(x_2) \). Inside the integral sign \( \hat{t}_2^*(x_2) \) is the tangent at \( x_2 \) of the ray from \( x_2 \) to \( x \). Thus \( \hat{t}_2^*(x_2) \) is a function of \( x \) as well as \( x_2 \). \( F_1^P \) is, as just noted, given by (17a), (18a), or (19a); in each of those equations there appears \( \hat{t}_2^*(x_1) \) which is the tangent at \( x_1 \) to the ray from \( x_1 \) to \( x \). Thus \( \hat{t}_2^*(x_1) \), and hence \( F_1^P \), is a function of \( x \) as well as \( x_1 \). The factor \( F_1^P \cdot R_{12}^{PP} \cdot \hat{t}_2^* \) means \( F_1^P(x) \cdot R_{12}^{PP}(x) \cdot \hat{t}_2^*(x) \) where \( R_{12}^{PP} \) is itself given as a function of \( \hat{t}_2^*(x) \), \( \hat{t}_2^*(x) \) and \( \hat{n}(x) \) by equations (9b, c), and \( \hat{n}(x) \) is the downward pointing unit normal to \( \Sigma \) at \( x \).

Note that \( \rho \) and \( \alpha \) inside the integral in (23), and \( \rho \) and \( \mu \) in equation (9b) are evaluated above the interface \( \Sigma \) and these quantities may vary along \( \Sigma \). The quantity \( PP \) in (9b) depends on velocities both above and below \( \Sigma \) and these too may vary. \( T_1^P \) and \( T_2^P \) are the \( P \)-wave travel times from \( x_1 \) to \( x \) and from \( x_2 \) to \( x \), respectively. If the integral is for three dimensions then according to equation (15a) \( B_1^P = 4\pi \alpha_1 \sqrt{dA_1(x)/d\Omega_1} \) and \( B_2^P = 4\pi \alpha_2 \sqrt{dA_2(x)/d\Omega_2} \). Here \( dA_1(x) \) is the cross-sectional area at \( x \) of the ray tube which subtends solid angle \( d\Omega_1 \) at \( x_1 \), and \( dA_2(x) \) is the cross-sectional area at \( x \) of the ray tube which subtends solid angle \( d\Omega_2 \) at \( x_2 \). Thus \( B_1^P \) and \( B_2^P \) can be obtained by shooting rays from
x₁ and from x₂ and differentiating according to equation (21). If the integral is for two
dimensions then B₁² is given by (16a) and B₂² by a similar expression and these expressions
can be obtained by differentiating according to equation (20). In two dimension dΣ = ds
where s is the curvilinear distance along Σ, but for three dimensions dΣ = f(s₁, s₂) ds₁ ds₂
where f is a function and s₁ and s₂ are the coordinates for Σ introduced just before equation
(21). Thus in either case dΣ is the element of area on Σ.

To obtain an expression for the PS reflected wave we use relation (13) to get u₁⁺, relation
(14) to get u₂⁺, and then substitute (10a) into (5) to obtain

\[ u₁^{PS}(x₂) = \frac{1}{\sqrt{\rho₁ \alpha₁ \rho₂ \beta₂}} \int _{Σ(x)} \frac{F₁^P}{\rho \sqrt{\alpha \beta}} \left\{ b₂ \hat{b}_2(x₂) + \hat{c}_2 \hat{c}_2(x₂) \right\} \]

\[ \exp[i \omega (T₁^P + T₂^F)] \times \frac{B₁² B₂²}{B₁² B₂²} dΣ(x). \]

(24)

For the SP reflected wave, proceeding in a similar manner, we find

\[ u₁^{SP}(x₂) = \frac{1}{\sqrt{\rho₁ \beta₁ \rho₂ \alpha₂}} \int _{Σ(x)} \frac{(F₁^P b₁ + F₂^P \hat{c}_1)}{\rho \sqrt{\alpha \beta}} \cdot R₁^{SP} \cdot \hat{t}_2 t_2(x₂) \]

\[ \exp[i \omega (T₁^P + T₂^F)] \times \frac{B₁² B₂²}{B₁² B₂²} dΣ(x). \]

(25)

and for the SS reflected wave we get

\[ u₁^{SS}(x₂) = \frac{1}{\sqrt{\rho₁ \beta₁ \rho₂ \alpha₂}} \int _{Σ(x)} \frac{(F₁^P b₁ + F₂^P \hat{c}_1)}{\rho \beta} \cdot R₁^{SS} \cdot \left\{ b₂ \hat{b}_2(x₂) + \hat{c}_2 \hat{c}_2(x₂) \right\} \]

\[ \exp[i \omega (T₁^S + T₂^S)] \times \frac{B₁² B₂²}{B₁² B₂²} dΣ(x). \]

(26)

Here R₁^{SS}, for example, is well defined in terms of \( \hat{t}_2(x) \), \( \hat{t}_2(x) \) and \( \hat{n}(x) \) by relations
(12b–h).

Equations (23)–(26) may be used when the surface Σ has corners or when the receiver is
located at a caustic, but they contain the asymptotic interaction coefficients and so they
give only the reflected field but not head waves or diving waves. Fortunately, computational
experiments using more exact methods (e.g. Fuchs 1971) have shown that even in most
refraction experiments the largest amplitude arrivals are generally post-critical reflections,
are less often refractions, and are almost never head waves. Refractions are discussed briefly
in Section 6 and will be treated in more detail elsewhere.

4.2 A MULTI-LAYERED MEDIUM

Up to this point we have assumed that the medium contains no interfaces other than Σ.
Now suppose the medium has above Σ two other interfaces A and B, as shown in Fig. 6.
Focusing effects associated with the curvature of interfaces A and B are automatically
included in relations (13) and (14) through the factors dA/dΩ or dl/dθ in equations (15)
or (16). However, to account for the loss of energy through reflections and mode changes
at A and B, relations (13) and (14) must be modified by the inclusion of plane wave trans­
m ission coefficients. Equation (13) becomes

\[ u₁^P(x) = \frac{\hat{t}_1^P(PP)_{LA}(PP)_{LB} F₁^P}{\sqrt{\rho \alpha \rho_1 \alpha_1}} \exp(i \omega T₁^P) \]

(27)
in which \((PP)_{1A}\) and \((PP)_{1B}\) are P-wave transmission coefficients through interfaces \(A\) and \(B\), respectively, and \(\rho_1\) and \(\alpha_1\) refer to the source while \(\rho\) and \(\alpha\) refer to the point \(x\) on \(\Sigma^+\), the upper side of \(\Sigma\). Equation (14) becomes

\[
\begin{align*}
\mathbf{u}^S_1(x) &= \{b_1(x) b_1(x_B^+) + \hat{c}_1(x) \hat{c}_1(x^+)\} \cdot \{\mathbf{w}^S_{1B}(\hat{V})_{1B} \mathbf{w}^S_1 + \hat{n}_{1B}(HH)_{1B} \hat{n}_{1B}\} \\
&+ \{b_1(x_B^+) b_1(x_B^+) + \hat{c}_1(x_B^+) \hat{c}_1(x_B^+)\} \cdot \{\mathbf{w}^S_{1A}(\hat{V})_{1A} \mathbf{w}^S_1 + \hat{n}_{1A}(HH)_{1A} \hat{n}_{1A}\} \\
&+ \{b(x_A^+) F^+_1 + \hat{c}(x_A^+) F^+_1\} \exp \left( \frac{i \omega \tau^+_1}{\sqrt{\rho \bar{\rho} \rho_1 \bar{\rho}_1}} B^+_1 \right).
\end{align*}
\]  

(28)

in which \((\hat{V})_{1A}\) and \((HH)_{1A}\) are, respectively, the downward SV and SH transmission coefficients through interface \(A\) with \((\hat{V})_{1B}\) and \((HH)_{1B}\) playing similar roles for interface \(B\). The vectors \(\mathbf{w}^S_{1A}\), \(\mathbf{w}^S_{1A}\), and \(\hat{n}_{1A}\) are an obvious extension of the notation introduced in Appendix A (Fig. A1) and are all well defined in terms of \(i^S(x_A^+)\) and the material parameters immediately above and below interface \(A\). The operator \(b_1(x) b_1(x_B^+) + \hat{c}_1(x) \hat{c}_1(x^+)\), for example, takes account of the curvature and torsion of the ray in getting from the bottom of \(B\) to the top of \(\Sigma\). If the material parameters change only across \(A\), \(B\), and \(\Sigma\) and are otherwise constant then these operators have no effect and may be suppressed. Equation (28) would then become

\[
\begin{align*}
\mathbf{u}^S_1(x) &= \{b_1(x_B^+) b_1(x_B^+) + \hat{c}_1(x_B^+) \hat{c}_1(x_B^+)\} \cdot \{\mathbf{w}^S_{1A}(\hat{V})_{1A} \mathbf{w}^S_1 + \hat{n}_{1A}(HH)_{1A} \hat{n}_{1A}\} \\
&+ \{b(x_A^+) F^+_1 + \hat{c}(x_A^+) F^+_1\} \exp \left( \frac{i \omega \tau^+_1}{\sqrt{\rho \bar{\rho} \rho_1 \bar{\rho}_1}} B^+_1 \right).
\end{align*}
\]  

(29)

In view of equation (27) our expression (23) for the reflected P-wave must now be modified to include a factor \((PP)_{1A}(PP)_{1B}(PP)_{2A}(PP)_{2B}\). Equations (24)-(26) must also be modified in obvious ways to include new factors like those obtained in going from (14) to (28). These modifications are straightforward but the resulting equations are very long and will be omitted. Henceforth we will assume that (23)-(26) have been modified whenever necessary to include the effects of interfaces shallower than \(\Sigma\). These modifications can of course include the effects of multiple reflections and mode changes anywhere above \(\Sigma\). All of the integrals (23)-(26) we have written have the form

\[
\mathbf{u}(x) = \int_{\Sigma(x)} \mathbf{u}_1 \cdot \mathbf{R}_{12} \cdot G_2 \, d\Sigma(x)
\]  

(30)

in which \(G_2\) is the elastic Green's tensor obtained from (13) or (14) by suppressing the 'a' in \(F^a, F^b, F^c\). Here \(\mathbf{u}_1\) is the wave from the source down to \(\Sigma^+\) and \(G_2\) is the wave from the
receiver down to $\Sigma^-$ (since the normal to $\Sigma$ points down we use $\Sigma^-$ to denote the upper side of $\Sigma$ and $\Sigma^+$ to denote its lower side). The interaction coefficient $R_{12}$ in (30) is chosen to agree with the modes of $u_1$ and $G_2$ on $\Sigma^-$ without regard to the mode of these waves in the shallower portions of their respective paths. For example, with reference to Fig. 6, if $u_1$ is the wave which travels from $x_1$ to $B$ as an $S$-wave and from $B$ to $\Sigma$ as a $P$-wave, and $G_2$ is the wave which travels from $x_2$ to $B$ as a $P$-wave and from $B$ to $\Sigma$ as an $S$-wave then $R_{12}$ in (30) is $R_{12}^{pS}$ given by equations (10b–g). $u_1$ and $G_2$ may even represent waves which have been reflected one or more times from $\Sigma$ before interacting, as shown in Fig. 7.

Figure 7. (a) The generalized ray whose response is to be synthesized; (b) first choice for $u_1$ and $u_2$; (c) a second choice for $u_1$ and $u_2$. Here the $AB$ multiple has been incorporated into $u_1'$ instead of $u_1$.

4.3 ATTENUATION AND DISPERSION

In an attenuating medium the Lamé parameters $\lambda$ and $\mu$ become frequency dependent and complex; hence so do the seismic velocities $\alpha$ and $\beta$. All the quantities in the geometrical optics expressions (13) and (14) remain well defined except for travel time $T$ and $dA/d\Omega$ (or $dl/d\theta$). To define these quantities we assume that the geometrical optics raypath is the same for all frequencies. Then $dA/d\Omega$ is frequency-independent; also, we obtain a frequency-dependent, complex travel time for the $P$-wave, say, by evaluating the ray path integral $T^P(\omega) = \int ds/\alpha(\omega)$ over the same path for each frequency. This definition of $T^P(\omega)$ accords with Fermat's principle, which states that, on any geometrical optics path, travel time is stationary with respect to small perturbations in the path. Thus the travel time $\int ds/\alpha(\omega)$ is nearly the same over our fixed path as it is over the true, frequency-dependent, path.

Even integrating over a frequency-independent path, evaluation of the travel times in (23)–(26) would be time consuming if many frequencies were needed. However, suppose the seismic quality factor $Q$ is spatially homogeneous between the interfaces of the multi-layered model shown in Fig. 6. Then, even if the velocity itself is not spatially homogeneous there, we may write for the layer between, say, $A$ and $B$

$$\frac{1}{\alpha(r, \omega)} = \frac{1}{\alpha_0(r)} \left[ 1 - \frac{1}{\pi Q_{AB}} \ln \left( \omega/\omega_0 \right) \right] \exp \left( \frac{i}{2Q_{AB}} \right) \quad (31)$$

where we have assumed $\omega > 0$ and $(1/\pi Q_{AB}) \ln (\omega/\omega_0) < 1$ (e.g. Kanamori & Anderson 1977). Here $Q_{AB}$ and $\omega_0$ are fixed parameters and $\alpha_0(r)$ is independent of frequency. Thus for the leg of the ray path between $A$ and $B$

$$\int ds = \left[ 1 - \frac{\ln (\omega/\omega_0)}{\pi Q_{AB}} \right] \exp \left( \frac{i}{2Q_{AB}} \right) / \alpha_0 \quad (32)$$

that is, all the frequency-dependent factors may be taken outside the integral. Doing likewise
for the other legs of the path we find that the total travel time $T_f(x)$ in equation (13) may be written

$$T_f(x_1, \omega) = f_{1A}(\omega) T^p_{1A}(x) + f_{AB}(\omega) T^p_{AB}(x) + f_{B\Sigma}(\omega) T^p_{B\Sigma}(x)$$

(33)

where $f_{1A}$, for example, is the coefficient of the integral on the right side of (32). Here $x$ is, as usual, a point on $\Sigma$. The significance of (33) is that if we are willing to store the three functions $T^p_{1A}(x)$, $T^p_{AB}(x)$, $T^p_{B\Sigma}(x)$ separately (instead of storing their sum, as we would in the lossless case) then we can quickly recover $T_f(x, \omega)$ for each frequency. Of course, if the quality factor $Q$ is the same in each layer then in (33) $f_{1A} = f_{AB} = f_{B\Sigma}$ and we need only save the sum of the travel times, just as in the lossless case.

4.4 Limitations of the Method

As noted above an advantage of Kirchhoff–Helmholtz (KH) theory over geometrical optics is that the former gives a correct response when the receiver is located on a caustic whereas the latter does not. However, this advantage also obtains with other methods for synthesizing reflections. The EWKBJ (Frazer & Phinney 1980) or Maslov method (Maslov 1965; Chapman & Drummond 1982) allows the receiver to be located on a caustic and so does the Gaussian beam method (e.g. Červeny 1983). These two methods are probably superior to KH theory for modelling refracted waves in a medium without interfaces (Sinton & Frazer 1981; Haddon 1982, 1983). For modelling reflections, however, KH theory is probably superior to both of these other methods because of its more correct rendering of shadow arrivals and diffractions. To see why this is so consider the situation shown in Fig. 8, where a fault in a reflective interface has caused the primary reflection wavefield to divide into two branches separated by a shadow zone. Without loss of generality we may for convenience take the velocity to be unity above the reflector and take the reflector to be flat except at the fault. Travel-time curves for the synthetics that would be obtained with various methods are also shown in Fig. 8. For reference: ABE is the reflection branch that would be obtained; in the absence of a fault, from an infinite length horizontal reflector passing through $P$; LHJ is the reflection that would be obtained from a single infinite length horizontal reflector passing through points $Q$ and $R$. The KH method gives branches ABD, CB, GHJ and HI with the arrivals on branches CB and HI opposite in polarity to arrivals on the other branches. Hilterman’s (1970) experiments with physical models and sources shows that these arrivals

![Figure 8. Travel-time curves of arrivals given by different methods of constructing synthetic seismograms for a model consisting of a faulted interface. See text for discussion.](image-url)
and polarities are correct. Geometrical optics gives only branches AB and HJ and so is correct only in the limit of infinite frequency. The EWKBJ/Maslov method gives branches ABF and KHJ; BF is a straight line tangent to ABE at x₁ and KH is a straight line tangent to LHJ at x₂. The Gaussian beam method also gives branches ABF and KHJ. Branches BF and KH are incorrect, although for low frequencies the large beam width makes this difficult to detect (e.g. Červený 1983, figs 8 and 9). To see why the Maslov and Gaussian beam methods give spurious branches BF and KH note that branch BF is associated with the single ray (plane wave) which travels from the source to the upthrown corner of the fault P hence to x₁, and the branch KH is associated with the ray which leaves the source at the same angle but just misses the fault corner and is reflected at R up to x₂. Branch BF and branch KH are straight and parallel because they are associated with essentially the same plane wave. It has often been pointed out that the fundamental limitation of KH theory is that the reflecting point must be contained in an area of surface whose linear dimensions and radii of curvature are large compared to a wavelength. The example of Fig. 8 indicates that the key word in this statement is 'contained' since KH theory works well when the reflecting point is on the boundary of such an area but the other ray methods (geometrical optics, Maslov, Gaussian beams) work only when the reflecting point is near the centre of such an area.

In any comparison of asymptotic methods correctness is, of course, a relative term. For instance, a very careful comparison of data with KH synthetics for arrivals on branches CB and HI in Fig. 8 would reveal a lack of agreement in the phase shift at low frequencies due to the fact that the region of the fault from Q to R is not truly shadowed but instead is lit by energy diffracted from the corner P. Also, depending on the nature of the velocity jump across the interface in Fig. 8, we would expect to see a head wave arrival from either the upthrown side of the interface or the downthrown side. The KH theory presented here will not give this arrival because it assumes only a single interaction with the interface. This lack of head waves will be particularly serious in a situation like that shown in Fig. 9 where the upthrown side of the interface is rounded off. If velocity increases across the interface then the incident field will excite a rather strong whispering gallery wave which will radiate upwards; but this wave will be absent from our KH reflection synthetics. Stephen (1984) has observed such a wave in marine data and modelled it using a finite difference method. Such waves could also be modelled using the boundary integral equation method (e.g. Cole, Kosloff & Minster 1978).

The most significant limitation of the KH theory outlined above is one to which the Maslov and Gaussian beam methods are not subject. It is encountered when the wavefields $u_1$ and $u_2$ in the integral (3) cannot be represented everywhere on $\Sigma$ by the geometrical optics equations (13) and (14). Consider the situation shown in Fig. 10. There the velocity structure is such that the $u_f^p$ rayfield (from the source) has two caustics on $\Sigma$. The geometrical optics representation of $u_f^p$ on $\Sigma$ is a sum of three terms of the form (13). At each of the caustics on $\Sigma$, two of the three terms will be singular due to a zero of their

![Figure 9](image-url)
spreading factors. When the representation for \( u_1'' \) is substituted into (22a) the resulting integrand is therefore a sum of three terms each of which has a different phase function and each of which is singular at one or two points on \( \Sigma \). It can be seen that in general, if \( u_1 \), the wave from the source to \( \Sigma \), is a sum of \( m \) terms of the form (13) and \( u_2 \), the wave from the receiver to \( \Sigma \), is a sum of \( n \) terms of the form (13) then the resulting integrand of (22a) is a sum of \( mn \) terms each of which has a different phase function and is singular at one or two points on \( \Sigma \). Let us denote such a term by \( f_k \exp(i\phi_k) \), \( 1 < k < mn \); then \( \phi_k = \omega(T_{1i} + T_{2j}) \) for some \( i \) and \( j \) in the ranges \( 1 < i < m \) and \( 1 < j < n \), respectively.

Each reflection arrival from \( \Sigma \) is associated with a stationary point of some function \( \phi_k \). Thus if no \( \phi_k \) is stationary near the singularities of its \( f_k \) we can compute all of our arrivals by limiting the range of integration to exclude the singularities. However, limiting the range of integration is unsatisfactory in several respects: it requires determining the location of stationary points before integrating; it gives synthetics with unreliable amplitudes, or truncation phases; and there is no guarantee that a stationary point of some \( \phi_k \) will not coincide with a singularity of \( f_k \). Another approach to this problem was taken by Sinton & Frazer (1981), who treated the singularities of each \( f_k \) as integrable and integrated over them. This procedure makes it unnecessary to locate the stationary points of the \( \phi_k \) before integrating; however, it too is unsatisfactory in several respects: the integrable singularities cause small spurious arrivals; if a stationary point of \( \phi_k \) coincides with a singularity of \( f_k \) then the amplitude of the arrival associated with that stationary point is unreliable; and, finally, not all the singularities of geometrical optics formulae are integrable. In a variation on this approach Sen & Frazer (1983) proposed that the integrable singularities could be removed from each term \( f_k \exp(i\phi_k) \) by convolving each \( f_k(x) \) with a smoothing operator whose width is a decreasing function of frequency. This enlarges the domain of \( f_k(x) \) and so the phase function \( \phi_k(x) \) must be extrapolated to cover this enlarged domain. This procedure has a number of advantages: the spurious phases associated with the singularities are no longer present; the integrand is still a sum of terms \( f_k \exp(i\phi_k) \) each of which has a well-defined phase function \( \phi_k \); we know that physically the singularities do not exist at finite frequencies, therefore any method of quelling them is better than not quelling them at all; on the other hand it seems clear that if we are to average \( f_k \) in order to quell its singularities we ought to average it over a wavefront instead of a surface on which \( \phi_k \) also varies. If we averaged over a wavefront our frequency-dependent smoothing operator would then play a role similar to the beam amplitude profile in the Gaussian beam theory. Deregowski & Brown (1983) have used such a scheme in numerical calculations; they average over the surface containing the receivers instead of a wavefront, and do not bother to extend the domain of \( \phi_k(x) \).
5 One-fold path integrals

Earlier we noted that the KH integral for the generalized ray shown in Fig. 7(a) could be written in two ways. In the first way we write

\[ \tilde{a}_2 \cdot \mathbf{u}_1(x_2) = \int_{\Sigma} \mathbf{u}_1 \cdot R_{12}^{pp} \cdot \mathbf{u}_2 \, d\Sigma \]  \hspace{1cm} (34)

using the interaction coefficient given by (9b) and the geometrical optics expressions for \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) shown in Fig. 7(b). In the second way we write

\[ \tilde{a}_2 \cdot \mathbf{u}_1(x_2) = \int_{\Sigma} \mathbf{u}_1' \cdot R_{12}^{SS} \cdot \mathbf{u}_2' \, d\Sigma \]  \hspace{1cm} (35)

using the interaction coefficient given by (12b) and the geometrical optics expressions for \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) shown in Fig. 7(c). If none of the rayfields in Fig. 7 has a caustic on \( \Sigma \) then (34) and (35) are equivalent. However, it may happen that \( \mathbf{u}_1 \) or \( \mathbf{u}_2 \) has a caustic on \( \Sigma \) whereas \( \mathbf{u}_1' \) and \( \mathbf{u}_2' \) do not. Then equation (35) will have an integrand consisting of a single term of the form \( \exp(i\omega \phi) \) where \( f \) is regular, whereas equation (34) will have an integrand of the form \( \sum_k f_k \exp(i\phi_k) \) where each \( f_k \) is singular. That is, equation (35) will be accurate and easy to evaluate by the method of this paper but equation (34) will be plagued by all the difficulties described in Section 4.4. Frazer (1983) showed that (34) and (35) are both one-fold path integrals that can be derived from the same nine-fold path integral by different applications of the method of stationary phase. [Frazer (1983) referred to these path integrals as Feynman path integrals (Feynman & Hibbs 1964), but this is incorrect. An actual application of the Feynman technique to the wave equation is given by Schulman (1981, p. 164).] Details of the theory of multi-fold integrals for reflection problems will be the subject of a separate paper but we summarize the theory here in order to exhibit the relation between (34) and (35) and to show how to find other one-fold integrals for the same generalized ray.

For simplicity consider once more the P-wave shown in Fig. 6(a). The path shown in Fig. 6(a) is a geometrical optics path; actually the energy in the P-wave reflected from \( \Sigma \) may be regarded as travelling from the source to every point on \( A \), from every point on \( A \) to every point on \( B \), from every point on \( B \) to every point on \( \Sigma \) then from every point on \( \Sigma \) upward to every point on \( B \), from every point on \( B \) to every point on \( A \) and from every point on \( A \) to the receiver. Paths such as the one shown in Fig. 11(a) are thus legitimate.

The integral giving the contribution of all such paths may be written

\[ \tilde{a}_2 \cdot \mathbf{u}_1(x_2) = \int_A dA \int_B dB \int_\Sigma d\Sigma \int_B dB' \int_A dA' \frac{f(x_1, x_A, x_B, x_\Sigma, x_B', x_A', x_2)}{R_{1A}R_{AB}R_{B\Sigma}R_{\Sigma B}R_{BA}R_{A2}} \times \exp \left\{ i\omega \left( \frac{R_{1A}}{\alpha_A} + \frac{R_{AB}}{\alpha_B} + \frac{R_{B\Sigma}}{\alpha_\Sigma} + \frac{R_{\Sigma B}}{\alpha_B} + \frac{R_{BA}}{\alpha_A} + \frac{R_{A2}}{\alpha_A} \right) \right\} \]  \hspace{1cm} (36)

where for brevity we have taken the velocity to be constant between interfaces. In this expression \( R_{1A} = || x_1 - x_A ||, R_{AB} = || x_A - x_B ||, R_{B\Sigma} = || x_B - x_\Sigma ||, R_{\Sigma B} = || x_\Sigma - \)
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Figure 11. (a) One of the paths summed in the five-fold path integral (36) for the P-wave of Fig. 6(a); (b) the one-fold integral that would be obtained if the first four integrals in (36) were evaluated simultaneously by the method of stationary phase. Similarly for: (c) the first, second, third and fifth integrals; (d) the first, second, fourth and fifth integrals; (e) the first, third, fourth and fifth integrals; (f) the second, third, fourth and fifth integrals.

Fortunately we do not actually have to write out the exact form of (36) and perform the stationary phase evaluation of four integrals to get whichever one-fold integral we seek. We can use the KH formalism to obtain the one-fold integral directly. For example, to obtain the one-fold integral which uses the \( u_1 \) and \( u_2 \) shown in Fig. 11(c) note first that since \( u_1 \) has been reflected upward from \( \Sigma \) it appears to emanate from a virtual source distribution below interface \( B \).

By an analysis very similar to that given in Section 2 we obtain

\[
\hat{n} \cdot (\tau_1^P \cdot u_2^P - \tau_2^P \cdot u_1^P) = u_1^P \cdot D_{12}^{PP} \cdot u_2^P
\]  

(37a)

in which

\[
D_{12}^{PP} = \left( \lambda (t_1^{P \cdot \hat{n}} - n t_1^{P \cdot \hat{n}}) + 2 \mu (t_1^{P \cdot \hat{n}} - t_2^{P \cdot \hat{n}}) \right) \cdot t_2^{PP} \frac{i\omega}{\alpha} (PP)_2 \cdot t_2^P.
\]  

(37b)

To make each term in (37b) well-defined we denote the \(-\hat{n}_B \) (upper) side of surface \( B \) by \( B^- \) and the \(+\hat{n}_B \) (lower) side of \( B \) by \( B^+ \). Since \( \lambda, \mu \) and \( \alpha \) are discontinuous across \( B \) we denote values of \( \lambda \) on the upper side of \( B \) by \( \lambda(x_B^-) \) and values of \( \lambda \) on the lower side of \( B \) by \( \lambda(x_B^+) \) and similarly for \( \mu \) and \( \alpha \). In (37b) \( \lambda = \lambda(x_B^-), \mu = \mu(x_B^-), \alpha = \alpha(x_B^-) \), \((PP)_2 \) is for
Figure 12. Using KH theory to calculate the one-fold integral for Fig. 11(c). We take \( B \subset \partial V \) and note that \( u_1 \) appears to emanate from outside \( V \). As usual the integration over \( \partial V \setminus B \) is neglected.

A P-wave incident from above \( B \) with direction of propagation \( \tau^P_2 \) (see Appendix A), and

\[
\tau^P_2 = \alpha(x_B^*) \sigma + \sqrt{1 - \alpha(x_B^*)^2} \sigma^2 \hat{n};
\]

(37c)

\[
\sigma = \tau^P_2 \cdot (1 - \hat{n})/\alpha(x_B^*), \quad \sigma = || \sigma ||, \quad \sigma = \sigma/\sigma.
\]

(37d)

Then

\[
\hat{a}_2 \cdot u_1(x_2) = \int_B \mathbf{u}^P_1 \cdot \mathbf{D}^{PP} \cdot \mathbf{u}^P_2 dB
\]

(37e)

is the one-fold integral for Fig. 11(c).

It is interesting to compare (37b) with equation (9b). In deriving (37b) we used a plane wave transmission coefficient to transmit \( u_2 \) downward through surface \( B \), then used the \( PP \) interaction equation (8); in deriving (9b) we used a plane wave reflection coefficient to reflect \( u_1 \) upward from \( \Sigma \) and then used equation (8). We could have obtained a different looking expression for (9b) by reflecting \( u_2 \) instead of \( u_1 \) and we can obtain an alternative expression for (37b) by transmitting \( u_1 \) instead of \( u_2 \). Doing so yields

\[
\hat{n} \cdot (\tau^{PP}_1 \cdot u^P_2 - \tau^{PP}_2 \cdot u^{PP}_1) = u^P_1 \cdot E^{PP}_{12} \cdot u^P_2
\]

(38a)

in which

\[
E^{PP}_{12} = \frac{i \omega}{\alpha} \tau^{PP}_1 \cdot \left\{ \lambda (\tau^{PP}_1 \hat{n} - \hat{n} \tau^P_1) + 2 \mu (\hat{n} \tau^P_1 - i \tau^P_1 \hat{n}) \right\}
\]

(38b)

where \( \lambda = \lambda(x_B^*), \mu = \mu(x_B^*), \alpha = \alpha(x_B^*), (\hat{P}\hat{P})_2 \) is for a P-wave incident from below \( B \) with direction of propagation \( \tau^P_1 \), and

\[
\tau^{PP}_1 = \alpha(x_B^*) \sigma \hat{\sigma} \cdot \sqrt{1 - \alpha(x_B^*)^2} \sigma^2 \hat{n}
\]

(38c)

\[
\sigma = \tau^{PP}_1 \cdot (1 - \hat{n})/\alpha(x_B^*), \quad \sigma = || \sigma ||, \quad \hat{\sigma} = \sigma/\sigma.
\]

(38d)

Then

\[
\hat{a}_2 \cdot u_1(x_2) = \int_B \mathbf{u}^P_1 \cdot E^{PP}_{12} \cdot \mathbf{u}^P_2 dB
\]

(38e)
Figure 13. The generalized ray in (a) may be written as a nine-fold path integral (equation 39) or a one-fold integral (KH integral) using any one of the geometrical optics ray pairs in (b)-(j).

is also the one-fold integral for Fig. 11(c). Although (37b) and (38b) look quite different they take the same values on neighbourhoods of stationary points of (37e) and (38e). Thus they are identical within the accuracy of the rest of our theory. The other interaction coefficients for transmission, \(D_{12}^{PS}, D_{12}^{SP}, D_{12}^{SS}\) (or equivalently \(E_{12}^{PS}, E_{12}^{SP}, E_{12}^{SS}\)) are straightforward to obtain using equation (8) and (9) and will be omitted.

We return briefly to the more complicated generalized ray of Fig. 7(a), repeated in Fig. 13(a). The full-fold path integral for this ray is

\[
a_2u_1(x_2) = \int dA \int dB \int d\Sigma \int dB' \int d\Sigma' \int dB'' \int dA' \int dB''' \int dB'''' \int dA'''
\]

\[
x \times \exp \left( i \omega \left( \frac{R_{1A}}{\alpha_A} + \frac{R_{AB}}{\beta_B} + \frac{R_{B\Sigma'}}{\beta_\Sigma} + \frac{R_{\Sigma'B'}}{\beta_\Sigma} + \frac{R_{\Sigma''B}}{\alpha_\Sigma} + \frac{R_{B'A}}{\beta_B} + \frac{R_{B''A}}{\beta_B} \right) \right)
\]

\[
\times (R_{1A} R_{AB} R_{B\Sigma'} R_{\Sigma'B'} R_{\Sigma''B} R_{B'A} R_{B''A} R_{A''A} R_{A''2})^{-1}
\]

where again for brevity we have assumed that density and velocities are constant between interfaces. In this expression \(R_{1A} = || x_1 - x_A ||, R_{AB} = || x_A - x_B ||, \) and so forth, and \(f\) is a product of interaction coefficients. Full fold integrals such as (39) and (36) are always robust with respect to internal caustics but they are time-consuming to compute (clearly!) and are usually not needed. If for one of the geometrical optics pairs \((u_1, u_2)\) in Fig. 13(b–j) neither \(u_1\) nor \(u_2\) has a caustic, then the KH integral which uses that pair will be just as robust. In order to determine whether such a pair exists it is only necessary to trace two sets of rays. For the problem in Fig. 13 one set of geometrical optics rays, the \(u_1\)-set, say, is traced from the source along paths similar to \(u_1\) in Fig. 13(j). Another set of geometrical optics rays, the \(u_2\)-set, is traced from the receiver along paths similar to \(u_2\) in Fig. 13(b). If for one of the (virtual) interface \(\{A, B, \Sigma, \Sigma', B', A', B'', A''\}\) the \(u_1\)-set has no caustics on or before \(C\) and the \(u_2\) set has no caustics on or before \(C\) then \(C\) is a good
surface of integration for a KH integral. Of course if no good surface $C$ exists then we can consider going to a two-fold path integral. We do not discuss two-fold path integrals here except to note that for the generalized ray of Fig. 13(a) there are $\binom{9}{2} = 36$ possible two-fold integrals, and a great many sets of rays must be traced to verify visually the goodness of even one of them. Note, however, that in Fig. 13 we have not exhausted the possibilities for one-fold integrals since we are free to choose a surface of integration across which velocity does not change. With reference to Fig. 13 suppose: $C$ and $C'$ are adjacent virtual surfaces; that $u_1$ has caustics on $C'$ but not $C$, and that $u_2$ has caustics on $C$ but not $C'$. Then there may be a good surface of integration between $C$ and $C'$. Also, every one-fold integral is implicitly an integration over the focal spheres of the source and receiver. Thus we are free to allocate different portions of the focal sphere to different one-fold integrals and then sum these integrals to get the complete response. Since the $u_1$- and $u_2$-ray sets introduced above give the correspondence between the reflector surface and the focal spheres, inspection of these ray sets will indicate which portions of the focal spheres should be allocated to which one-fold integrals.

6 Refracted waves

As a simple example we consider the situation shown in Fig. 14. We take the material parameters to be linear both above and below the surface $\Sigma$ but discontinuous across $\Sigma$.

Thus for $P$-wave velocity: above $\Sigma$, $\alpha(r) = \alpha(r - r_a) \cdot w_a$ and below $\Sigma$, $\alpha(r) = (r - r_b) \cdot w_b$ for some constant vectors $r_a, w_a, r_b,$ and $w_b$. Fig. 14(a) shows a geometrical optics path between $x_1$ and $x_2$. To synthesize the $P$-wave associated with this path we may use the full-fold path integral

$$\tilde{a}_2 \cdot u_1(x_2) = \int_{\Sigma} d\Sigma \int_{\Sigma} d\Sigma' f(x_1, x_{\Sigma}, x'_{\Sigma}, x_2) \frac{\exp\left[i\omega(T_{1\Sigma} + T_{\Sigma\Sigma'} + T_{\Sigma'2})\right]}{B_{1\Sigma}B_{\Sigma\Sigma'}B_{\Sigma'2}}$$

in which $T_{1\Sigma}, T_{\Sigma\Sigma'}, T_{\Sigma'2}$ are the travel times from $x_1$ to $x_{\Sigma}$, $x_{\Sigma}$ to $x'_{\Sigma}$, and $x'_{\Sigma}$ to $x_2$, respectively; $B_{1\Sigma}, B_{\Sigma\Sigma'}, B_{\Sigma'2}$ are spreading factors; and $f(x_1, x_{\Sigma}, x'_{\Sigma}, x_2)$ is a product of interaction coefficients. [At a stationary phase point of (40) $f$ will be proportional to $PP(x_d)PP(x_u)$ where $x_u$ and $x_d$ are the upward and downward points, respectively, of the
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geometrical ray in Fig. 14(a). Application of the method of stationary phase to the integration over \( \Sigma' \) in equation (40) yields the one-fold path integral (Kirchhoff-Helmholtz integral)

\[
\hat{a}_2 \cdot u_1(x_1) = \int_\Sigma u_{1} \cdot E_{12}^{pp} \cdot u_2
\]

where \( u_1 \) and \( u_2 \) are shown in Fig. 14(c) and \( E_{12}^{pp} \) is given by (38b). On the other hand, application of the method of stationary phase to the integration over \( \Sigma' \) in (40) yields the one-fold integral

\[
\hat{a}_2 \cdot u_1(x_2) = \int_\Sigma' u_{2} \cdot E_{12}^{pp} \cdot u_1
\]

where \( u_1 \) and \( u_2 \) are shown in Fig. 14(d). To see whether (41) or (42) is robust (i.e. whether neither \( u_1 \) nor \( u_2 \) has caustics on \( \Sigma \)) we trace one set of rays from the source, down through \( \Sigma \), through their turning points and back up to \( \Sigma \); and a similar set of rays from the receiver, down through \( \Sigma \), through their turning points and back up to \( \Sigma \).

Integrals (40)–(42) are for waves that interact only twice with \( \Sigma \), but a similar procedure can be used for refracted waves that reflect off the bottom of the interface. For example, a wave with three turning points in the lower medium will give rise to a four-fold integral analogous to (40) and four equivalent one-fold integrals. The tracing of one set of rays from the receiver and one set of rays from the source will reveal whether any of these one-fold integrals are robust. It is clear that the total interaction of the incident field with the interface can be written formally as an infinite series of multi-fold integrals (one for each generalized ray), of which integral (40) is a single term. However, numerical evaluation of this series would certainly be a very inefficient way of computing the total response.

7 Discussion

In this paper we have attempted to show how Kirchhoff-Helmholtz (KH) theory can be extended to elastic media, especially multi-layered elastic media. Throughout we have assumed that the frequencies in the signal that is to be synthesized are sufficiently high that, were it not for the presence of caustics and diffractions, our methods could be replaced by geometrical optics. Thus all of our results are asymptotic. However, the experience of many workers over the past two decades in modelling body wave amplitudes and arrival times with geometrical optics suggests that such asymptotic theories are useful. Also, comparison of such theories with both experiments (e.g. Hilterman 1970) and more accurate methods (e.g. Choy et al. 1980) reveals that they generally work well at frequencies far lower than the mathematics would lead one to expect.

The extension of KH theory to the case of a single interface between two elastic media, carried out in Section 4.1, is relatively straightforward, consisting mainly in the inclusion of an elastic reflection coefficient and the use of the more general KH formula given by equation (5). The main limitation of this extension is that the reflected wave is assumed to have interacted only once with the boundary, so that head waves, for example, are not included. Section 6 shows that waves which interact numerous times with the interface can be computed using KH theory but that additional KH integrals are required to compute them. The contributions of these other waves cannot be included in the KH integral for the primary reflection. Another limitation of the extension arises in consequence of our use of a
plane wave reflection coefficient. Computational experiments in Sen & Frazer (in preparation) show that rapid changes in the reflection coefficient with angle of incidence can give rise to small erroneous arrivals in the KH synthetic seismogram.

The extension of KH theory to a multi-layered medium, in Section 4.2, is accomplished by including plane wave transmission coefficients for intermediate interfaces. With reference to Fig. 6, note that even though the energy flow is from $x_1$ down to $\Sigma$, then up to $x_2$, we use downward transmission coefficients for both the downward and upward paths through interfaces $A$ and $B$. (The reason is that for reasons of convenience we calculate the spreading factor $B_2^f$ on $\Sigma$ by tracing rays from the receiver to $\Sigma$. If we were to use upward transmission coefficients for the upward leg then to be correct we would have to calculate $B_2^f$ by shooting rays upward from each point of $\Sigma$ to a neighbourhood of the receiver.) The procedure given in Section 4.2 works well even if the receiver is located on a caustic but breaks down if either the source wavefield or the receiver wavefield has a caustic on the reflector $\Sigma$, for $\Sigma$ is the surface of integration. The nature of the breakdown is discussed in Section 4.4. Numerical examples will be given in Sen & Frazer (in preparation), and have already been given for a similar situation by Frazer & Sinton (1984).

The breakdown in the method of Section 4.2 can often be avoided by using a surface of integration other than the reflector. As shown in Section 5 all such one-fold integrals are derivable from a multifold path integral. For a medium consisting of homogeneous irregular layers the most general path integral that might be required to synthesize a primary reflection is the one with a single integration over the reflector and two integrations over each intermediate interface; we refer to this integral as the full-fold path integral. With the full-fold integral we associate a generalized ray or path like the one shown in Fig. 11(a). This generalized ray obeys Snell's law between interfaces but is non-Snell at each interface. If one of the integrations in the full-fold integral is evaluated by stationary phase then the result is a lower-fold integral whose associated generalized ray now has a Snell-type interaction at one interface. Continuing this procedure leaves a one-fold integral whose generalized ray has a Snell-type interaction with every interface but one. Seen from this point of view, the KH integral over the reflector is just one member of a family of one-fold integrals. Fortunately, as shown in Section 5, all of these one-fold integrals can be derived directly with the KH theory of Section 4.2; it is not necessary actually to perform stationary phase on a full-fold integral. Frazer (1983) showed an example of the breakdown of the KH integral and calculated the reflection using a one-fold integral over an intermediate interface. Other examples will be given in Sen & Frazer (in preparation).

Unfortunately there are many velocity models that will give reflections for which no member of the family of one-fold integrals is robust. In such situations a multifold integral must be used. A detailed derivation of the full-fold path integral, including generalized transmission coefficients, will be given elsewhere. It is interesting to note that similar path integrals have already been used for the synthesis of refracted waves in a smoothly varying acoustic medium by Haddon (1983) and Zherniak (1983). Haddon (private communication) has stated that the amount of ray tracing required makes this method prohibitively expensive. However, in the refraction problem treated by Haddon (1983), velocity could not be held constant between surfaces of integration and rays had to be traced numerically from surface to surface. For the reflection problem, one can choose models with velocity constant between surfaces of integration so that ray paths are piecewise straight and easily calculated.

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References


Appendix A

Here we give invariant formulae for calculating the local direction of propagation and the displacement of the scattered waves created by a locally plane wave incident on a smooth boundary. Frequent reference is made to Vol. I of Aki & Richards (1980) and these references are abbreviated as AR. Equations (5.32) on p. 144 of AR are of SH-waves and we shall replace the symbol $S$ on their left sides by $H$. Thus, for example, we write $HH$ instead of $SS$. Similarly, equations (5.89), which begin on p. 150 of AR, are of $P$ and $SV$-waves so, when referring to these equations, we use $V$ instead of $S$ and write, for example, $PV$ instead of $PS$. As shown in Fig. A1, the smooth surface $\Sigma$ has unit normal $\hat{n}$ with material parameters $\alpha$, $\beta$, and $\rho$, on its $+\hat{n}$ side and parameters $\alpha$, $\beta$, $\rho$ on its $-\hat{n}$ side. The surface identity tensor on $\Sigma$ is $I = 1 - \hat{n}\hat{n}$ where $I$ is the identity tensor for the underlying Euclidean 3-space. The formulae we derive here are for the case of a wave incident on the $-\hat{n}$ side of $\Sigma$.
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Figure A1. Directions of propagation and directions of displacement for: (a) an incident P-wave; and (b) an incident S-wave.

Table 1A. Incident P-wave. Given quantities are: \( \hat{n} \), the normal to \( \Sigma \); \( \hat{t}_P \), the direction of propagation of the incident P-wave; \( \alpha, \beta, \rho \), the material parameters on the \( \Sigma^- \) side of \( \Sigma \): \( \alpha_\rho, \beta_\rho, \rho_\rho \), the material parameters on the \( \Sigma^+ \) side of \( \Sigma \); and \( u^P = A^P \hat{t}_P \) the displacement of the incident P-wave. Derived quantities are \( 1_1 = I - n_1 \)), \( a = (\hat{t}_P \cdot \hat{n}) / a \), \( a = 1_1 a_1 \), and \( \epsilon = \hat{n} / a \). In the AR formulae \( \cos i_1 = \hat{t}_P \cdot \hat{n} \), \( \cos i_2 = \hat{t}_P \cdot \hat{n} \), and \( \cos j_1 = \hat{t}_P \cdot \hat{n} \); and the branch to be taken for each root in column 2 is \( \text{Im}(\sqrt{\cdot}) > 0 \).

<table>
<thead>
<tr>
<th>Wave type</th>
<th>Direction of propagation</th>
<th>Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incident ( P )</td>
<td>( \hat{t}_P )</td>
<td>( u^P = A^P \hat{t}_P )</td>
</tr>
<tr>
<td>Reflected ( P )</td>
<td>( \hat{t}_R = \hat{t}_P \cdot (\hat{n} - \hat{n}) )</td>
<td>( u^P = A^P \hat{t}_P )</td>
</tr>
<tr>
<td>Reflected ( S )</td>
<td>( \hat{t}<em>R = \beta</em>\rho \hat{n} - \sqrt{1 - \rho^2 \hat{n}} )</td>
<td>( u^P = A^P \hat{t}_P )</td>
</tr>
<tr>
<td>Transmitted ( P )</td>
<td>( \hat{t}<em>T = \alpha</em>\rho \hat{n} + \sqrt{1 - \rho^2 \hat{n}} )</td>
<td>( u^P = A^P \hat{t}_P )</td>
</tr>
<tr>
<td>Transmitted ( S )</td>
<td>( \hat{t}<em>T = \beta</em>\rho \hat{n} + \sqrt{1 - \rho^2 \hat{n}} )</td>
<td>( u^P = A^P \hat{t}_P )</td>
</tr>
</tbody>
</table>

as shown in Fig. A1. To use these formulae for a wave incident from \( \Sigma^+ \) one replaces \( \hat{n} \) by \( -\hat{n} \) in each formula containing \( \hat{n} \).

Appendix B

When seismic reflection or refraction data are gathered along a line perpendicular to the strike of the geological structure then these data are often modelled by synthetic seismograms computed using the two-dimensional (2-D) wave equation. In any method of computing synthetics which has a ray theoretical foundation (e.g. geometrical optics, Maslov theory, Gaussian beams, Kirchhoff–Helmholtz) these 2-D solutions are easily converted to approximate 3-D solutions by assuming that the medium has cylindrical symmetry about an axis (generally vertical) through either the source or the receiver. If the departures of the medium from stratification are greatest in the vicinity of the source then it is best to choose this axis through the receiver and vice versa. The switch from one axis to the other can be
Table A2. Incident S-wave. Given quantities are: \( \mathbf{n} \), the unit normal to \( \Sigma; \mathbf{t} \), the direction of propagation of the incident S-wave; \( \alpha, \beta, \rho \), the material parameters on the \( -\mathbf{n} \) side of \( \Sigma; \alpha^+, \beta^+, \rho^+ \), the material parameters on the \( +\mathbf{n} \) side of \( \Sigma; \alpha^-, \beta^-, \rho^- \), the displacement of the incident S-wave. Derived quantities are:

\[
\begin{align*}
I_1 &= \mathbf{t} - \mathbf{n}, \\
\varphi &= (\mathbf{i}_S \cdot I_1) / |I_1|, \\
\eta &= \mathbf{t} \cdot (\mathbf{i}_S - \mathbf{n}), \\
\mu^+ &= \mathbf{t} \cdot (\mathbf{i}_S + \mathbf{n}).
\end{align*}
\]

In the AR formulae

\[
\cos \varphi = -i \eta \cdot \rho^+ \cdot \mathbf{t}, \\
\cos \varphi = i \mu^+ \cdot \mathbf{t}, \\
\cos \varphi = i \eta \cdot \rho^- \cdot \mathbf{t}, \\
\cos \varphi = i \mu^- \cdot \mathbf{t},
\]

and the branch to be taken for each root in column 2 is \( \Im(\sqrt{\varphi}) > 0 \).

### Wave type

<table>
<thead>
<tr>
<th>Wave type</th>
<th>Direction of propagation</th>
<th>Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incident S</td>
<td>( \mathbf{t} )</td>
<td>( u^S = A V^S \mathbf{S} + A H^S \hat{\mathbf{n}} )</td>
</tr>
<tr>
<td>Reflected P</td>
<td>( \mathbf{t}^S = \mathbf{t} \cdot (\mathbf{h} \hat{\mathbf{n}} - \mathbf{n}) )</td>
<td>( \mathbf{u}_R^P = A_R^P \mathbf{t}_R^P )</td>
</tr>
<tr>
<td>Reflected S</td>
<td>( \mathbf{t}^S = \mathbf{t} \cdot (\mathbf{h} \hat{\mathbf{n}} - \mathbf{n}) )</td>
<td>( \mathbf{u}_R^S = A_R^S \mathbf{t}_R^S + A_R^S H^S \hat{\mathbf{n}} )</td>
</tr>
<tr>
<td>Transmitted P</td>
<td>( \mathbf{t}^S = \mathbf{t} \cdot (\mathbf{h} \hat{\mathbf{n}} - \mathbf{n}) )</td>
<td>( \mathbf{A}_R^P = V P A V )</td>
</tr>
<tr>
<td>Transmitted S</td>
<td>( \mathbf{t}^S = \mathbf{t} \cdot (\mathbf{h} \hat{\mathbf{n}} - \mathbf{n}) )</td>
<td>( \mathbf{A}_R^S = V V A V )</td>
</tr>
</tbody>
</table>

made continuously so as to accommodate, for example, deep departures from stratification beneath the source and shallow departures from stratification near the receiver.

Consider first the geometrical optics equations (13) and (14). The spreading factors \( B^P \) and \( B^S \) are given for 3-D by equations (15) and for 2-D by equations (16). To convert a 2-D solution to a 3-D solution for a P-wave, say, we multiply our 2-D solution by the ratio of \( \frac{\partial \alpha}{\partial \phi} \) and \( \frac{\partial \alpha}{\partial \theta} \), i.e. by the conversion factor

\[
C_\theta^P = \frac{\sqrt{8 \pi \omega \alpha_1}}{4 \pi \alpha_1 \sqrt{dA/d\Omega}} \exp \left(-i \pi/4\right)
\]

To simplify this relation let \( x(\theta, \phi, T) \) be a point on the wavefront which contains the receiver \( x_2 \). Here \( \theta, \phi \) are spherical polar coordinates at the source \( x_1 \) and \( \theta \) is the same polar angle which appears in the quantity \( dA/d\theta \). On this wavefront let \( dA \) be the patch of area that is the cross-section of the ray tube that subtends solid angle \( d\Omega = \sin \theta d\theta d\phi \) at the source. Then assuming the medium is cylindrically symmetric about the polar axis, it follows that \( dA = |\delta x/\delta \theta \times \delta x/\delta \phi|\ d\theta d\phi = (dA/d\theta)(d\theta)(x \cdot d\phi) \) where \( x \) is the horizontal distance between the source and receiver. Dividing \( dA \) by \( d\Omega \) we obtain

\[
dA/d\Omega = (dA/d\theta)(x/\sin \theta)
\]

and substitution of this quantity into (B1) yields

\[
C_\theta^P = \exp \left(-i \pi/4\right) \frac{\omega}{2 \pi x} \left(\frac{\sin \theta}{\alpha_1}\right)^{1/2} \left(\frac{\alpha_1}{\sin \theta}\right)^{1/2}
\]

The conversion factor for S-waves is the same except that \( \alpha_1 \) is replaced by \( \beta_1 \).

Suppose now that instead of assuming cylindrical symmetry about an axis through the source we assume it holds about an axis through the receiver. We may take advantage of
Kirchhoff–Helmholtz reflection seismograms – 1

seismic reciprocity to write formulae analogous to (15), (16) and (17) for our response and then make an argument the same as that just given to find

\[ C_2^p = \exp \left( -i\pi/4 \right) \left( \frac{\omega}{2\pi x} \right)^{1/2} \left( \frac{\sin \theta_2}{\alpha_2} \right)^{1/2} \]  \hfill (B4)

where \( \theta_2 \) is the angle which the geometrical ray from \( x_1 \) to \( x_2 \) makes with the axis of cylindrical symmetry at the receiver. In a stratified medium the axis of symmetry is normal to the stratification and the final factors in (B3) and (B4) are then both equal to ray parameter \( p \). In this case, as expected, \( C_2^p = C_2^p \approx (wp/2)H^{(1)}_0(wp)\exp(iwp) \) where \( H^{(1)}_0 \) is the Hankel function of the first kind of order zero.

![Figure B1. Use of approximate 3-D/2-D conversion factors. The use of \( C_1^p \) means a virtual reflector \( \Sigma'_1 \) and the use of \( C_1^p \) means a virtual profile \( \Sigma'_2 \). (a) Here \( C_1^p \) is better than \( C_2^p \), (b) here \( C_2^p \) is better than \( C_1^p \).](image)

Although equations (B3) and (B4) were derived using geometrical optics they may be applied to any solution involving a generalized ray which has a well-defined angle with a polar axis at either the source or the receiver. Consider how to apply them to the KH equations (23)–(26). If departures from stratification are greater in the vicinity of the receiver, as shown in Fig. B1(a) then we put \( C_1^p \) under the integral sign in (23)–(26). (Note that \( x \) in equation (B3) for \( C_1^p \) is the horizontal distance from \( x_1 \) to \( x_2 \), not the distance from \( x_1 \) to \( x \in \Sigma \).) On the other hand, if departures from stratification are greater in the vicinity of the source, as shown in Fig. B1(b), then \( C_1^p \) should be used instead. The examples shown in Fig. B1 are extreme. In general, the integrands of (23)–(26) will contain many arrivals not all of which can be associated with a single neighbourhood. To account for this we suggest the use of an average factor which will act more like \( C_1^p \) when \( \sin \theta_2 \) is small and act more like \( C_2^p \) when \( \sin \theta_2 \) is small. One example of such a factor is

\[ C^p = \exp \left( -i\pi/4 \right) \left( \frac{\omega}{2\pi x} \right)^{1/2} \left( \frac{\sin \theta_2 \cos \theta_2}{\alpha_1} + \frac{\sin \theta_2 \cos \theta}{\alpha_2} \right)^{1/2} \]  \hfill (B5)
APPENDIX B

Kirchhoff–Helmholtz reflection seismograms in a laterally inhomogeneous multi-layered elastic medium – II. Computations

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Summary. High-frequency reflection and refraction seismograms for laterally variable multi-layered elastic media are computed by using the frequency domain elastic Kirchhoff–Helmholtz (KH) theory of Frazer and Sen. Both source and receiver wavefields are expanded in series of generalized rays and then elastic (KH) theory is applied to determine the coupling between each source ray and each receiver ray at each interface. The motion at the receiver is given as a series of integrals, one for each generalized ray. We use geometrical optics and plane wave reflection and transmission coefficients for rapid evaluation of the integrand. When the source or the receiver ray field has caustics on the surface of integration geometrical ray theory breaks down and this gives rise to singularities in the KH integrand. We repair this using methods suggested by Frazer and Sen.

Examples of reflection seismograms for 2-D structures computed by elastic KH theory are shown. Those for a vertical fault scarp structure are compared with the seismograms obtained by physical modelling. Then OBS data obtained from the mid-America trench offshore Guatemala area are analysed by computing KH synthetics for a velocity model that has been proposed for that area. Our analysis indicates the existence of a small low-velocity zone off the trench axis.

No head wave arrivals are obtained in our KH synthetics since we do not consider multiple interactions of a ray with an interface. The nearly discontinuous behaviour of elastic $R/T$ coefficients near the critical angle causes small spurious phases which arrive later than the correct arrivals.

1 Introduction

Synthetic seismograms are now a very important tool in the interpretation of field seismograms. There exist numerous ways of computing these synthetics. The algorithms for computing synthetic seismograms for stratified earth models such as Cagniard-de Hoop (e.g. Helmberger 1968), reflectivity (Fuchs & Muller 1971) and WKBJ methods (Richards 1973; Chapman 1978) have reached a very advanced state of development (Aki & Richards 1980; Kennett 1983) and most recent developments in this line may fairly be regarded as extensions. For example,
Chapman (1978) improved the speed of the WKBJ method by interchanging the order of the double integral and Frazer (1983a) showed how the WKBJ method could be used with frequency-dependent velocities. Some of the principles of these methods are outlined in a review paper by Mooney (1983). Over the last decade, geological interest has largely shifted to portions of the Earth, such as active margins, where seismic velocities are often not stratified and the methods noted above do not generally work. In response to this need great effort has gone into the development of methods of computing synthetic seismograms for laterally inhomogeneous velocity models.

One of the most popular methods of studying high-frequency wave propagation in inhomogeneous media is geometrical ray theory. Červený and his coworkers (Červený, Molotkov & Pšenčík 1977) have brought the theory of geometrical optics in elastic media (Babich & Alekseev 1958) to a very advanced state. Unfortunately no geometrical optics method works correctly when the receiver is located at a caustic. Frazer & Phinney (1980) and Sinton & Frazer (1982) extended the WKBJ method to continuous velocity functions which are nearly stratified. Chapman & Drummond (1983) showed that the restriction to nearly stratified media could be removed by combining the extension of Frazer & Phinney (1980) with geometrical optics according to a prescription given some years earlier by Maslov (1965). Recently Červený (1983) and others have applied the method of Gaussian beams to the propagation of seismic waves. Unlike geometrical optics, all these methods work when the receiver is located on a caustic but none of them work when the shape of the reflector and location of the source are such that the reflected field has shadows. More precisely, they do not work for a reflector which has an abrupt change in slope. Frazer & Sen (1985) discussed the performance of these methods relative to the Kirchhoff method when the reflector has a near vertical fault such that the two reflection branches are separated by a shadow.

Kirchhoff–Helmholtz theory has long been used by exploration geophysicists (e.g. Hilterman 1970, 1975; Trorey 1977) to model reflection data. In their method the surface of integration is always a physical reflector. Recently Scott & Helmberger (1982) extended Hilterman’s (1970) method to the case of a separate source and receiver and a single reflector with an arbitrary shape. Sinton & Frazer (1981), Haddon (1982, 1983) and Frazer & Sinton (1984) used KH theory to compute synthetic seismograms for continuous laterally-variable velocity functions by putting surfaces of integration between source and receiver. Haddon & Buchen (1981) used a similar technique to synthesize PKP for a stratified earth model. Their methods are not very convenient when the velocity function is discontinuous. More recently Carter & Frazer (1983) and Deregowski & Brown (1983) used KH theory to compute reflections from 2-D acoustic models by tracing rays from each point of a subsurface reflector up to the surface containing the receivers and then using the principle of seismic reciprocity to predict the downward propagating shot energy from the upward ray trace. In this paper, we are concerned with the computation of synthetic seismograms for media consisting of elastic layers with irregular boundaries using the elastic KH theory of Frazer & Sen (1985), hereinafter referred to as paper 1.

In paper 1, Frazer & Sen (1985) derived the elastic KH formulation for the motion at a receiver at a point in a multi-layered elastic medium. Their integral representation of the motion can be reduced to the form

$$u(\omega) = \int_{\Sigma} d\Sigma f \exp(i\omega T) \quad (1)$$

where $f$ is a function of angle-dependent reflection coefficients, geometrical spreading etc., $\omega$ is the angular frequency and $T$ is the travel time from the source to the receiver via the interface $\Sigma$. The integration is carried over the surface $\Sigma$ (see Figs 1 and 2) which, in general, is a surface of material discontinuity. We may note that for a 2-D medium, the surface integral in (1) reduces to
Kirchhoff–Helmholtz synthetics

a line integral. The travel time $T$ may be a complex function when one wants to incorporate attenuation and dispersion in the medium by using complex velocity functions. The total response at the receiver is given by the sum of a series of integrals of the form (1), one for each generalized ray. In this paper we discuss techniques for evaluating such integrals numerically and show examples of synthetic seismograms from 2-D multi-layered elastic media.

2 Theory

Paper 1 discussed the multi-layered elastic KH theory in detail. Here we shall summarize the theory and reproduce only those definitions and results necessary for computations. The pressure detected by a pressure sensor at a point $x_2$ due to a source at $x_1$ is

$$P_1(x_2) = K(x_2) \int_{\Sigma} \mathbf{n} \cdot (\mathbf{r}_1 \cdot \mathbf{u}_2 - \mathbf{r}_2 \cdot \mathbf{u}_1) \, d\Sigma.$$  (2)

If the detector at $x_2$ measures motion in the direction $\mathbf{a}_2$, then the integral which gives the motion at $x_2$ can be written as

$$\mathbf{a}_2 \cdot \mathbf{u}_1(x_2) = \int_{\Sigma} \mathbf{n} \cdot (\mathbf{r}_1 \cdot \mathbf{u}_2 - \mathbf{r}_2 \cdot \mathbf{u}_1) \, d\Sigma,$$  (3)

where $\mathbf{r}_1$ and $\mathbf{u}_1$ are the stress tensor and displacement vector, respectively, due to the source at $x_1$, and $\mathbf{r}_2$ and $\mathbf{u}_2$ are the stress tensor and displacement vector respectively due to a fictitious source at $x_2$. In the case of integral (2) this source has equivalent body force density (Burridge & Knopoff 1964) $\nabla \delta(x-x_2)$ and in the case of integral (3) it has equivalent body force density $\mathbf{a}_2 \delta(x-x_2)$. $P_1$ and $K$ are pressure and bulk modulus respectively. $\Sigma$ is the surface of material discontinuity along which two inhomogeneous elastic media are in welded contact. $\mathbf{n}$ is the downward pointing unit normal at the boundary. Using geometrical optics formulae equation (3) can be converted to the form

$$u_1^{P'}(x_2) = \frac{1}{\sqrt{\rho_1 \alpha_1 \alpha_2}} \int_{\Sigma(x)} i_2^P(x_2) F_1^P \frac{i_2^P}{i_1^P} R_1^P \cdot B_2^P \exp\left(i\omega(T_1^P + T_2^P)\right) d\Sigma(x),$$  (4)

which corresponds to a $P$ to $P$ reflection. $R_1^P$ is the interaction coefficient or elastic KH reflection coefficient defined in paper 1 (equation 9b). For a point force of equivalent body force density, $f_1 = \mathbf{a}(\omega) \delta(x-x_1)$, we have $F_1^P = \mathbf{a}(\omega)(x-x_1)$. In a 2-D medium,

$$B_2^P = \sqrt{8\pi\omega \alpha_2} dl_2/d\theta_2 \exp(-i\pi/4) \quad \text{and} \quad B_1^P = \sqrt{8\pi\omega \alpha_1} dl_1/d\theta_1 \exp(-i\pi/4),$$  (5)

in which $\rho_1$ and $\alpha_1$ are the density and $P$-wave velocity, respectively, at point $x_1$, $\rho_2$ and $\alpha_2$ are the density and $P$-wave velocity at $x_2$ and $\rho$ and $\alpha$ are the density and $P$-wave velocity at a point $x$ on $\Sigma$. $T_1^P$ and $T_2^P$ are the $P$-wave travel times from the source and receiver, respectively, down to a point $x$ on the interface $\Sigma$. $i_1^P$, $i_2^P$, $dl_1$, $dl_2$, $d\alpha_1$, $d\alpha_2$ are shown in Fig. 1. Expressions similar to equation (4) for $P$ to $S$, $S$ to $P$ and $S$ to $S$ reflections are given in paper 1 (equations 24, 25 and 26). Equation (4) applies to $P$ to $P$ reflection from a single interface assuming that there are no interfaces other than $\Sigma$. When we consider a medium which has some other interface above $\Sigma$ (Fig. 2) we need to include plane wave transmission coefficients to take into account the loss of energy through reflection and mode changes at each interface shallower than $\Sigma$. Then the response of $\Sigma$ at a receiver is given by an expression of the form

$$u_1(x_2) = \int_{\Sigma(x)} u_1 \cdot R_1 G_2 d\Sigma(x)$$  (6)

where $G_2$ is the Green's tensor from the receiver down to $\Sigma^-$ and $u_1$ is the wave from the source down to $\Sigma^-$. Both these quantities are computed using geometrical optics and ordinary plane
wave reflection and transmission coefficients. $R_{12}$ is the generalized KH interaction coefficient introduced above. It is chosen to agree with the modes of $u_1$ and $G_2$ on $\Sigma^-$ without regard to the mode of these waves in the shallower portions of their respective paths. In subsequent sections we use equations of the form (4) and (6) to compute synthetic seismograms from different geological structures.

3 Numerical computation

The first step in our computation is tracing rays from each source and each receiver down to each interface. In most of the examples shown in the sequel, we consider media consisting of homogeneous, elastic layers with curved interfaces. Ray tracing in such media is fairly simple since the rays are piecewise straight lines. Gebrarde's (1976) velocity parameterization scheme has been used to define the interfaces in two dimensions. Gebrarde used normalized inverse tangent functions to define an arbitrarily shaped curved interface. According to this algorithm, a curve in the $xz$ plane is defined by the equation

$$z = z_0 + \sum_{k=1}^{n} \Delta z_k \left( 0.5 + \frac{1}{\pi} \tan^{-1}(x-x_k)/b_k \right)$$

(7)

where different structures are modelled by adding different terms to a normal depth $z_0$. Each term within the summation represents a flexure with centre at the $x$-coordinate $x_k$ and throw $\Delta z_k$. The half-width of the flexure is $2b_k$. This algorithm is very easy to code and many realistic geological structures such as pinchouts, faults, anticlines etc. can be modelled easily by suitably combining flexures of different parameter. We compute quantities such as $i_1^p$, $R_{12}^{pp}$, $T_1^p$, $\sigma_1/\theta_1$ etc. (see Fig. 1) for each point on the surface $\Sigma$ where the rays from the source intersect the surface. Similar quantities are also computed at points on $\Sigma$ at which the rays drawn from the
Kirchhoff–Helmholtz synthetics

The functions $\frac{d\sigma_1}{d\theta_1}$ and $\frac{d\sigma_2}{d\theta_2}$ are computed by a method of numerical differencing as used by Frazer & Sinton (1984). We replace $\frac{d\ell_1}{d\theta_1}$ and $\frac{d\ell_2}{d\theta_2}$ by their equivalent forms.

$$
\frac{d\ell_1}{d\theta_1} = (\hat{n} \cdot \ell'_1(x)) \frac{d\sigma_1}{d\theta_1} ; \quad \frac{d\ell_2}{d\theta_2} = (\hat{n} \cdot \ell'_2(x)) \frac{d\sigma_2}{d\theta_2}
$$

(8)

where $\hat{n}$ is the downward unit normal at $x$ and $d\sigma_1$ and $d\sigma_2$ are explained in Fig. 1. For models consisting of homogeneous layers, $\frac{d\ell_1}{d\theta_1}$ and $\frac{d\ell_2}{d\theta_2}$ for the shallowest interface can be replaced by $|x_1-x|$ and $|x_2-x|$ respectively. For tracing rays, the take-off angle ($\theta_1$ for source, $\theta_2$ for receiver) is used as the independent variable. Any one of the three variables $\theta_1$, $\theta_2$ or $\theta$ may be used as a variable of integration. Usually $\theta_1$ or $\theta_2$ is a more convenient variable of integration than $\theta$. Source and receiver rays are drawn independently for various values of take-off angles by varying $\theta_1$ and $\theta_2$ independent of each other. Hence source and receiver rays do not always intersect at one point on the surface of integration. Therefore if $\theta_1$ is chosen as the variable of integration, then for a point on the surface of integration intersected by a ray from the source, quantities such as $\frac{d\sigma_2}{d\theta_2}$ for the receiver ray have to be obtained by linear interpolation of the corresponding parameters of the neighbouring receiver rays unless the point is in the region of geometrical shadow of the receiver rays in which case no integration needs to be carried out. A similar procedure has to be followed for source rays in case $\theta_2$ is chosen as the variable of integration. We trace rays in such a way that the surface of the reflector is well defined by its intersection with the ray field, i.e. so that irregularities of the reflector are not spatially aliased. Of course, complicated structures may cause the formation of shadow zones on some reflectors. A method for treating such cases will be discussed below. In order to compute the arrivals of different modes such as P–S, S–P, S–S etc., one needs to trace separate sets of source and receiver rays. Having computed the ray data, the next step in our computation is to evaluate integrals of the form (4) or (6). We have so far assumed that the integrand is regular everywhere on the surface of integration, i.e. neither the source nor the receiver has any caustics. If either the source or the receiver ray field has a caustic then one of the terms $(d\sigma_1/d\theta_1)$ and $(d\sigma_2/d\theta_2)$ which appears in the denominator of the integrand becomes zero. This makes the integrand singular at caustics. In the sequel we will discuss various methods of repairing this breakdown of geometrical optics at caustics on the surface of integration. All our integrals can be expressed in the form of equation (1) which is a typical oscillatory integral encountered in wave propagation problems. If $\omega$ is large then the phase factor will be rapidly oscillating and the use of the trapezoidal rule for numerical evaluation of the integral would require an extremely small step size, making the computation very expensive. Numerical evaluation of such an integral can be carried out with a generalized Filon method quadrature formula (Frazer 1977; Frazer & Gettrust 1984). With this method the integration step size becomes essentially independent of the frequency $\omega$ and is limited by the variation of the slowly varying part of the integrand. This makes the integrand singular at caustics. In the sequel we will discuss various methods of repairing this breakdown of geometrical optics at caustics on the surface of integration. All our integrals can be expressed in the form of equation (1) which is a typical oscillatory integral encountered in wave propagation problems. If $\omega$ is large then the phase factor will be rapidly oscillating and the use of the trapezoidal rule for numerical evaluation of the integral would require an extremely small step size, making the computation very expensive. Numerical evaluation of such an integral can be carried out with a generalized Filon method quadrature formula (Frazer 1977; Frazer & Gettrust 1984). With this method the integration step size becomes essentially independent of the frequency $\omega$ and is limited by the variation of the slowly varying part of the integrand. In the integrand of equation (1), the function $f$ is slowly varying and for large values of $\omega$ the phase function is rapidly oscillating which has a self-cancelling effect on the integral except near the point(s) of stationary phase. Such points correspond to the ray paths of geometrical ray theory. An additional contribution to the integral may occur from points at which either $f$ or the phase term has a discontinuity.

The algorithm for computing KH synthetics in the frequency domain can be summarized as follows:

(1) For a particular model, ray data such as $T$, $d\ell/d\theta$ etc. are computed at points on each reflector for rays drawn from both source and receiver down to each interface.

(2) A series of integrals, one for each generalized ray of the form of equation (1) are evaluated numerically at different finite frequencies using a generalized Filon’s method quadrature
formula. Integrations are carried out only in the portions of the reflectors which are lit by rays from both source and receiver. All these integrals are combined into the final seismogram.

(3) An inverse Fourier transform (FFT) is applied to the frequency series thus obtained and is then convolved with a source function to obtain the final time domain synthetics for the model for a particular source and a specific receiver location.

This algorithm works extremely well as long as the source and receiver ray fields on the surface of integration are regular. Unfortunately the velocity structure in a medium may be such that the source or receiver ray field or both may have caustics on the surface of integration. At each of the caustics the KH integral will be singular due to a zero of a spreading factor. This can be repaired by one of the three methods given below:

(1) Treat the singularities as integrable and integrate over them (Sinton & Frazer 1981; Frazer & Sinton 1984). A change of variable in the integrals helps to remove the singularity. In the Appendix we derive expressions to be used for evaluating the integral near the caustics. The interval of integration is divided into a number of subintervals in each one of which the integrand is regular. However, truncation of the surface of integration causes arrivals of small spurious phases and the amplitude information obtained by this method may not be very reliable when a caustic coincides with a point of stationary phase.

(2) Convolve the function with a smoothing operator whose width is a decreasing function of frequency (Sen & Frazer 1983). Smoothing has to be done over the wavefront instead of a surface of integration over which the phase also varies. Derekowski & Brown (1983) used a similar smoothing operator over rays adjacent to the caustics. None of these approaches has a clear mathematical basis. However, it can be shown by more exact methods that the amplitude at caustics is finite at finite frequencies and hence such smoothing is justified physically.

(3) A single fold KH integral (e.g. equation 4) can be regarded as a special case of a more general multi-fold KH integral (Frazer 1983b) where each integral corresponds to one interaction of a ray with an interface. Paper 1 explains how this concept can be utilized to choose the surface of integration arbitrarily and avoid the surface on which either the source or the receiver ray field has caustics. In a subsequent section we illustrate this with an example.

Time-domain algorithm: For time-domain computations we can use an approach similar to Chapman's WKBJ algorithm (Chapman 1978; Dey-Sarkar & Chapman 1978). Our frequency-domain response is given by equation (1) as,

\[ u(\omega) = \int_{\Sigma(x)} f(x) \exp\{i\omega T(x)\} \, dx. \]

Then,

\[ \hat{u}(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega \int_{\Sigma} f(x) \exp\{i\omega T(x) - i\omega t\} \, dx \]

\[ = \int_{\Sigma} f(x) \delta(t - T(x)) \, dx \]  \hspace{1cm} (9)

which is singular wherever \( \delta T/\delta x = 0 \). To get around this difficulty we may note that we really never make observations of \( \hat{u}(t) \) but we observe \( \hat{u}(t)^{*} A(t) \) instead where \( A(t) \) is the impulse response of a more or less narrow band physical system. Thus instead of trying to synthesize \( \hat{u}(t) \), it is better to compute a smooth or filtered form of \( \hat{u}(t) \). This is obtained by convolving the time series with a boxcar window of length twice the digitizing interval (Dey-Sarkar & Chapman 1978) which is written as

\[ \hat{u}_s(t) = \hat{u}(t)^{*} \frac{1}{\Delta t} B(t/\Delta t) \]  \hspace{1cm} (11)
where \( \Delta t \) is the digitizing interval and the ‘box-car function’ is defined by the equation
\[
B(t) = \frac{1}{2} [H(t+1) - H(t-1)]
\]  
where \( H(t) \) is the Heaviside step function. Thus the smoothed version of \( \hat{u}(t) \) is
\[
\hat{u}_s(t) = \frac{1}{\Delta t} \int f(x) B\left(\frac{t-T(x)}{\Delta t}\right) dx
\]
which has no singularities. For the 2-D problem, where \( \Sigma \) is a line, numerically we have
\[
\hat{u}_s(t) = \frac{1}{\Delta t} \sum f(T^{-1}(t)) \|T^{-1}(t+\Delta t) - T^{-1}(t-\Delta t)\|
\]
where the summation is over the number of solutions of \( T(x) = t \). Expression (14) can be used for direct time-domain computation of KH synthetic seismograms. Whereas the time-domain computations are relatively faster, the computation in the frequency domain enables one to take into account the attenuation and dispersion in the medium by allowing the velocity to be frequency-dependent and complex as explained in paper 1.

These procedures can also be applied to the computation of diving ray (refraction) arrivals in a medium in which the surface of integration separates two layers such that at least in one of the layers, velocity changes continuously with depth. In order to be consistent with the sign of \( (d\alpha_1 / d\theta_1) \) and \( (d\alpha_2 / d\theta_2) \) in different situations, it is necessary to adhere to some sign conventions for \( \theta_1 \) and \( \theta_2 \) for different source-receiver locations and for reflected and refracted rays. These are summarized in Fig. 3.

4 Example seismograms

In this section we show examples of KH synthetic seismograms computed for different models in order to bring out the features of the KH algorithm. All the seismograms are for 2-D models and they consist of P-wave reflections or refractions computed using our frequency domain algorithm. For converting 2-D synthetics to 3-D we adopted the procedure suggested in appendix B of paper 1.

The first in our series of example models is the simplest one viz. a flat layer over a half-space. Fig. 4(a) shows the model along with the source-receiver spread and the rays drawn from the source. For computing the synthetics shown in Fig. 4(b), we assumed the half-space below the layer to be rigid and used a value of one for reflection coefficients at all angles of incidence. The synthetic seismograms show that the same shape of the pulse is retained for all values of the range. The set of synthetics shown in Fig. 4(c) were computed for the same medium by assuming both the layer and the half-space to be acoustic and using an angle-dependent fluid-fluid reflection coefficient. It is observed that the \( \delta \)-like pulse shape of the seismogram changes with increasing range. This is caused by the phase change due to a reflection coefficient which is a complex number at post-critical angles of incidence. A linear combination of the \( \delta \)-like term and a second term which is the Hilbert transform of the first produces the seismograms at larger ranges (Aki & Richards 1980, p. 157). For very large ranges, the second term becomes dominant. In addition to the change in pulse shape of the reflected wave at the post-critical angles we see small spurious phases (indicated by dots in Fig. 4c) arriving at a time later than the correct reflection arrivals. These phases have relatively large amplitude in the pre-critical range and are extremely small in the post-critical range. Next, in Fig. 4(d), we consider both the layer and the half-space to be elastic and compute synthetic seismograms for P to P reflection arrivals using our KH algorithm. Elastic reflection/transmission coefficients were used so as correctly to take into account the
energy loss due to phase conversion. The series of synthetics thus computed looks very different from the one for the fluid-fluid case in the post-critical range, because of the larger contribution of the Hilbert transformed part. Small spurious phases arriving later than the correct P to P reflection arrivals also show up for the elastic-elastic case. These spurious phases are caused by the nearly discontinuous behaviour of the elastic/acoustic (Fig. 4e) reflection coefficient near the critical angle. It may be noted that no head wave arrivals are observed in any of these synthetic seismograms.

Fig. 5(a) shows a layer over a half-space such that the interface has a vertical fault. The rays drawn from the source located at a point on the surface do not light the entire reflector but leave a shadow zone QR on the dowthrown side of the fault. The reflected ray field consists of two branches separated by a shadow as shown in Fig. 5(a). Geometrical optics would not show any arrival in this shadow zone which is true only in the limit of infinite frequency. However, at finite frequencies, both points P and R on the reflector act as point scatterers which form hyperbolas in the X-T plane as suggested by diffraction theory. We computed KH synthetic seismograms for this model for the source-receiver spread shown in Fig. 5(a) and compared the synthetics with the seismograms obtained by a physical model experiment done at the Seismic Acoustic Laboratory of the University of Houston. The details of the physical modelling procedures and apparatus

![Figure 3](image-url)  
**Figure 3.** Sign conventions for $\theta_1$ and $\theta_2$ for different reflection/refraction problems. The arrows show positive directions of $\theta_1$ and $\theta_2$. (a), (b) and (c) have the same convention for reflection problem for one and two layer cases. (d), (e) and (f) show the need for different sign convention for different refraction problems. $A$ and $B$ are the surfaces of first-order velocity discontinuity in all the cases.
used at SAL are described in the book by MacDonald, Gardner & Hilterman (1983). Piezo-electric transducers were used for sources and receivers and the fault model was made with plexiglass which was placed inside the model tank filled with water. We used a geometrical scale factor of 12000 which means 1 inch of the model scales to 1000 ft and a time-scale factor of 5000 which means 0.2 µs in the model scales to 1 ms. This gives a velocity scaling factor of 2.4. Thus corresponding to a true P-wave velocity of 4847 ft s\(^{-1}\) through water at 65°F, the scaled P-wave velocity through water is 11 633 ft s\(^{-1}\). Similarly the true P-wave velocity through clear plexiglass is 9000 ft s\(^{-1}\) and its scaled velocity is 21 600 ft s\(^{-1}\). Using these parameters we computed synthetic seismograms for the vertical fault model and for the source–receiver spread marked in Fig. 5(a). Since the rays from the source do not reach all the portions of the reflector, we restricted the computation of the integral only to the portions of the reflector lit by the source, i.e. integrations were carried out only over the portions of the reflector emphasized in Fig. 5(a). Fig. 5(b) shows the impulse response of the fault obtained by our KH algorithm. We assumed the half-space below the faulted layer to be rigid and used a value of one for reflection coefficients at all angles for computing these seismograms. The branch AB corresponds to
reflection from the upthrown side of the fault. The branch CBD corresponds to diffraction from the corner \( P \) marked in Fig. 5(a). The amplitudes in the branch BC are small and of opposite phase compared to the reflection arrivals in the branch AB. Similarly the branch EF corresponds to reflection from the downthrown side of the fault while the branch GEH shows hyperbolic diffraction arrivals from the point scatterer R marked on the downthrown side of the fault (Fig. 5a). The arrivals in the branch EH are small and negative compared to the reflection arrivals in branch EF. The points B and E marked by arrows correspond to points at which reflection and diffraction arrivals are time-coincident. Fig. 5(c) shows the seismogram obtained by the model tank experiment. Reflections from the upthrown and downthrown sides of the fault and the positive amplitude diffraction arrivals from the edges (P and R in Fig. 5a) can be identified very well but the reversed polarity diffraction arrivals are too small to be detected. Next we convolved the impulse response functions with a source function similar to the one used in the model tank experiment. Acoustic/elastic reflection coefficients were also included in computing the seismograms. The KH seismograms thus obtained are shown in Fig. 5(d). The comparison of the KH seismograms for both small and large offsets with those obtained by the physical model experiment is quite good. The negative amplitude diffractions are impossible to distinguish from the primary reflection arrivals for three reasons: (i) they are very small in amplitude; (ii) their arrival time is very close to those of the primary reflection arrivals, and (iii) the source pulse is highly oscillatory in nature. The positive amplitude diffraction arrivals make the primary reflections look continuous even in the shadow zone both in the model data and K-H synthetics. A careful examination, however, would show that diffractions and reflections have distinctly different moveouts.

![Reflection from a flat horizontal layer over a half-space](image)

**Figure 4.** Reflection from a flat horizontal layer over a half-space. (a) Reflected ray field for the rays drawn from the source and the source-receiver geometry. (b) KH synthetics for rigid surface. (c) KH synthetics for acoustic case. (d) KH synthetics for elastic case. (e) Plot of modulus of reflection coefficient versus angle of incidence. Note the discontinuous behaviour near the critical angle.
Kirchhoff–Helmholtz synthetics

Figure 4 – continued
The next example is intended to show the effect of caustics on a surface of integration. For this purpose we chose a model consisting of two layers over a half-space. The first interface has a small sloped fault structure while the second interface is entirely flat. Fig. 6(a) shows the source-receiver spread and the ray field of the reflection from the upper discontinuity, for a source located at $x=30$ km. The ray field associated with the primary reflections from the deeper interface is shown in Fig. 6(b). Both reflections have caustics and triplications on the surface containing the receivers so geometrical optics, sometimes referred to as asymptotic ray theory, could not be used to synthesize these reflections. In order to synthesize the first reflection we used KH theory in the form of an integral over the first reflector (equation 4). Since the shallowest layer is homogeneous, neither the source wavefield shown in Fig. 6(c) nor any of the receiver wavefields, one of which is shown in Fig. 6(d), has a caustic on the upper reflector. Thus KH theory works very well for the first reflection as can be seen from synthetic seismograms shown in Fig. 6(e). The reflection from the deeper interface is more of a problem. The source ray field shown in Fig. 6(c) has no caustics on the deeper interface but the ray field from the receiver at 34.0 km, shown in Fig. 6(d) does have a caustic there. This reflection can be synthesized by one of the non-standard KH integrals introduced in paper I. There it was shown that the reflection from the deeper interface can be synthesized using any one of three different, but in theory equivalent, KH integrals. One of these involves an integral over the reflector itself. The other two involve

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**Figure 5.** Reflections from a vertical fault. (a) Reflected ray field and the source-receiver geometry. The portion QR of the reflector is not lit by rays. (b) Impulse response of the model. The arrows mark the positions where the reflection and diffraction arrivals are time coincident. (c) and (d): comparison of experimental data with KH synthetics.
integrals over the shallower reflector. In the first non-standard KH integral we use the values of the source wavefield on its way down through the first interface and the values of the receiver wavefield on its way up through that interface after reflection. In the second non-standard KH integral we use the values of the source wavefield on its way up through the first interface (after reflection from the second interface) and the values of the receiver wavefield on its way down through the first interface. Fig. 6(b) shows that geometrical optics can be used to evaluate the integrand of the second non-standard integral without resorting to the device of integrable singularities. Synthetics calculated this way are shown in Fig. 6(e) and the part of the record section near the receiver caustic is shown enlarged in Fig. 6(f). We next computed synthetics for this reflection with a standard KH integral over the deeper reflector, by treating the caustics as integrable singularities of the receiver wavefield. This is the procedure explained in Section 3, and in the Appendix. Synthetics calculated this way are shown enlarged in Fig. 6(g). The portion of the synthetics not shown is the same as Fig. 6(e). Usually we expect to see spurious phases caused by a singularity which, although integrable, acts like a point source (Frazer & Sinton 1984). In the present case, the caustics are at a fairly large distance from the stationary phase (primary reflection) points and their contribution is too small to be detected.

Paper 1 discussed the theory of computing refracted rays in media for which velocity may vary within layers as well as across them. As an example we consider a model for which the medium above the surface of material discontinuity is homogeneous and below the surface the velocity
Figure 6. Two-layer model: (a) Reflected ray field from the shallow and (b) deep interface both showing triplication. (c) Source ray field which is regular everywhere on both the reflectors. (d) Rays drawn from one of the receiver locations show caustics on the deeper reflector. (e) Synthetics computed by a modified KH approach (see text). (f) Enlarged portions of synthetics shown in (e). (g) Enlarged portion of record section computed by standard KH algorithm.
changes continuously with depth (Fig. 7a). The source is located in the half-space, below the surface of discontinuity, and the receivers are located in the upper medium. The model shows a first-order velocity discontinuity at a depth of 3.0 km where velocity changes discontinuously from 1.5 to 3.5 km s⁻¹. In addition, it shows an iso-velocity line with a velocity of 5.5 km s⁻¹ at a depth of 12.0 km. Along any vertical line velocity is obtained by a linear interpolation between the two nearest iso-velocity lines. Rays were traced both from the source and each receiver down to the interface and integration was carried along the surface of material discontinuity. Unlike the previous examples, the source rays are curved while the receiver rays are straight lines. For computing the diving ray arrivals the signs of $\theta_1$ and $\theta_2$ were chosen in accordance with the convention outlined in Fig. 3(e). Synthetic seismograms showing the refracted arrivals are shown in Fig. 7(b). To model the effects of energy loss at the discontinuity correctly we used
generalized elastic transmission coefficients which are functions of both the angle of incidence and the angle of transmission (Frazer & Sen 1985). The pulse shapes of the synthetics shown in Fig. 7(b) and the agreement of the synthetics with geometrical optics amplitudes (not shown) indicate how well the method works.

In the following example we have applied a number of the techniques demonstrated above to model a part of one of the seismic refraction lines recorded by an ocean bottom seismometer in the vicinity of the Middle America trench offshore Guatemala. The Hawaii Institute of Geophysics deployed a network of OBSs to record shot profiles for a study of the tectonics and geology of the area. The purpose of the experiment was to determine the velocity structure in the lower slope region of the subduction complex. The OBS locations and locations of the shot lines are marked in Fig. 8(a). Most of these lines have been interpreted by Ambos & Hussong (1984) by 2-D ray tracing. Their results show a gradual increase of velocity away from the trench axis in the landward direction and the existence of a prominent reflector (a first-order velocity discontinuity) marking the plate interface (Fig. 8b). We have used here only the vertical component data of that part of shot line 3 which is landward of the trench axis. This line was recorded by OBS 'D' (Fig. 8a). These data are displayed in record section form in Fig. 8(d). The amplitudes were corrected for shot size and geometrical divergence and were normalized to a seismic trace situated at the range of 3.3 km. The following formula was used:

$$AMP = (R/R_0)^{1.2}(W_0/W)^{0.66}$$
Kirchhoff–Helmholtz synthetics

where $R_0$ and $W_0$ are the range (km) and shot size (kg) for the standard trace to which all the traces were normalized. Each trace was multiplied by an appropriate ‘AMP’ factor. The effect of instrument automatic gain control was also removed. We restricted our analysis to a horizontal range of 15 km from the trench axis since beyond this range the spatial sampling of the data is very poor and the amplitude information is not very reliable. Ambos & Hussong (1984) analysed this data set by tracing rays from a shot located at the OBS location up to the surface and matching the travel time with the data within the range of 5–15 km. They marked the arrivals of two distinct phases: (i) a refracted (turning ray) arrival discernible up to a horizontal range of approximately 17 km and (ii) a reflection (post-critical) continuing up to a horizontal range of approximately 11 km. Using this model we computed the synthetics shown in Fig. 8(c). These synthetics do not show the relative amplitude behaviour of the observations in Fig. 8(e), particularly the decay of the amplitude of the first arriving refraction phase beyond a range of 10 km, though the travel time shows an extremely good match. Our proposed model and the ray trace are shown in Fig. 8(d). For these models, we used the ray trace method of Sinton & Frasier (1982). The general trend of the model is similar to the one proposed by Ambos & Hussong. We require a gradual increase in velocity landward of the trench axis in addition to a 0.5–1.2 km thick low-velocity zone between the ocean bottom and the first-order discontinuity marking the plate interface. The first turning rays (marked AA’ in Fig. 8e) appear up to 10.8 km range and then turn into low-velocity sediments and reappear at approximately 20 km. This second branch of turning rays (marked CC’ in Fig. 8e) is a reverse branch and terminates at 12.3 km range. The reflection branch (marked BB’ in Fig. 8e) terminates at a range of 11.6 km. For computing KH synthetic seismograms for this model we chose the ocean bottom as the surface of integration. We traced rays from each one of the receivers down to the ocean bottom and also from the source up to the ocean bottom surface. We may note here that the principle of reciprocity was applied and the source and receiver location was interchanged. Three KH integrals, one for each one of the phases (AA’, BB’, CC’) were computed. As discussed in the previous example, generalized reflection and transmission coefficients were also used in this case. Fig. 8(e, f) shows a comparison of KH synthetic seismograms with the data and it can be seen that a fairly good agreement of travel time and amplitude was obtained although the travel-time match is slightly degraded from the initial model. Since it was not possible to retrieve a unique source function from the data, the waveforms computed by synthetic seismograms show a slight mismatch with those of the data. We interpret that the high amplitudes in the 9–13 km range are caused by focusing of the rays. The small spurious arrivals in our K–H synthetics are caused by the truncation of the surface of integration. The main feature of our model differing from that of Ambos & Hussong (Fig. 8b) is a low-velocity zone with a gradual increase of velocity in the landward direction.

To obtain a geologically plausible explanation for the velocity inversion zone seemingly present within the subduction complex, we may note that the top of the subducting Cocos plate consists of approximately 400 m of sedimentary material (Aubouin et al. 1982). These sediments, as well as the turbidites ponded in the trench axis area that may also be subducted, have a high fluid content and are of low velocity (1.7 km s$^{-1}$; Aubouin et al. 1982) prior to subduction. During subduction, these sediments compact and dewater (Burst 1976) and thus increase in velocity, although they may still be of lower velocity than the sedimentary and igneous material comprising the overlying plate (e.g. Hole 494A, DSDP Leg 67). Thus it is reasonable to propose a low-velocity sandwich of subducting sediments which increases in velocity and decreases in thickness with increasing overburden landward of the trench.

At this stage we may note the following facts: (a) no reverse profile was available for the refraction line 3 (OBS D), (b) knowledge of the source function was not adequate, (c) spatial resolution of the data beyond a horizontal range of 13 km was poor.

Hence one may not regard our proposed model as unique so that the geologic and tectonic
Figure 8. (a) Location of the Middle America Trench (MAT) experiment P-M = Polochic, Motagua and associated left-lateral faults marking the North America–Caribbean plate boundary in Guatemala. TR = Tchauantepec Ridge, a prominent offshore ridge separating the Mexican and Guatemala portions of the Middle America Trench. The main diagram indicates the location of seismic lines and OBSs. The 600 m bathymetry contour roughly defines the trench axis location. (b) Ambos & Hussong model. (c) KH synthetic seismograms for the model in (b). (d) Proposed model for Line 3. OBS 'D' used in the analysis. (e) and (f) Comparison of refraction data with KH synthetic seismograms for the model in (d).
Kirchhoff–Helmholtz synthetics

Figure 8 – continued
interpretation is of necessity limited. Nonetheless, the delineation of 2-D features of no more than several hundred metres of relief, using these synthetics, shows the potential resolution of our KH synthetics for the 2-D case.

5 Discussion and conclusions

In this paper we have shown how elastic KH theory can be used to compute synthetic seismograms for models of geological interest where the elastic properties vary laterally. Use of a generalized Filon method (GFM) quadrature formula enables the rapid computation of KH integrals in the frequency domain. Computation time depends on the complexity of the model and is proportional to the number of layers in the model. Complex geological structures require a large number of rays to be traced so that the reflectors are lit uniformly by the rays and irregularities in the reflectors are well sampled on the scale of a wavelength of the signal. In our example trench model, the computation of the series of synthetic seismograms (Fig. 8f) required less than 5 min of CPU time (not including the time for ray-tracing) on a Harris H800 computer which has a Whetstone number of 1470.

One advantage of the KH technique is that it is able to model the frequency-dependent diffraction behaviour from corners present on a reflector whereas other methods (Maslov theory, Gaussian beams) cannot. Since the KH seismogram computation is fairly rapid, it can be used for routine analysis of high-frequency reflection/refraction seismograms from areas of complex geology. The problem of caustics on a reflector can often be avoided by choosing some other surface of integration on which both the source and receiver ray fields are regular. The alternative surface need not necessarily coincide with any surface of material discontinuity. In many cases, one can find a surface of integration on which both the source and receiver ray fields are regular; however, if this is not possible, one can use the device of integrable singularities or multi-fold KH theory. Computation of synthetic seismograms using multi-fold KH theory will be the subject of a separate paper.

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References


Kirchhoff–Helmholtz synthetics


Appendix

Here we derive formulae to be used for evaluating our integral

\[ I = \int \frac{f(x)}{\Sigma} \exp \left\{ B\phi(x) \right\} \, dx \]  

at caustics where the integrand is singular. The function \( f(x) \) contains the factors \((d\sigma_1/d\theta_1)^{-1/2}\) and \((d\sigma_2/d\theta_2)^{-1/2}\) and it can be seen that \( d\sigma_1/d\theta_1 \) vanishes when the source wavefield has a caustic on \( \Sigma \) and \( d\sigma_2/d\theta_2 \) vanishes when the receiver wavefield has a caustic on \( \Sigma \). In either of these cases the travel-time function \( \phi(x) \) becomes multivalued. Frazer & Sinton (1984) have discussed in detail the behaviour of \( \sigma \) and \( d\sigma/d\theta \) with the variation of \( \theta \) in different situations and have shown how the integration over \( \Sigma \) can be expressed as a sum of integrals on overlapping subsets of \( \Sigma \) such that \( \phi(x) \) is single valued on each subset. Since we are treating only the 2-D case, in the integral (A1) \( x \) denotes the distance of a point on the surface of integration measured from a reference point. We assume that either \( d\sigma_1/d\theta_1 \) or \( d\sigma_2/d\theta_2 \) vanishes at one of the end points of an interval of integration (Fig. A1) and first consider the case where \( x_2 \) is the right end point, i.e. \( x_1 < x_2 \) (Fig. A1a). We can assume that near \( x_2 \), the function \( f(x) \) in equation (A1) behaves like \( A(x_2 - x)^{-1/2} \) and the phase function \( \phi(x) \) is smoothly varying and can be written as \( \phi(x) = C + D(x_2 - x) \) where \( A, C \) and \( D \) are constants to be evaluated from the known values of \( f(x_1), \phi(x_1) \) and \( \phi(x_2) \). Thus (A1) takes the form

\[ I = \int_{x_1}^{x_2} f(x) \exp \left\{ B\phi(x) \right\} \, dx \]

= \[ A \exp(BC) \int_{x_1}^{x_2} (x_2 - x)^{-1/2} \exp \left\{ BD(x_2 - x) \right\} \, dx \]

= \[ A \exp(BC) \int_{x_1}^{x_2} u^{-1/2} \exp(BD) \, (-du) \quad \text{where } u = x_2 - x \]

= \[ A \exp(BC) \sum_{n=0}^{\infty} \frac{1}{n!} (BD)^n \int_{x_1}^{x_2} u^{-1/2} u^n \, du \]

= \[ A \exp(BC) \sum_{n=0}^{\infty} \frac{(BD)^n}{n!(n+1/2)} \int_{x_1}^{x_2} u^{n+1/2} \, du \]

\[ I = A \exp(BC) \sum_{n=0}^{\infty} \frac{(BD)^n(x_2 - x_1)^{n+1/2}}{n!(n+1/2)} \]  

(A2)

Figure A1. (a) \( d\sigma_1/d\theta_1 \) approaches zero at \( x_2(x_2 > x_1) \). (b) \( d\sigma_2/d\theta_2 \) approaches zero at \( x_1(x_2 > x_1) \).
We next consider the case where the caustic is at the left end point $x_1$ (Fig. A1b). The functions $f(x)$ and $\phi(x)$ can be assumed to take the following forms

\[ f(x) = A(x-x_1)^{-1/2}; \quad \phi(x) = C + D(x-x_1). \]

Hence (A1) can be written as

\[
I = \int_{x_1}^{x_2} f(x) \exp\{B\phi(x)\} \, dx
\]

\[
= A \exp(BC) \int_{x_1}^{x_2} (x-x_1)^{-1/2} \exp\{BD(x-x_1)\}
\]

\[
= A \exp(BC) \int_{0}^{x_2-x_1} u^{-1/2} \exp(BDu) \, du; \quad \text{where } u = x-x_1
\]

\[
= A \exp(BC) \sum_{n=0}^{\infty} \frac{(BD)^n}{n!} \int_{0}^{x_2-x_1} u^n u^{-1/2} \, du
\]

\[
I = A \exp(BC) \sum_{n=0}^{\infty} \frac{(BD)^n (x_2-x_1)^n u^{n+1/2}}{n! (n+1/2)} \quad (A3)
\]

which is the same as (A2); however, notice that $A$, $C$, $D$ are defined quite differently.