Solid Earth - Geodesy



# DEVELOPMENT OF A NEW THEORY FOR DETERMINATION OF GEOPOTENTIAL FROM THE ORBITAL MOTION OF ARTIFICIAL SATELLITES

Development of a New Theory for Determination of
Geopotential from the Orbital Motion
of Artificial Satellites. (PhD Thesis)
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A DISSERTATION SUBMITTED TO THE GRADUATE DIVISION OF THE
UNIVERSITY OF HAWAII IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN GEOSCIENCES-SOLID EARTH GEOPHYSICS

JUNE 1967

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## ABSTRACT

A new theory has been developed to exploit the satellite data particularly the position vector and the relative velocity of a satellite in the problem of obtaining the terrestrial gravity field with special consideration to its localised anomalous features. The new theory makes use of the fact that the dynamical variable Hamiltonian, associated with the satellite motion is time-invariant in the ideal case when all the perturbing forces are neglected. With this as a working premise, it is possible to take into account the effects of perturbing forces such as lunar attraction, air drag, radiation pressure and solar attraction. The ideal case ignoring all the perturbing forces, here called the 'simplified theory' and the more factual case allowing for the effect of the important perturbing forces, here called the 'extended theory' are both discussed in detail. The potential function of the earth appears additively in the Hamiltonian function and can be determined from observations of the position vector and the relative velocity of a satellite at a number of points along a small segment of the orbit. Minimally, there must be as many observations as there are unknown coefficients in the expansion of geopotential but an abundance of measurements is desirable for the application of the least squares method. In case the position vector and the relative velocity of a satellite are not available as directly observed quantities, the equations can be expressed in terms of the orbital elements of the satellite. The theory emphasises the local features of the gravity field by allowing for the fact that a satellite

gives information weighted primarily by conditions in its immediate proximity and thus provides expressions for describing the gravitational potential of regions immediately below its orbit. Theoretically, it appears possible to cover the surface of the earth by overlapping expressions of this type and hence to obtain an adequate description of the gravity field of the earth. The equations of condition obtained when the theory is developed to include the effects of lunar attraction and air drag, are shown to remain valid when all the important perturbations; i.e., lunar attraction, air drag, radiation pressure, solar attraction, etc., are taken into consideration. The method of setting up the equations of condition appears to have the advantage of eliminating the necessity of quantifying the perturbing factors, thus enabling us to avoid some of the poorer approximations involved in the process. The new theory appears to offer the possibility of exploiting the 'short wavelength sensing potentiality' of the low altitude satellites which cannot be used with advantage in the perturbation theory. If the geopotential coefficients can be determined to a fairly high degree of accuracy, the theory theoretically has the potential for determining the time-variant part of the earth's gravity field and may be used to give some idea as to the differential rotation of the core and mantle if the core has a radial asymmetry of mass distribution as one resulting from convection currents within the core.

For purposes of comparison, a short review of the existing method to determine the geopotential using perturbation theory, is included as well as the results obtained by some other investigators in the field.

CHAPTER I

INTRODUCTION

One of the major scientific objectives of the artificial satellite program was the better determination of the geopotential coefficients used for defining the geoid. Although surface gravity measurements permit the geoid to be determined, the fact that over 70% of the earth's surface is water and approximately 30% of the land surface is of difficult access has severely limited gravity coverage. Other limiting factors include uncertain vertical and horizontal control which restricts the reliability of the derived free air gravity anomalies required, and poor position control at sea which imposes a severe limitation in determining the Eötvös correction necessary for obtaining observed gravity values at sea. As a ship moves independently of the earth, its true motion relative to that of the earth is a significant factor in determining gravity and without precise navigation control to determine true course heading and speed, no reliable estimate of the Eötvös correction can be made.

Although celestial navigation is adequate for traversing the oceans, it is an inferior system for determining accurate geocentric positions because of refraction effects and its susceptibility to local departures of the vertical, and the restrictions imposed by cloud coverage. Recent use of VLF and satellite navigation, however, is overcoming some of these disadvantages.

Electronic navigation aids such as LORAN are also restricted in their usefulness by their limited coverage of the oceans and the uncertainties in base station positions which may be on different geodetic datums and also subject to local departure of the vertical effects.

Satellites appeared to offer a solution to most of these problems in that the orbit which is governed by the earth's external gravity field could be established with high precision by having a suitable network of tracking stations whose geocentric positions could be worked out independently from orbital data for a number of satellites and simultaneous transit observations between tracking stations without recourse to observations influenced by the local gravity field.

By using perturbation theory to analyze tracking data it has been possible to determine the geopotential coefficients for the Earth out to the 15th degree. Although the complete set of coefficients for the 15th degree fit is classified, the published literature includes zonal harmonics to the 14th degree and tesseral harmonics to the 10th degree plus some higher degree tesserals determined from the resonance of satellites. However, there is some question about the absolute reliability of some of the higher degree tesseral coefficients derived so far which show a dependence on the type of tracking data used and the assumptions made by different investigators. Even for an 8th degree fit some disagreement exists in some of the higher degree tesseral and zonal coefficients. In view of these discrepancies, the writer has used a mean set of coefficients (out to 8th degree) based on the values obtained by several investigators to determine a generalized representation of the geoid and the earth's gravity field.

A comparison of two of the most pronounced anomalous features of the satellite results with the available surface gravity information of those regions shows that the satellite results agree with the surface gravity in general. They define broad areas of anomalous gravity where detailed gravity surveys and other pertinent geophysical studies should be undertaken. From a geodetic standpoint, the generalised picture of the geoid and the earth's gravity field has value in computing the effect of local anomalous masses defined by local gravity surveys on the local geoidal heights. However, a detailed comparison of the satellite and gravimetric results shows that there are significant differences in detail. The satellite results are too generalised and do not contain the short wavelength variation of the terrestrial measurements--a feature of great importance in many geophysical applications. Comparisons of the satellite and terrestrial gravity results carried out by other investigators (Kaula, 1966; Khan and Woollard, 1967) indicate the same general results.

As pointed out above, many of the geophysical applications require that satellite data be used to define the earth's gravity field to a greater degree of detail than has been done up to now. There are inherent limitations in the perturbation theory to accomplish this. The low order harmonic representation of the gravity field does not give the required degree of detail, and the accurate determination of the higher degree tesserals to obtain a higher degree harmonic representation of the earth's gravity field is complicated in the perturbation theory for reasons explained later.

In order to achieve the degree of detail required in the definition of the earth's gravity field, the method employed to obtain it must be sensitive to local anomalous features. This appears possible theoretically because the motion of a satellite at any instant is more ~~e~~affected by the nearer mass anomalies than the distant one, and there are a number of dynamical variables associated with the satellite motion which can be used to exploit this relation. With this as a premise, a new theory is presented which gives expressions descriptive of the local gravity field in the region being traversed by the satellite. Since the new theory envisages a local description of the gravity field in contrast to the global description given by perturbation theory as being used now, it is expected that it will define the gravity field to a better degree of detail with due emphasis on local anomalies. Hopefully this description of the gravity field will give some of the subtler short wavelength fluctuations exhibited by terrestrial gravimetry and hence will be adequate for many geophysical applications. It may be emphasized that the new theory is entirely different from the existing perturbation theory in scope, method and principle.

In the second chapter, a short review of the perturbation theory is presented along with the results obtained from its application. In the third chapter we discuss the development of the new theory, outlining its physical bases, its simplified form and its extension to include the effects of the different perturbing forces. The equations of condition are developed both for the 'simplified theory' and for the 'extended theory' and the domain of application in each case is defined as far as possible.

It should be noted that the new theory has not been tested as yet; hence at this stage it is possible to discuss only the theoretical aspects of its applicability. It will be only after examination of the results obtained from the new theory that its real scope and limitations can be defined.

# CHAPTER II

## PERTURBATION THEORY FOR THE DETERMINATION OF GEOPOTENTIAL

### 1. Review of the Perturbation Theory

Let R be the disturbing potential defined as that part of the geopotential which should be added to the potential of spherical attraction to define the total geopotential. The equations for the variation of orbital elements in terms of the disturbing potential R are (Smart, 1961):

$$\dot{a} = \frac{2}{\bar{n}a} \frac{\partial R}{\partial M}$$

$$\dot{e} = \frac{1}{\bar{n}a^2 e} \left[ (1-e^2) \frac{\partial R}{\partial M} - (1-e^2)^{\frac{1}{2}} \frac{\partial R}{\partial w} \right]$$

$$\dot{M} = \bar{n} - \frac{1-e^2}{\bar{n}a^2 e} \frac{\partial R}{\partial e} - \frac{2}{\bar{n}a} \frac{\partial R}{\partial a} \tag{1}$$

$$\dot{\Omega} = \frac{1}{\bar{n}a^2 (1-e^2)^{\frac{1}{2}} \sin i} \frac{\partial R}{\partial i}$$

$$\dot{w} = \frac{(1-e^2)^{\frac{1}{2}}}{\bar{n}a^2 e} \frac{\partial R}{\partial e} - \frac{\cot i}{\bar{n}a^2 (1-e^2)^{\frac{1}{2}}} \frac{\partial R}{\partial i}$$

$$\frac{di}{dt} = \frac{1}{\bar{n}a^2 (1-e^2)^{\frac{1}{2}}} \left[ \cot i \frac{\partial R}{\partial w} - \operatorname{cosec} i \frac{\partial R}{\partial \Omega} \right]$$

where

$\Omega$ = right ascension of the ascending node

$a$ = semi-major axis of the satellite orbit

$\bar{n}$ = mean motion

$e$ = eccentricity

$i$ = inclination

$w$ = argument of perigee

$M$ = mean anomaly defined by $M = \bar{n}\,(t - T_0)$

and

$T_0$ = time of perigee passage

The disturbing potential R can be expressed in terms of spherical harmonics as:

$$(2) \qquad R = \frac{GM}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm} (\sin \phi)$$

where

$\phi$, $\lambda$, $r$ = latitude, longitude and the geocentric distance of point being considered

$a_e$ = earth's equatorial radius

$M$ = mass of the earth

$G$ = universal constant of gravitation

$C_{nm}$, $S_{nm}$ = conventional geopotential coefficients

$P_{nm}$ $(\sin \phi)$ = conventional associated Legendre's functions.

The general term $R_{nm}$ of this expansion is:

$$(3) \qquad R_{nm} = \frac{GM}{r} \left(\frac{a_e}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm} (\sin \phi)$$

By an essentially geometrical transformation, the general term $R_{nm}$ can be expressed in terms of the Keplerian elements. The final result of this conversion is (Kaula, 1961; Khan and Woollard, 1967):

$$(4) \qquad R_{nm} = \frac{GMa_e^n}{a^{n+1}} \sum_{p=0}^{n} F_{nmp}\,(i) \sum_{q=-\infty}^{+\infty} G_{npq}\,(e)\, Z_{nmpq}\,(w, M, \Omega, \theta)$$

or denoting one term of the above expression by $R_{nmpq}$, we get:

$$R_{nmpq} = \frac{GMa_e^n}{a^{n+1}} F_{nmp}(i)\; G_{npq}(e)\; Z_{nmpq}\; (w, M, \Omega, \theta) \tag{5}$$

where

$$F_{nmp}(i) = \sum_t \frac{(2n-2t)}{2^{2n-2t}\,(n-m-2t)!n!} \binom{n}{t} \sin^{n-m-2t} i$$

$$\sum_{s=0}^{m} \binom{m}{s} (-1)^k \cos^s i \sum_c \binom{n-m-2t+s}{c} \cdot \binom{m-s}{p-t-c} (-1)^c$$

$$Z_{nmpq}\; (w, M, \Omega, \theta) = \left[\begin{matrix} C_{nm} \\ -S_{nm} \end{matrix}\right]\begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \cos\,[(n-2p)\; w +$$

$$(n-2p+q)\; M + m(\Omega-\theta)] + \left[\begin{matrix} S_{nm} \\ C_{nm} \end{matrix}\right]\begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \sin\,[(n-2p)\; w +$$

$$(n-2p+q)\; M + m\; (\Omega-\theta)]$$

and the limits of the above summations are defined as follows:

p - summation:

$$0 < p < n$$

c - summation:

$$\begin{matrix} p-t < m-s, & 0 & & n-m-2t+s, & p-t > n-m-2t+s \\ & & <c< & & \\ p-t > m-s, & p-t-m+s & & p-t\quad, & p-t < n-m-2t+s \end{matrix}$$

t - summation:

$$0 < t < \begin{matrix} p, & p < k \\ k, & p > k \end{matrix}\;; \qquad k = \begin{matrix} \frac{n-m}{2}, & \text{for } n-m \text{ even} \\ \frac{n-m-1}{2}, & \text{for } n-m \text{ odd} \end{matrix}$$

The development of the functions $G_{npq}(e)$ depends upon whether the perturbation is long periodic or short periodic. For a long-period perturbation, $n-2p+q=0$. For this case the function $G_{npq}(e)$ becomes:

$$G_{np(2p-n)}(e) = \frac{1}{(1-e^2)^{n-\frac{1}{2}}} \sum_{d=0}^{p'-1} \binom{n-1}{n+2d-2p'} \binom{n+2d-2p'}{d}$$
$$\cdot \left(\frac{e}{2}\right)^{n+2d-2p'}$$

$$p' = \begin{cases} p & \text{for } p < \frac{n}{2} \\ n-p & \text{for } p > \frac{n}{2} \end{cases}$$

For short periodic terms, $n-2p+q \neq 0$ and the development of $G_{npq}(e)$ becomes complicated. The resulting expressions are, however, of the form:

$$G_{npq}(e) = (-1)^{|q|} \left[1 + \frac{e^2}{(1+\sqrt{1-e^2})^2}\right]^n \left[\frac{e}{1+(1-e^2)^{\frac{1}{2}}}\right]^{|q|}$$

$$\sum_{k=0}^{\infty} P_{npqk}\, Q_{npqk} \left[\frac{e}{1+(1-e^2)^{\frac{1}{2}}}\right]^{2k}$$

where

$$P_{npqk} = \sum_{r=0}^{h} \binom{2p'-2n}{h-r} \left[\frac{(n-2p'+q')\ [1+(1-e^2)]^{\frac{1}{2}}}{2}\right]^r \frac{(-1)^r}{r!}$$

and

$$Q_{npqk} = \sum_{r=0}^{\ell} \binom{-2p'}{\ell-r} \left[\frac{(n-2p'+q')\ [1+(1-e^2)]^{\frac{1}{2}}}{2}\right] \frac{1}{r!}$$

The summation limits in the above expressions are defined as:

$$h \begin{cases} = k+q' & \text{for } q' > 0 \\ = k & \text{for } q' < 0 \end{cases}$$

$$\ell \begin{cases} = k & \text{for } q' > 0 \\ = k-q' & \text{for } q' < 0 \end{cases}$$

$$\begin{matrix} p' = p \\ q' = q \end{matrix} \quad \text{for } p < \frac{n}{2}$$

$$\begin{matrix} p' = \ell-p \\ q' = -q \end{matrix} \quad \text{for } p > \frac{n}{2}$$

Differentiate Eq. (5) with respect to various orbital elements and substitute the result in Eq. (1). This would give the time rate of change of orbital elements. The resulting equations giving the time variation of the orbital elements corresponding to $R_{nmpq}$ would then be:

$$\frac{d\Omega_{nmpq}}{dt} = \frac{GMa_e^n F'_{nmp} G_{npq} Z_{nmpq}}{\bar{n}a^{n+3}(1-e^2)^{\frac{1}{2}} \sin i}$$

$$\frac{dw_{nmpq}}{dt} = GMa_e^n \left[\frac{(1-e^2)^{\frac{1}{2}}}{e} F_{nmp} G'_{npq} - \frac{\cot i}{(1-e^2)^{\frac{1}{2}}} F'_{nmp} G_{npq}\right] \frac{Z_{nmpq}}{\bar{n}a^{n+3}}$$

$$\frac{di_{nmpq}}{dt} = \frac{GMa_e^n F_{nmp} G_{npq} Z'_{nmpq}}{\bar{n}a^{n+3}(1-e^2)^{\frac{1}{2}} \sin i} [(n-2p) \cos i-m]$$

(6)

$$\frac{da_{nmpq}}{dt} = \frac{2GMa_e^n F_{nmp} G_{npq} Z_{nmpq}}{\bar{n}a^{n+2}} (n-2p+q)$$

$$\frac{de_{nmpq}}{dt} = \frac{GMa_e^n F_{nmp} G_{npq} Z'_{nmpq}}{\bar{n}a^{n+3}e} [(1-e^2) (n-2p+q)$$

$$- (1-e^2)^{\frac{1}{2}} (n-2p)]$$

$$\frac{d\ M^*_{nmpq}}{dt} = \frac{GMa_e^n F_{nmp} Z_{nmpq}}{\bar{n}a^{n+3}} \left[2(n+1)G_{npq} - \frac{1-e^2}{e} G'_{npq}\right]$$

In the foregoing expressions:

$M^*$ = perturbation of the mean anomaly = $\int_{T_0}^{t} \bar{n}\ dt-\bar{n}(t-T_0)$

$$F'_{nmp} = \frac{dF_{nmp}}{di}$$

$$G'_{npq} = \frac{dG_{npq}}{de}$$

and

$Z'_{nmpq}$ = the derivative of $Z_{nmpq}$ with respect to its argument.

If it is now assumed that the dominant perturbations of the orbital elements are secular, it is possible to integrate the equations of motion to obtain the integrated changes in the orbital elements caused by the perturbing potential function. To get the expressions for the integrated changes, consider

$$\int Z' \, dt = \overline{Z}' = \frac{Z}{\frac{d}{dt}\,[\text{argument of } Z']}$$

$$\int Z \, dt = \overline{Z} = \frac{Z^{i}}{\frac{d}{dt}\,[\text{argument of } Z]}$$

where $Z^{i}$ = the integral of Z with respect to its argument.

Note that the integration in the above way is valid only if the orbital elements appearing in the argument of Z are linear functions of time.

Substitution of Eq. (5) in the equations of motion and the subsequent integration with respect to time gives the desired expressions for the integrated perturbations due to the effect of $R_{nmpq}$ as follows:

$$\Delta\Omega_{nmpq} = GMa_e^n \frac{F'_{nmp}\, G_{npq}\, Z^{i}_{nmpq}}{\overline{na}^{n+3}\,(1-e^2)^{\frac{1}{2}} \sin i[(n-2p)\dot{w}+(n-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]}$$

$$\Delta w_{nmpq} = GMa_e^n \frac{}{\overline{na}^{n+3}}$$

$$\frac{[e^{-1}(1-e^2)^{\frac{1}{2}}F_{nmp}G'_{npq} - \cot i(1-e^2)^{-\frac{1}{2}}F'_{nmp}G_{npq}]Z^{i}_{nmpq}}{[n-2p)\dot{w}+(n-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]}$$

$$\Delta i_{nmpq} = GMa_e^n \frac{F_{nmp} \; G_{npq} \; [(n-2p)\cos i - m]}{\bar{n}a^{n+3}(1-e^2)^{\frac{1}{2}} \sin i}$$

$$\frac{Z_{nmpq}}{[(n-2p)\dot{w}+(n-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]}$$

$$\Delta a_{nmpq} = GMa_e^n \frac{2F_{nmp} \; G_{npq} \; Z_{nmpq} \; (n-2p+q)}{\bar{n}a^{n+2}[(n-2p)\;\dot{w}+(n-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]}$$

(7) $$\Delta e_{nmpq} = GMa_e^n \frac{F_{nmp} G_{npq} Z_{nmpq}}{\bar{n}a^{n+3} \; e}$$

$$\frac{[(1-e^2)(n-2p+q)-(1-e^2)^{\frac{1}{2}}(n-2p)]}{[(n-2p)\dot{w}+(n-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]}$$

$$\Delta M_{nmpq} = GMa_e^n \frac{F_{nmp} Z^i_{nmpq}}{\bar{n}a^{n+3}}$$

$$\frac{[2(n+1)G_{npq} - (1-e^2)e^{-1}G'_{npq}]}{[(n-2p)\dot{w}+(n-2p+q)\dot{M}+m(\dot{\Omega}-\dot{\theta})]}$$

For long-period variations, we must have $n-2p+q = 0$ or $q = 2p-n$. With this condition $Z_{nmpq}$ becomes $Z_{nmp(2p-n)}$ and is now independent of the terms in M. Further, if we are interested only in the effect of zonal harmonics, we have m=0 and for the long-period effects of the zonal harmonics, the functions $Z_{nmpq}$ become:

$$Z_{nop(2p-n)} \begin{cases} = C_{n0} \cos(n-2p)w & , \; n \text{ even} \\ = C_{n0} \sin(n-2p)w & , \; n \text{ odd} \end{cases}$$

For the long-period effects, $G_{npq}$ becomes $G_{np(2p-n)}$. While computing G, considerable labor can be saved by remembering that $G_{np(2p-n)} = G_{n(n-p)(n-2p)}$.

In the case of zonal harmonics, m=0 and $F_{nmp}$ becomes:

$$F_{n0p} = \sum_t \frac{(2n-2t)!}{2^{2n-2t}(n-2t)!n!} \binom{n}{t} \sin^{n-2t} i \; (-1)^k \sum_c \binom{n-2t}{c} \binom{0}{p-t-c} (-1)^c$$

The last binomial coefficient will be non-zero only if p-c-t = 0 or c = p-t. Thus for a particular value of p, there is only one value of c corresponding to every value of t and the c-summation can then be substituted by that value. Thus $F_{n0p}$ finally becomes:

$$F_{n0p} = \sum_t \frac{(2n-2t)!}{2^{2n-2t}(n-2t)!n!} \binom{n}{t} \sin^{n-2t} i \; (-1)^k \binom{n-2t}{p-t} (-1)^{p-t}$$

Now examine Eq. (6). Put m=0 for convenience of discussion. Then the expressions for $\dot{\Omega}$ and $\dot{w}$ can be written as

$$\frac{d\Omega}{dt} = \sum_{n=2}^{\infty} \sum_{j=0}^{n-2} C_{n0} X_{1n} X_{1nj} \begin{matrix} \cos j w & j \text{ and } n \text{ even} \\ \sin j w & j \text{ and } n \text{ odd} \end{matrix}$$

(8)

and

$$\frac{dw}{dt} = \sum_{n=2}^{\infty} \sum_{j=0}^{n-2} C_{n0} X_{2n} X_{2nj} \begin{matrix} \cos j w & j \text{ and } n \text{ even} \\ \sin j w & j \text{ and } n \text{ odd} \end{matrix}$$

Where the X's can be obtained by comparing these expressions with the original Eq. (6).

Similar expressions can be written for the time-rate of change of other orbital elements and their integrated changes.

It can be verified easily that $X_{1ni}$ and $X_{2n1}$, the coefficients of the term sin w, contain a factor $\frac{1}{\sin i}$. Also $X_{2n1}$, the coefficient of sin w is the expression for $\dot{w}$, contains $\frac{1}{e}$. Hence this theory fails for sin i = 0 or i = 0 and e = 0, i.e., the Eq. (6) in invalid for orbits of zero inclination and zero eccentricity. This holds for Eq. (7) also.

The physical explanation for this is simple. For zero inclination the position of the ascending node and for a circular orbit, the position of the perigee cannot be defined and hence the above relations become meaningless.

In addition, all expressions in Eq. (7) (for the case of the secular or long-period effects of zonal harmonics) contain $\dot{w}$ in the denominator and therefore are not valid for $\dot{w} = 0$. To investigate this point a little further consider the effect of $C_{20}$ on perturbation in w. The expression for $\dot{w}$ in this particular case is

$$\dot{w} = K\, C_{20}\, (1 + \cos^2 i - 3/2 \sin^2 i)$$

where K is independent of i. For $\dot{w} = 0$, i is roughly 63°26'. This value of i is called the <u>critical inclination</u>. Thus the equations for integrated changes are not valid for orbits with critical inclination.

Note that in Eq. (8), the coefficient X represents the amplitudes of different perturbations of wave length $\frac{2\pi}{j}$. For j = 0, cos j w = 1 and hence the perturbations are secular. Note also that it is only for even values of n that the secular terms appear in the expressions. For odd values of n, the expressions contain only long-period terms.

Similar remarks can be made for other orbital elements.

To sum up, the even zonal harmonics give rise to secular and long-period changes in $\Omega$, $w$, and $M$ and long-period changes in $e$ and $i$. The odd zonal harmonics produce long-period perturbations in $\Omega$, $i$, $e$, $w$, and $M$.

In practice, the even zonal harmonics are determined from the secular motion of the right ascension of the ascending node and argument of perigee, and the odd zonal harmonics, from the long-period changes in the orbital elements.

Note that before the observed perturbations can be used in the above equations for determining zonal harmonic coefficients, they must be freed from the effects of variations from all other possible sources. To avoid a serious interplay of the interaction terms among the different perturbations, satellites greatly affected by air resistance, lunisolar attraction and radiation pressure should not be used. Also if the motion of the perigee is too slow, it may be hard to distinguish between the long period and secular effects.

An accurate determination of the tesseral and sectorial harmonic coefficients is rendered more difficult because of the short periods and the small amplitudes of the perturbations involved. The occurrence of common periodicities in perturbations arising from different harmonic terms adds to the difficulty. The resonance method has been said to be fairly sensitive in the determination of some tesseral harmonic coefficients but it can only be used for specific satellites.

The calculation of both the zonal and the tesseral harmonic coefficients using perturbation theory appears to show some degree of sensitivity to the types of perturbations used, the method of removing the perturbations originating from sources other than gravitational, the type, number and distribution of the satellite observations and the time interval of analysis.

## 2. Method of Computing the Gravity Field of the Earth

The Geoidal Undulation:

The geoidal undulation N at a point whose coordinates are (r, $\phi$, and $\lambda$) is given [Mueller, 1964] as:

$$N = \frac{GM}{a_e g_0} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[ \left(\frac{a_e}{r}\right)^{n+1} (\Delta C_{nm} \cos m\lambda + \Delta S_{nm} \sin m\lambda) P_{nm} (\sin \phi) \right] \tag{9}$$

where

$\phi$, $\lambda$, and r = latitude, longitude and the geocentric distance of the point of computation, respectively

$a_e$ = earth's equatorial radius

$g_0$ = normal gravity at the point of computation on the reference ellipsoid

M = mass of the earth

G = universal constant of gravitation

$C_{nm}$, $S_{nm}$ = conventional geopotential coefficients

$P_{nm}$ (sin $\phi$) = conventional associated Legendre's functions and

$$\Delta C_{nm} = C_{nm} - \text{reference } C_{nm}$$

$$\Delta S_{nm} = S_{nm} - \text{reference } S_{nm}$$

If the reference surface is so chosen that all of its geopotential coefficients called the 'reference $C_{nm}$'s' are zero except $C_{20}$ and $C_{40}$, then $\Delta C_{nm}$ and $\Delta S_{nm}$ can be replaced by the observed $C_{nm}$ and $S_{nm}$ except for n = 2, 4 and m = 0.

We can simplify Eq. (2) by making the following approximations:

$$\text{(i)} \ \frac{GM}{a_e^2} = g_0 \text{ which gives } \frac{GM}{a_e g_0} = a_e$$

$$\text{(ii)} \ r = a_e \text{ which gives } \left(\frac{a_e}{r}\right)^{n+1} = 1$$

The error introduced by approximation (i) arises from the fact that the equatorial value of gravity $g_e$ computed on the basis of a spherical earth with $g_e = \frac{GM}{a_e^2}$ using recent values of GM and $a_e$, is not the same as $g_0$, the theoretical value of gravity at the point of computation on the reference ellipsoid. The maximum magnitude of this error is about 0.5% and occurs at the poles. Except for a narrow equatorial belt where the sign of this error will depend on the relative magnitudes of $\frac{GM}{a_e^2}$ and theoretical equatorial gravity of the reference ellipsoid, the correction resulting from this error has to be subtracted from the computed value of N in order to arrive at the correct value.

The error due to assumption (ii) arises from the fact that $a_e$ is not equal to r as assumed. The total error e introduced in N from this source turns out to be:

$$e = \sum_n (n + 1) \frac{\Delta r}{a_e} N_n$$

where

$$N_n = \text{contribution to N due to the } n\underline{th} \text{ term}$$

and

$$\Delta r = a_e - r$$

Since $a_e > r$, the computed value N of the geoidal undulation is always less than its correct $N_c$. Thus

$$N_c = N + \sum_n (n + 1) \frac{\Delta r}{a_e} N_n$$

Note that the error (e) increases with increasing values of n. Thus the error becomes important if the harmonic analysis is carried to a fairly high degree. Although $N_n$ should decrease with increasing values of n and hence should diminish the error (e) in its own right, the error may become a sizable fraction of $N_n$ for higher degree terms, and for very high values of n may equal the partial contribution $N_n$ itself. However, this is more of a theoretical limitation than a practical one, for harmonic analyses under consideration are not likely to be carried to such high degree terms in near future.

The error e increases with $\Delta r$ which is a function of latitude. Thus the error is zero at the equator and a maximum at the poles for any given value of n. Note that e also increases with n.

Since it is inconvenient to calculate $N_n$, it is helpful to express e as the mean of the percentage errors of the individual terms, i.e.,

$$\text{Mean of the percentage error} = \frac{1}{n} \sum_n (n + 1) \frac{\Delta r}{a_e} \cdot 10^2\%$$

For harmonic analyses carried to the 8th degree, the maximum magnitude of this mean percentage error is about 2%. A more detailed discussion of these errors is given in Khan and Woollard (1967).

Neither of the above errors is significant in view of the present accuracy of the satellite results.

The above assumptions reduce Eq. (9) to the following form:

$$(10) \qquad N = a_e \sum_{n=2}^{\infty} \sum_{m=0}^{n} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm}(\sin \phi)$$

Equation (10) has been used to compute the results reported in this section.

The Gravity Anomaly:

The gravity anomaly ($\Delta g$) at any point can be computed from the following equation:

$$(11) \qquad \Delta g = g_s \sum_{n=2}^{\infty} \sum_{m=0}^{n} [(n - 1) .(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\, P_{nm} (\sin \phi)$$

where $g_s = \frac{GM}{a_e^2}$ .

As Eq. (11) incorporates the same assumptions as in Eq. (10) for the geoidal undulations, it is influenced by the same sources of error and in the same degree.

## 3. Discussion of Results

Data Used:

Table I gives the zonal harmonic coefficients obtained by Smith (1963, 1965), Kozai (1964), and King Hele, et. al (1965). Table II lists the tesseral harmonic coefficients obtained by Anderle (1966) Guier and Newton (1965) and Gaposhkin (1966). These values were chosen since they represent the most recent and presumably most reliable data available. Since some discrepancies existed in the corresponding values of the tesseral harmonic coefficients and some of the higher degree zonal harmonic coefficients in different sets, a 'mean solution' was obtained from a linear combination of these sets. The mean normalised coefficients are given in Table III.

Geoidal Undulations:

Table IV summarizes the magnitude and location of the maximum elevations and depressions of the different geoids obtained using the different sets of geopotential coefficients given in Tables I to III. It should be noted that Table IV is made up of three sections. In the first section each geoid is referred to a reference ellipsoid whose $C_{20}$ and $C_{40}$ parameters are those defined by the set of zonal coefficients

TABLE I: NORMALIZED ZONAL HARMONIC

COEFFICIENTS $C_{n0}$ OF THE GEOPOTENTIALS

| | Smith (1963, 1965) | Kozai (1964) | King Hele et al (1965) |
|---|---|---|---|
| n | $C_{n0}10^6$ | $C_{n0}10^6$ | $C_{n0}10^6$ |
| 2 | -484.172 | -484.174 | -484.172 |
| 3 | 0.923 | 0.963 | 0.967 |
| 4 | 0.567 | 0.550 | 0.507 |
| 5 | 0.054 | 0.063 | 0.045 |
| 6 | - 0.202 | - 0.179 | - 0.158 |
| 7 | 0.077 | 0.086 | 0.114 |
| 8 | 0.112 | 0.065 | - 0.107 |

TABLE II: NORMALIZED TESSERAL HARMONIC COEFFICIENTS

$C_{nm}$, $S_{nm}$ OF THE GEOPOTENTIAL

| | | Anderle (1966) | | Guier & Newton (1965) | | Gaposhkin (1966) | |
|---|---|---|---|---|---|---|---|
| n | m | $C_{nm}$ | $S_{nm}$ | $C_{nm}$ | $S_{nm}$ | $C_{nm}$ | $S_{nm}$ |
| | | $10^{-6}$ | $10^{-6}$ | $10^{-6}$ | $10^{-6}$ | $10^{-6}$ | $10^{-6}$ |
| 2 | 2 | 2.45 | -1.52 | 2.38 | -1.20 | 2.38 | -1.35 |
| 3 | 1 | 2.15 | 0.27 | 1.84 | 0.21 | 1.94 | 0.27 |
| | 2 | 0.98 | -0.91 | 1.22 | -0.68 | 0.73 | -0.54 |
| | 3 | 0.58 | 1.62 | 0.66 | 0.98 | 0.56 | 1.62 |
| 4 | 1 | -0.49 | -0.57 | -0.56 | -0.44 | -0.57 | -0.47 |
| | 2 | 0.27 | 0.67 | 0.42 | 0.44 | 0.33 | 0.66 |
| | 3 | 1.03 | -0.25 | 0.84 | 0.00 | 0.85 | -0.19 |
| | 4 | -0.41 | 0.34 | -0.21 | 0.19 | -0.05 | 0.23 |
| 5 | 1 | 0.03 | -0.12 | 0.14 | -0.17 | -0.08 | -0.10 |
| | 2 | 0.64 | -0.33 | 0.27 | -0.34 | 0.63 | -0.23 |
| | 3 | -0.39 | -0.12 | 0.09 | 0.10 | -0.52 | 0.01 |
| | 4 | -0.55 | 0.15 | -0.49 | -0.26 | -0.26 | 0.06 |
| | 5 | 0.21 | -0.59 | -0.03 | -0.67 | 0.16 | -0.59 |
| 6 | 1 | -0.08 | 0.19 | 0.00 | 0.10 | -0.05 | -0.03 |
| | 2 | 0.13 | -0.46 | -0.16 | -0.16 | 0.07 | -0.37 |
| | 3 | -0.02 | -0.13 | 0.53 | 0.05 | -0.05 | 0.03 |
| | 4 | -0.19 | -0.32 | -0.31 | -0.51 | -0.04 | -0.52 |
| | 5 | -0.09 | -0.79 | -0.18 | -0.50 | -0.31 | -0.46 |
| | 6 | -0.32 | -0.36 | 0.01 | -0.23 | -0.04 | -0.16 |
| 7 | 1 | 0.33 | 0.08 | 0.13 | 0.09 | 0.20 | 0.16 |
| | 2 | 0.35 | -0.19 | 0.46 | 0.06 | 0.36 | 0.16 |
| | 3 | 0.32 | 0.04 | 0.39 | -0.21 | 0.25 | 0.02 |
| | 4 | -0.47 | -0.24 | -0.14 | 0.00 | -0.15 | -0.10 |
| | 5 | 0.05 | 0.02 | -0.06 | -0.19 | 0.08 | 0.05 |
| | 6 | -0.48 | -0.24 | -0.45 | -0.75 | -0.21 | 0.06 |
| | 7 | | | 0.09 | -0.14 | 0.06 | 0.10 |

TABLE II: (Continued) NORMALIZED TESSERAL HARMONIC COEFFICIENTS

$C_{nm}$, $S_{nm}$ OF THE GEOPOTENTIAL

| | | Anderle (1966) | | Guier & Newton (1965) | | Gaposhkin (1966) | |
|---|---|---|---|---|---|---|---|
| n | m | $C_{nm}$ | $S_{nm}$ | $C_{nm}$ | $S_{nm}$ | $C_{nm}$ | $S_{nm}$ |
| 8 | 1 | | | -0.15 | -0.05 | -0.08 | 0.07 |
| | 2 | | | 0.09 | -0.04 | 0.03 | 0.04 |
| | 3 | | | -0.05 | 0.22 | -0.04 | 0.00 |
| | 4 | | | -0.07 | -0.04 | -0.21 | -0.01 |
| | 5 | | | 0.08 | 0.00 | -0.05 | 0.12 |
| | 6 | | | -0.02 | 0.67 | -0.02 | 0.32 |
| | 7 | | | 0.17 | -0.07 | -0.01 | 0.03 |
| | 8 | | | -0.15 | 0.09 | -0.25 | 0.10 |

TABLE III: NORMALIZED SPHERICAL HARMONIC COEFFICIENTS OF THE GEOPOTENTIAL $C_{nm}$, $S_{nm}$

(Mean Solution)

| n | m | $C_{nm}$ $10^{-6}$ | $S_{nm}$ $10^{-6}$ | n | m | $C_{nm}$ $10^{-6}$ | $S_{nm}$ $10^{-6}$ |
|---|---|---|---|---|---|---|---|
| 2 | 0 | -484.173 | -- | 6 | 3 | 0.15 | -0.02 |
| | 2 | 2.40 | -1.36 | | 4 | -0.18 | -0.45 |
| 3 | 0 | 0.951 | -- | | 5 | -0.19 | -0.58 |
| | 1 | 1.98 | 0.25 | | 6 | -0.12 | -0.25 |
| | 2 | 0.98 | -0.71 | 7 | 0 | 0.092 | -- |
| | 3 | 0.60 | 1.41 | | 1 | 0.22 | 0.11 |
| 4 | 0 | 0.541 | -- | | 2 | 0.39 | 0.01 |
| | 1 | - 0.54 | -0.49 | | 3 | 0.32 | -0.05 |
| | 2 | 0.34 | 0.59 | | 4 | -0.25 | -0.11 |
| | 3 | 0.91 | -0.22 | | 5 | 0.02 | -0.04 |
| | 4 | - 0.22 | 0.25 | | 6 | -0.38 | -0.31 |
| 5 | 0 | 0.054 | -- | | 7 | 0.07 | -0.02 |
| | 1 | 0.03 | -0.13 | 8 | 0 | 0.024 | -- |
| | 2 | 0.51 | -0.30 | | 1 | -0.11 | 0.01 |
| | 3 | - 0.33 | 0.0 | | 2 | 0.06 | 0.0 |
| | 4 | - 0.43 | -0.02 | | 3 | -0.04 | 0.11 |
| | 5 | 0.11 | -0.62 | | 4 | -0.14 | -0.02 |
| 6 | 0 | - 0.180 | -- | | 5 | 0.01 | 0.06 |
| | 1 | - 0.04 | 0.09 | | 6 | -0.02 | 0.49 |
| | 2 | 0.01 | -0.33 | | 7 | 0.08 | -0.02 |
| | | | | | 8 | -0.20 | 0.09 |

TABLE IV: COMPARISON OF GEOIDAL UNDULATIONS OBTAINED FROM

DIFFERENT SETS OF GEOPOTENTIAL COEFFICIENTS

| Ref. of the Geopotential Coeff. used to Compute the Geoidal Undulation | Maximum Height Above the Reference Ellipsoid | | | Maximum Depression Below the Ref. Ellipsoid | | | Total Range in Meters | Parameters of the Reference Ellipsoid |
|---|---|---|---|---|---|---|---|---|
| | Magnitude in Meters | Location Long. | Lat. | Magnitude in Meters | Location Long. | Lat. | | |
| Kozai (1964); Gaposhkin (1966) | +81 | 140° to 150° | 0° to 5°N | -98 | 75° to 80° | 0° to 10°N | 179 | $C_{20}$ & $C_{40}$ of Table I Col. 2 |
| Smith (1963, 65); Guier & Newton (1965) | +65 | 350° to 355° | 55°N to 60°N | -91 | 75° to 80° | 10°N to 15°N | 156 | $C_{20}$ & $C_{40}$ of Table I Col. 1 |
| King-Hele (1965); Anderle (1966) | +83 | 140° to 150° | 0° to 5°N | -98 | 70° to 80° | 5° to 10°N | 181 | $C_{20}$ & $C_{40}$ of Table I Col. 3 |
| Table III: Mean Coefficients | +70 | 145° to 155° | 5°S to 5°N | -95 | 75° to 80° | 5°N to 10°N | 165 | $C_{20}$ & $C_{40}$ of Table III |

TABLE IV: (Continued) COMPARISON OF GEOIDAL UNDULATIONS OBTAINED FROM DIFFERENT SETS OF GEOPOTENTIAL COEFFICIENTS

| Ref. of the Geopotential Coeff. used to Compute the Geoidal Undulation | Maximum Height Above the Reference Ellipsoid | | | Maximum Depression Below the Ref. Ellipsoid | | | Total Range in Meters | Parameters of the Reference Ellipsoid |
|---|---|---|---|---|---|---|---|---|
| | Magnitude in Meters | Location Long. | Lat. | Magnitude in Meters | Location Long. | Lat. | | |
| Kozai (1964); Gaposhkin (1966) | +94 | 360° to 355° | 60°N to 70°N | -128 | 75° to 80° | 0° to 10°N | 222 | International reference Ellipsoid |
| Smith (1963, 65); Guier & Newton (1965) | +101 | 345° to 355° | 60°N to 65°N | -118 | 75° to 80° | 5°N to 10°N | 219 | -do- |
| Mean Coefficients | +99 | 340° to 355° | 60°N to 70°N | -125 | 75° to 80° | 50°N to 10°N | 224 | -do- |
| Mean Coefficients | +69 | 135° to 165° | 50°S to 15°N | -93 | 65° to 75° | 0° to 20°N | 162 | Best fit Satellite Spheroid Polar Flattening = $\frac{1}{298.25}$ |

TABLE IV: (Continued) COMPARISON OF GEOIDAL UNDULATIONS OBTAINED FROM DIFFERENT SETS OF GEOPOTENTIAL COEFFICIENTS

| Ref. of the Geopotential Coeff. used to Compute the Geoidal Undulation | Maximum Height Above the Reference Ellipsoid | | Maximum Depression Below the Ref. Ellipsoid | | Total Range in Meters | Parameters of the Reference Ellipsoid |
|---|---|---|---|---|---|---|
| | Magnitude in Meters | Location Long. Lat. | Magnitude in Meters | Location Long. Lat. | | |
| | | Other Results | | | | |
| Uotila's geoid (1962) obtained from free air gravity anomalies | +60 | 130° to 150°, 10°S to 10°N | -60 | 60° to 80°, 10°N to 40°N | 120 | Ellipsoid with Flattening = $\frac{1}{298.24}$ |
| Kaula's map (1966) obtained from a combination of satellite and gravimetric data. (Geopotential coeff. for this case not given) | +76 | 135° to 165°, 15°S to 5°N | -90 | 65° to 85°, 10°S to 10°N | 166 | Ellipsoid with Flattening = $\frac{1}{298.25}$ |

TABLE IV: (Continued) COMPARISON OF GEOIDAL UNDULATIONS OBTAINED FROM DIFFERENT SETS OF GEOPOTENTIAL COEFFICIENTS

| Ref. of the Geopotential Coeff. used to Compute the Geoidal Undulation | Maximum Height Above the Reference Ellipsoid | | | Maximum Depression Below the Ref. Ellipsoid | | | Total Range in Meters | Parameters of the Reference Ellipsoid |
|---|---|---|---|---|---|---|---|---|
| | Magnitude in Meters | Location Long. | Lat. | Magnitude in Meters | Location Long. | Lat. | | |
| | | | | | 3 locations of equal value | | | |
| Zongolovich Geoid based on surface gravity data | +80 | 120° to 140° | 3°S to 12°N | -60 | 55° to 80° | 0 to 22°N | 140 | Russian Ellipsoid |
| | | | | | 300° to 340° | 18°S to 20°N | | |
| | | | | | 235° to 265° | 12°S to 35°S | | |

that are used to compute the individual geoid under consideration. Although each set of geoidal undulations was thus derived from a different reference surface, this difference is not significant because the $C_{20}$ and $C_{40}$ values for the different sets of zonal coefficients are in fairly good agreement. Figure 1 shows the geoidal undulations obtained in this manner using the mean coefficients.

In the second section each derived geoid is referred to the international reference ellipsoid. Figure 2 is a plot of the geoid determined on this basis using the mean coefficient values. For comparative purposes data are also given for a geoid derived using the mean coefficients and best-fitting satellite-derived reference spheroid with a polar flattening value of $\frac{1}{298.25}$. This solution is shown in Figure 3.

In the third section comparative geoidal data are given as obtained by Kaula (1966), Uotila (1962), and Zongolovich (1952).

Kaula (1966) used a combination of satellite and gravimetric data to obtain his results. Uotila (1962) and Zongolovich (1952) both used surface gravimetric data. For comparative purposes Zongolovich's (1952) geoidal map is shown in Figure 4.

From an inspection of Table IV it is seen that broadly speaking, the area of maximum geoidal depression defined by each geoid has much the same location, although there is some variation in magnitude values. The significantly lower magnitude found with the gravity solutions, Uotila (1962) and Zongolovich (1952), can be attributed to the paucity and poor distribution of the gravity data available. In the case of the area of maximum geoidal elevation there is a significant dependence on the reference ellipsoid used. However, the scatter is restricted to one or two areas, the Solomon Islands-New Guinea region and the North Atlantic area immediately south of Iceland. This would suggest the gravity anomaly in the two areas is of similar magnitude. Actually the available data indicate the free air anomaly values in the Solomons area is at least locally considerably higher than in the North Atlantic area. A 15th degree fit of the data presumably would give a consistent pattern with the Solomons region being the area of maximum geoidal rise.

It is to be noted that the geoidal undulations referred to the 'best-fitting satellite ellipsoid' (whose reference $C_{20}$ and $C_{40}$ are equal to the observed ones) show a consistently different pattern from those referred to the international reference ellipsoid. As the equatorial radius and flattening of the "satellite ellipsoid" are smaller than the corresponding parameters of the international reference ellipsoid, the geoidal undulations referred to the "satellite ellipsoid", appear to show some accentuation of equatorial "highs" and damping of polar "highs." However, this argument may hold only for the general pattern of these differences and not give a systematic change in magnitude. At this stage this point has not been investigated adequately.

The data of Table IV bring out one important point. Until recently it had been believed on the basis of gravimetric data that the maximum deviation of the geoid from the reference surface of the international ellipsoid was not more than 30 to 40 meters. As seen, the differences obtained are of the order of 100 meters or more for an 8th degree fit.

In connection with Figure 3 in which the geoidal undulations are referred to a 'best-fitting satellite derived spheroid' the reference geopotential ($V_1$) was defined by

$$V_1 = \frac{GM}{a_e} [1 + \sum_{n=2}^{\infty} (\frac{a_e}{r})^{n+1} C_n P_n (\sin \phi)]$$

where $C_n$ are the zonal harmonic coefficients and $P_n(\sin \phi)$ the Legendre's polynomials. As is obvious, it is an axially symmetrical surface but not equatorially symmetrical. The maximum geoidal deviations are +69 meters and -93 meters in this case and are of almost the same magnitude as those computed by setting both $C_{20}$ and $C_{40}$ equal to zero.

Gravity Anomalies:

Figure 5 gives the free air anomaly map obtained using the mean coefficients and referred to the International Gravity formula. Figure 6 is a similar map obtained by Kaula (1966) using a combination of surface gravity data and satellite gravity information. Although these maps show broad-scale agreement on some features, there are significant differences on others. These differences can be related, in part at least, to the difference in input data and the difference in some of the assumptions used.

The short wavelength component of the gravity field which is of interest to the geophysicist, is the one most poorly represented in these results, for as indicated earlier, with an 8th degree fit the results represent regional rather than local values. That they do accomplish this purpose is shown by a comparison of these results with the available surface gravity information expressed as free air anomalies. Figure 7 shows a free air anomaly map for the North Atlantic Ocean which takes in a portion of the gravity "high" defined south of Greenland and the gravity "low" defined in the eastern North Atlantic Ocean on all the satellite derived maps. Figure 8 shows regional variations in free air anomaly values in the Pacific Ocean in terms of areas having anomalies >+20 mgals, >-20 mgals, and with no dominant sign. The agreement of the satellite derived maps with the surface gravity anomaly maps is on the whole good, and as would be expected Kaula's map (Fig. 6) appears to be somewhat better, especially in the Atlantic Ocean, since his map was derived using the available surface gravity data.

It is clear, therefore, that the satellite data are useful in determining areas of anomalous mass associated with the earth, or in defining the gravity field for areas remote from a point in using Stokes' theorem. Many other important geophysical applications, however, pre-require a degree of detail in the earth's gravity field representation which is not available in the satellite results at present.

Moreover, the significance of the anomalous areas of gravity is not too clear. Because of the long wavelengths portrayed, the anomalous mass could be deep seated and associated with the earth's core, or represent the integrated effect of a number of shallow mass anomalies located in the upper mantle or crust. In either case there would also be a contribution from surface topography. The fact that the topographic effect only appears to be of secondary importance stresses the need for geophysical investigations in these areas. Some of them such as the positive anomaly area over the Mid-Atlantic Ridge are known to be characterized by anomalous geophysical relations:

> a sub-normal mantle velocity, pronounced magnetic anomalies, high heat flow along the crest of the ridge but sub-normal heat flow along the flanks.

However, it is difficult to reconcile these observations with the anomalous gravity field which conforms closely with the regional topographic relief and which the Bouguer anomalies indicate is compensated without postulating that the subnormal mantle velocity values are indicative of higher than normal density values or that there is deeper, as yet undiscovered, layering in the upper mantle. Worzel (1965) has shown three possible theoretical mass distributions, all in the upper 30 kms of the crust to explain the observed gravity relations over the mid-Atlantic Ridge. Cook (1962) has postulated that the apparent subnormal mantle velocities are due to a mixture of crustal and mantle materials as a result of convection with attendent high heat flow. While eminently reasonable for the mid-Atlantic Ridge, these explanations do not explain the relations in the Indian Ocean area where the satellite data define a broad negative anomaly area that appears to be related to a stable ocean basin region lying between a narrow volcanic ridge and a rise of the mid-Atlantic Ridge type which has many of the geophysical associations noted for the mid-Atlantic Ridge.

It is this lack of consistency between gravity and other geophysical relations on a regional scale that raises doubts as to interpretations

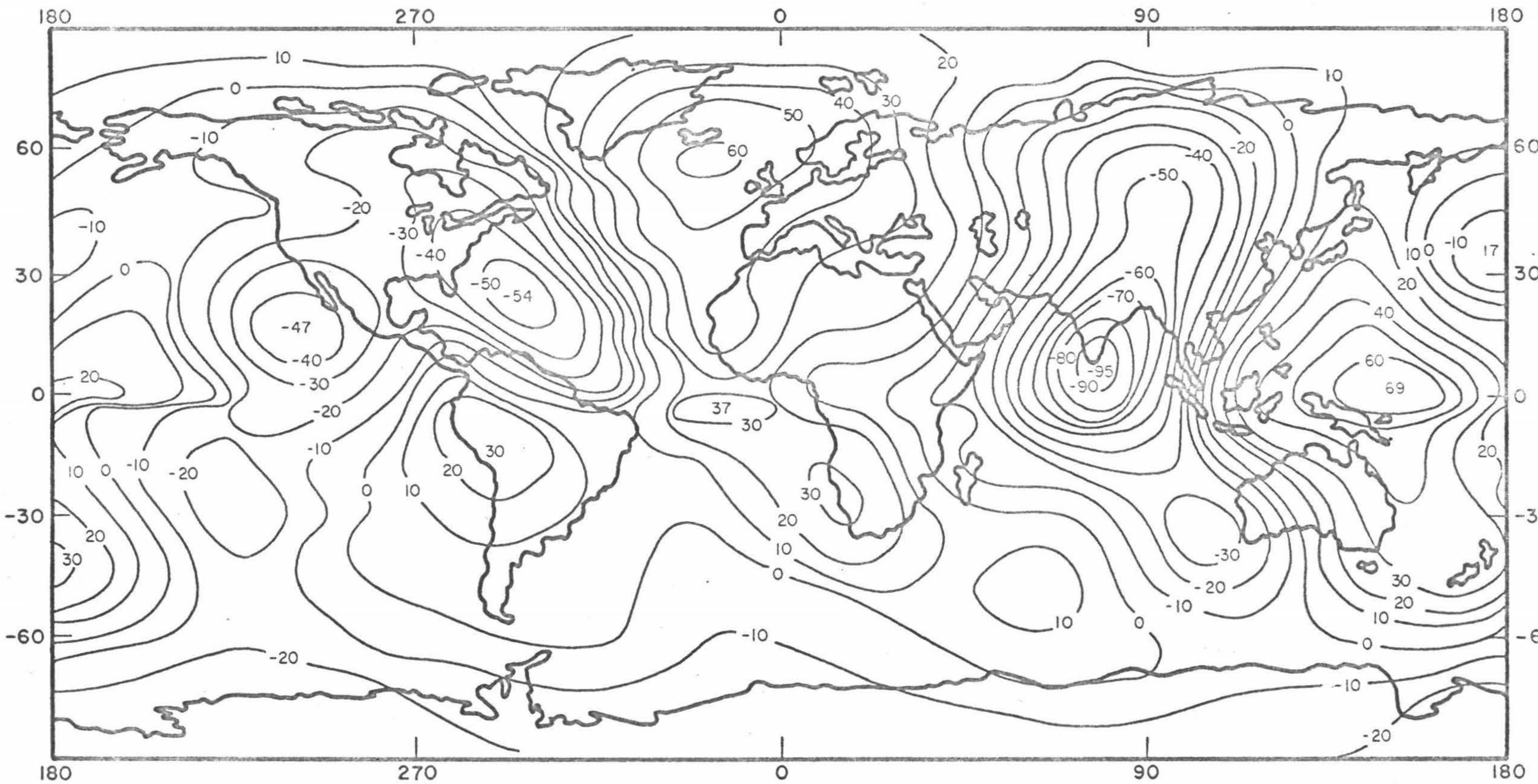


FIGURE 1. GEOIDAL UNDULATIONS IN METERS REFERRED TO AN ELLIPSOID WHOSE $C_{20}$ COEFFICIENTS COINCIDE WITH THE SATELLITE-OBSERVED ONES. COMPUTED USING MEAN ZONALS AND TESSERALS UP TO $P_{88}$ AS DEFINED IN THE TEXT. CONTOUR INTERVAL: 10 METERS

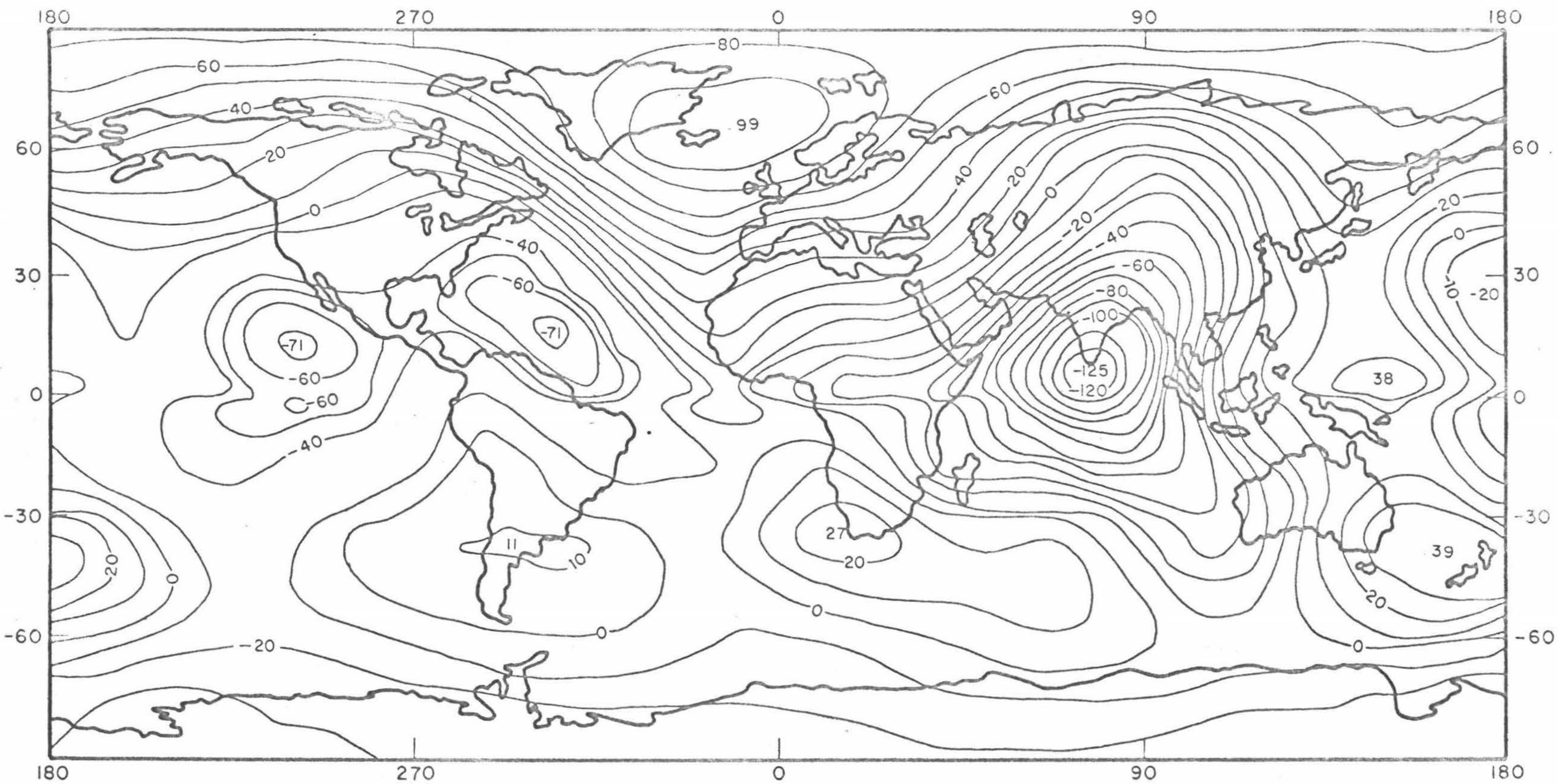


FIGURE 2. GEOIDAL UNDULATIONS IN METERS REFERRED TO THE INTERNATIONAL REFERENCE ELLIPSOID. COMPUTED USING MEAN COEFFICIENTS UP TO $P_{88}$ AS DEFINED IN THE TEXT.

CONTOUR INTERVAL: 10 METERS

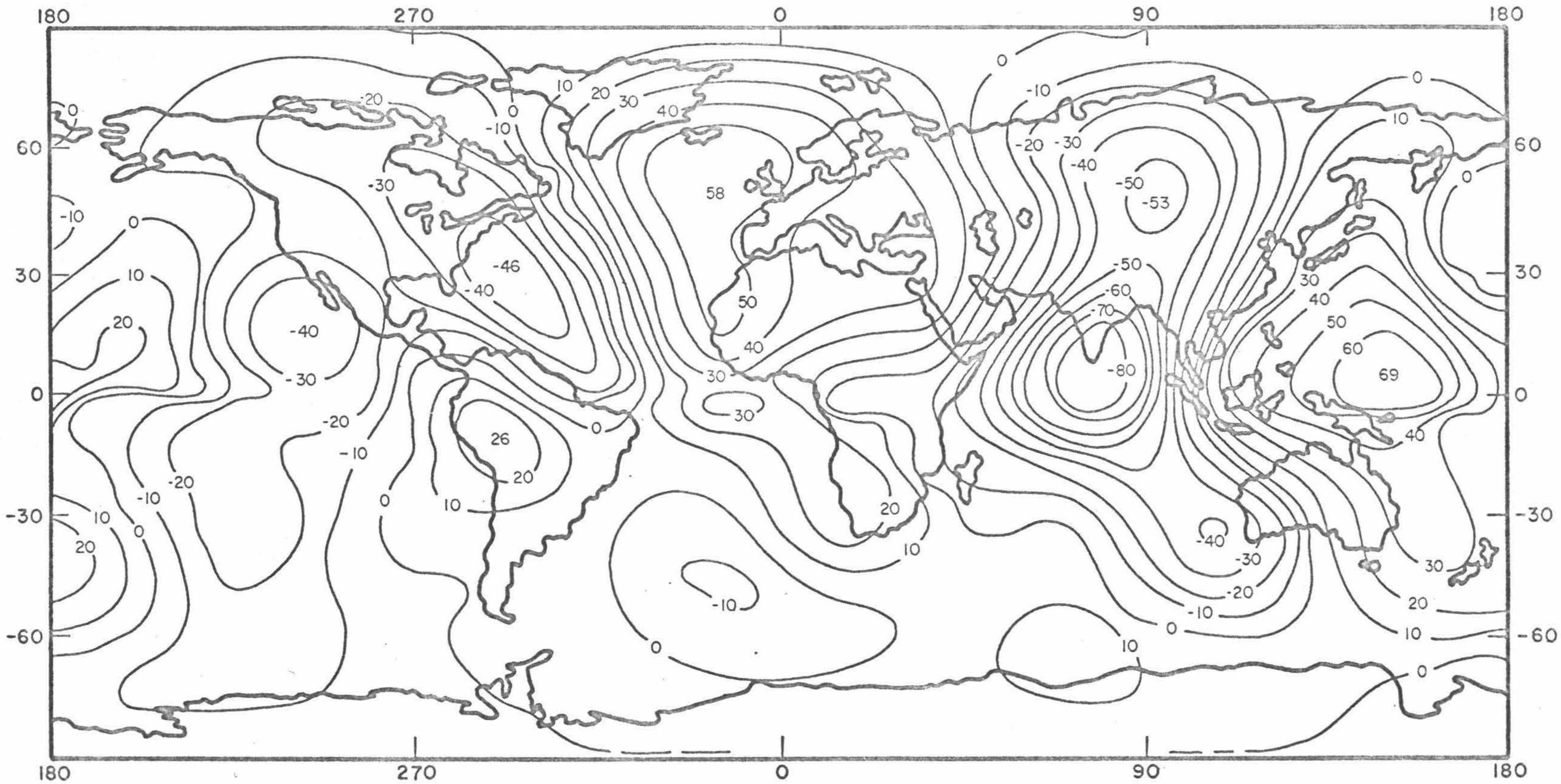


FIGURE 3. GEOIDAL UNDULATIONS IN METERS REFERRED TO A BEST-FITTING SATELLITE SPHEROID AS DEFINED IN THE TEXT. COMPUTED USING MEAN TESSERAL COEFFICIENTS UP TO $P_{88}$ AS DEFINED IN THE TEXT.

CONTOUR INTERVAL: 10 METERS

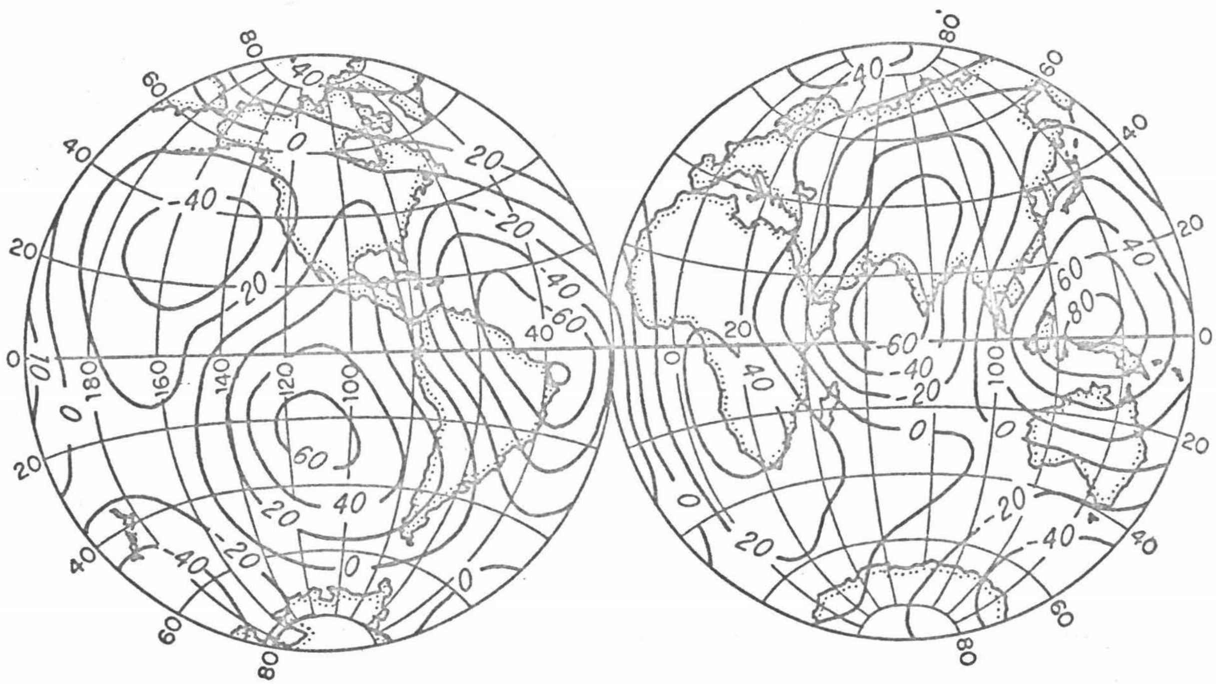

FIGURE 4. GEOIDAL UNDULATIONS IN METERS REFERRED TO THE RUSSIAN REFERENCE ELLIPSOID COMPUTED BY ZONGOLOVICH (1952) USING SURFACE GRAVITY INFORMATION

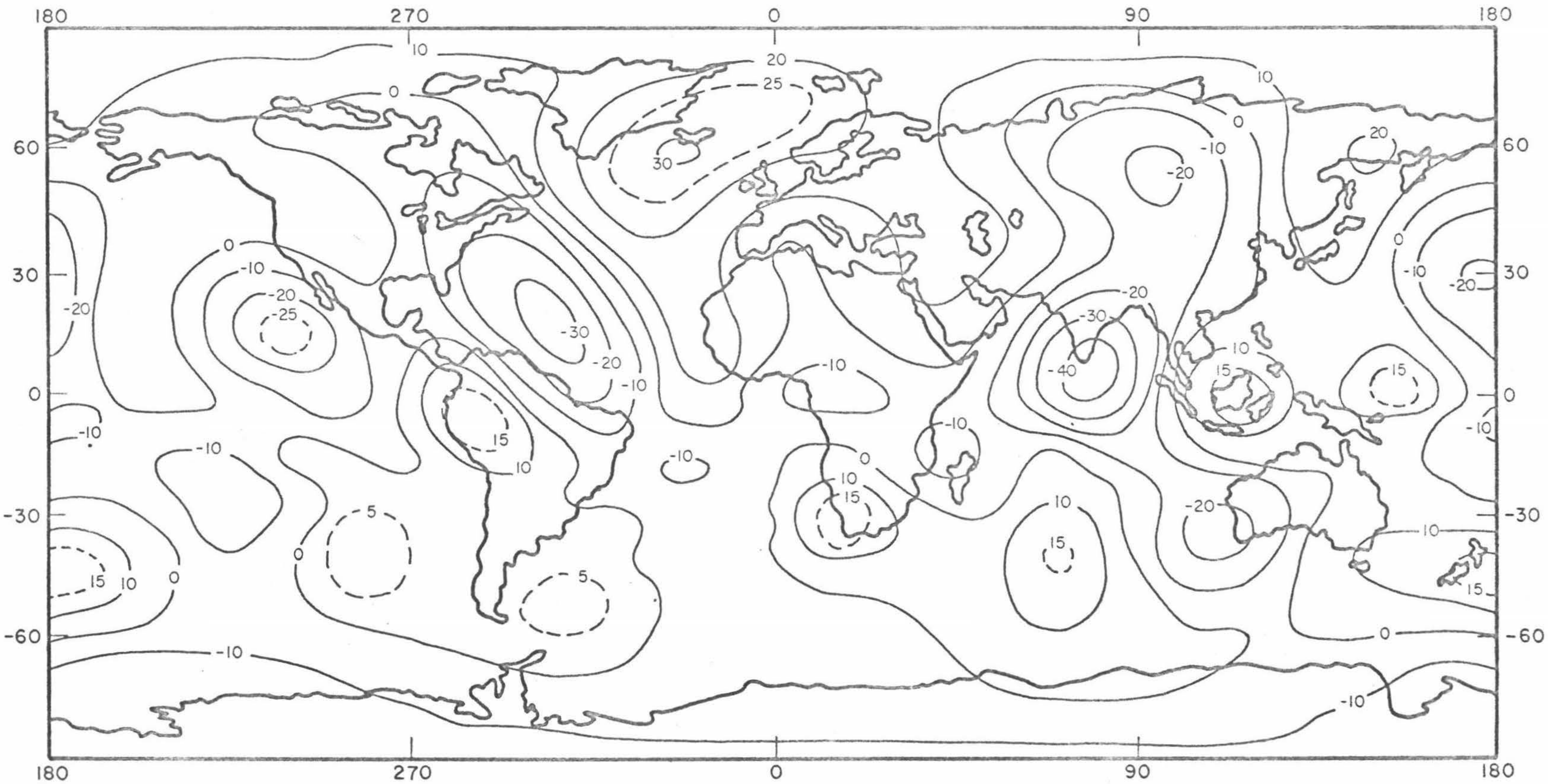


FIGURE 5. GRAVITY ANOMALIES IN MILLIGALS REFERRED TO THE INTERNATIONAL GRAVITY FORMULA. COMPUTED USING MEAN ZONALS AND TESSERALS UP TO $P_{88}$ AS DEFINED IN THE TEXT. CONTOUR INTERVAL BETWEEN SOLID LINES: 10 MILLIGALS; BETWEEN SOLID AND BROKEN LINES: 5 MILLIGALS

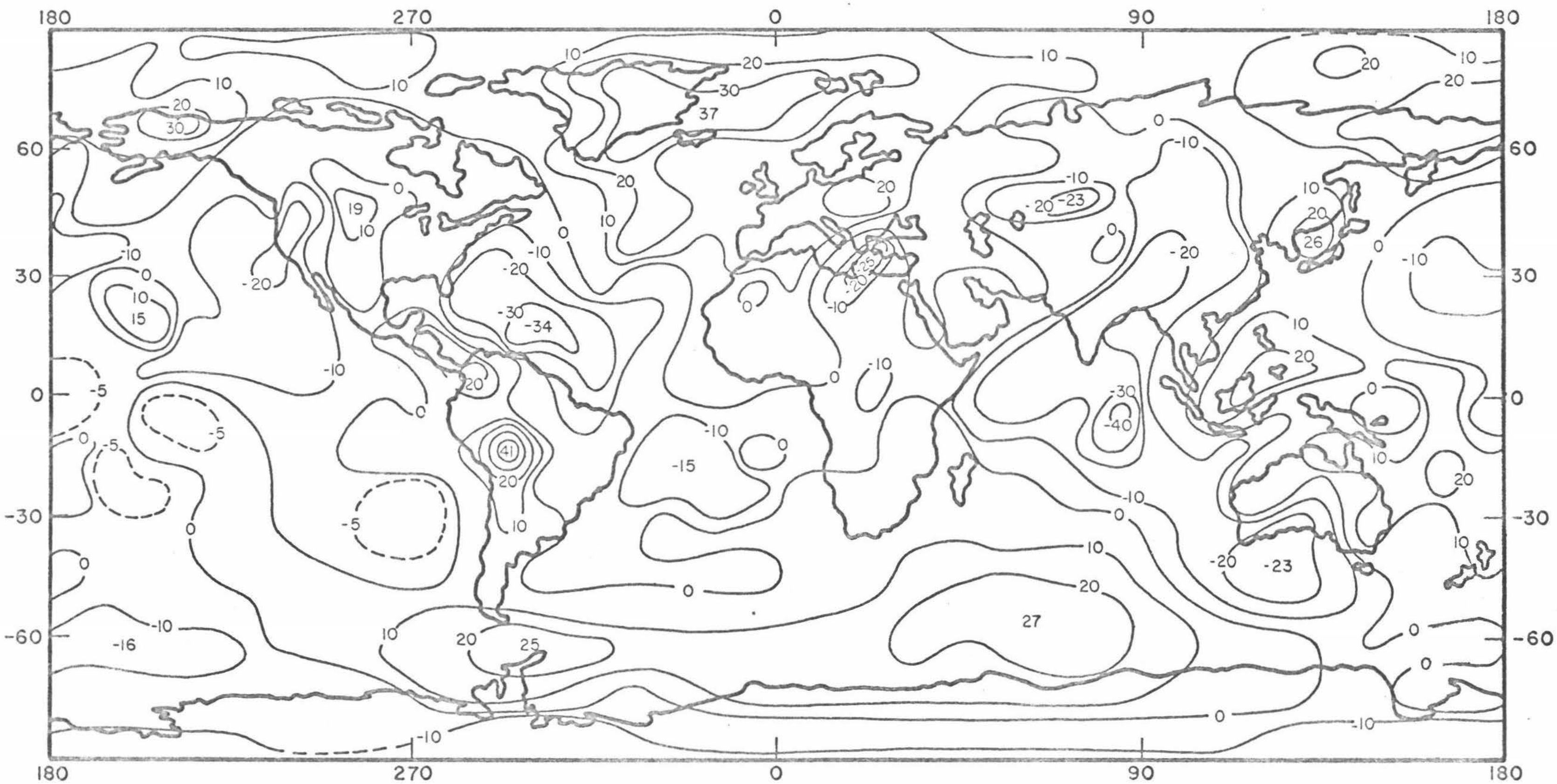


FIGURE 6. GRAVITY ANOMALIES IN MILLIGALS (TAKEN FROM KAULA, 1966). RESULTS COMBINE AVAILABLE SURFACE GRAVITY MEASUREMENTS WITH SATELLITE INFORMATION. CONTOUR INTERVAL BETWEEN SOLID LINES: 10 MILLIGALS; BETWEEN SOLID AND BROKEN LINES: 5 MILLIGALS

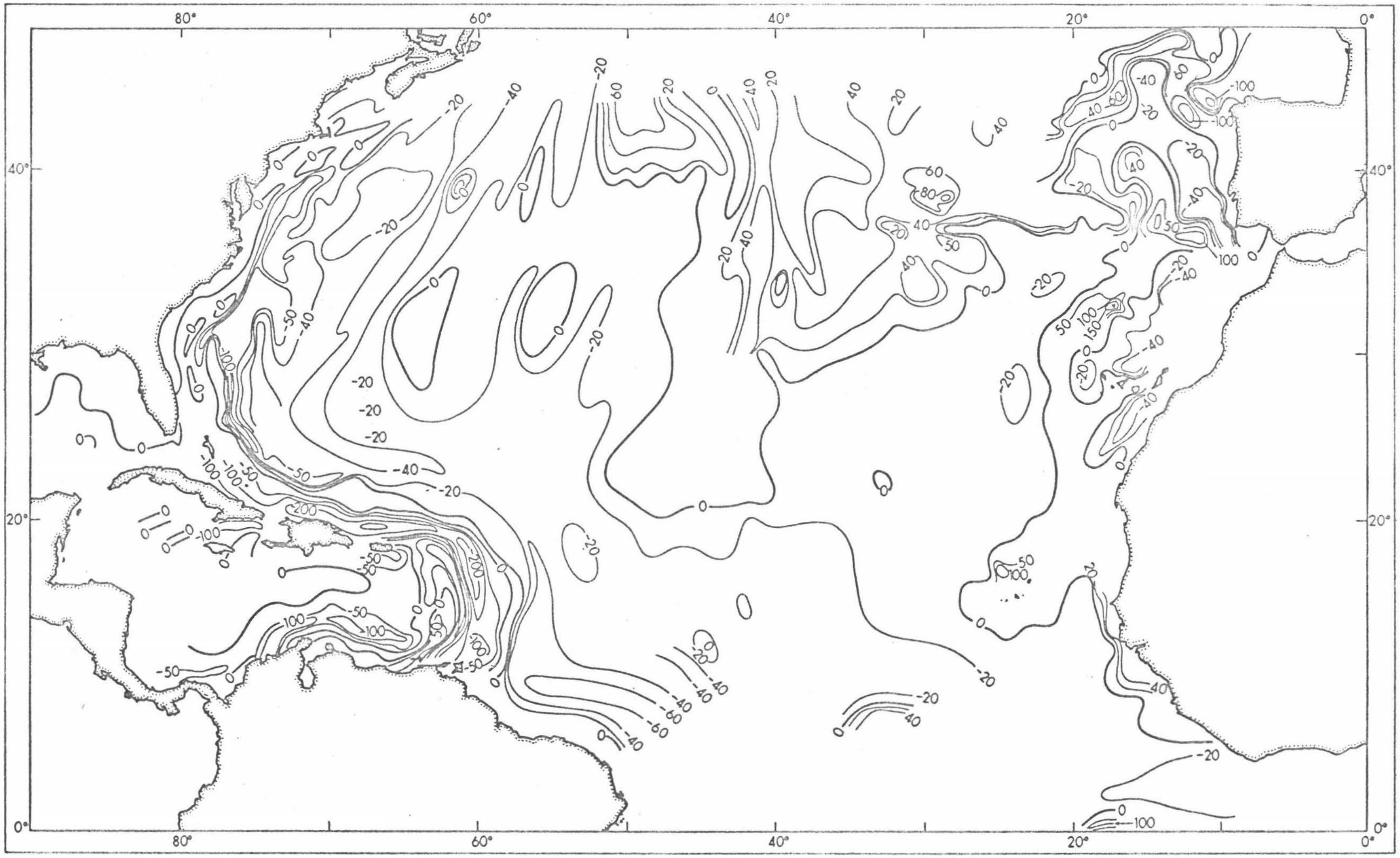


FIGURE 7. FREE AIR GRAVITY ANOMALIES - NORTH ATLANTIC OCEAN

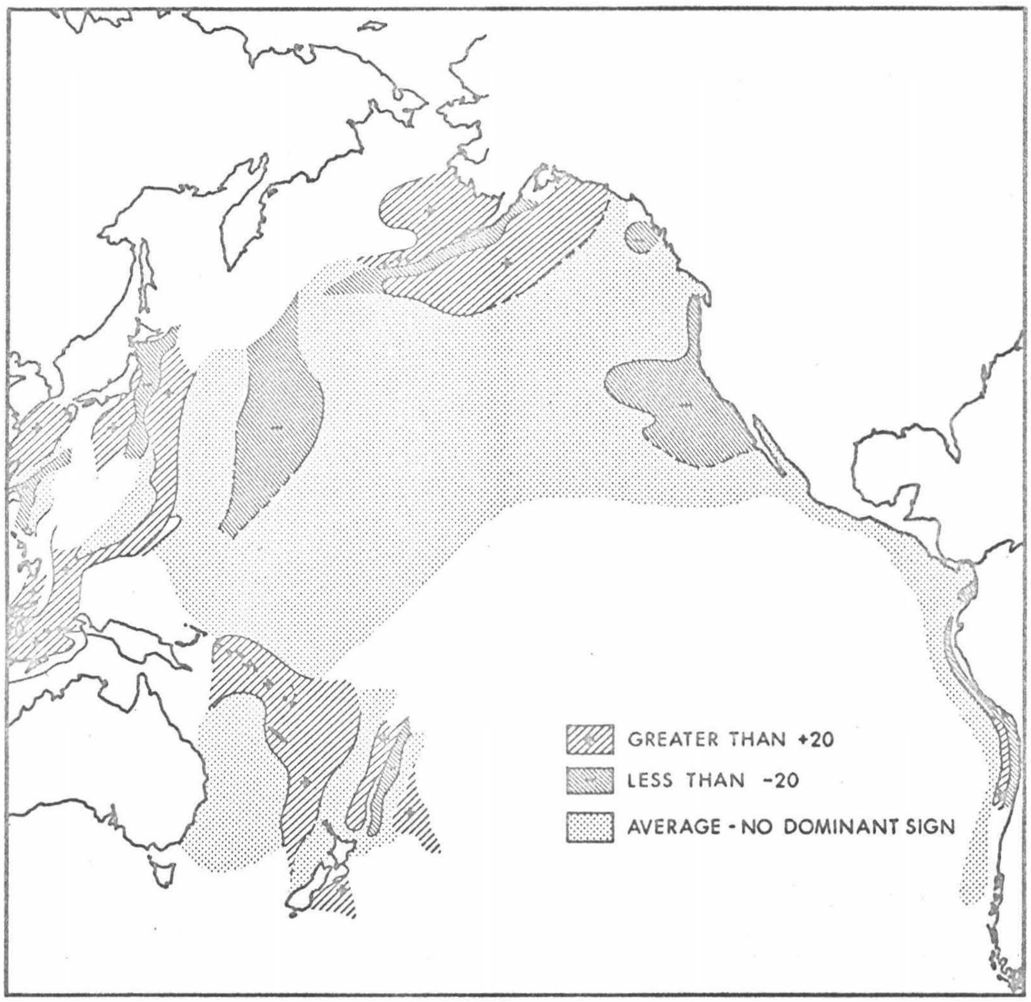


FIGURE 8. FREE AIR GRAVITY ANOMALIES - PACIFIC OCEAN

that have been placed on the data and point up the need for more extensive geophysical studies in areas of anomalous gravity. In this respect whereas the satellite gravity results have proven their worth by outlining, in general, the anomalous areas, they do not exhibit any of the local variations shown by gravimetric results and fail to furnish the degree of detail required in many of the detailed geophysical studies.

### 4. Limitations of the Perturbation Theory

The method of determination of the geopotential using the perturbation theory has certain inherent limitations. The observed perturbations of the orbital elements arise from a composite effect of all disturbing factors and in order to accurately determine the geopotential, it is necessary to isolate the portion arising solely from gravitational sources. This separation of gravitational component is rendered difficult because of the uncertainties involved in identifying the component perturbations with their parent sources. As of now, our knowledge of the atmospheric structure, the radiation pressure, etc., is too inadequate to enable us to determine accurate corrections for these factors. The uncertainty in the corrections for the non-gravitational factors results in the introduction of some interaction terms among the different perturbations which consequently can lead to an erroneous determination of the geopotential. Another problem is the separation of the perturbation effects of the individual harmonic terms. Common periodicities occur in the periodic motions originating from various harmonic terms. The decomposition of these periodic motions into component parts and the assignment of these components to their legitimate sources is likely to introduce further uncertainty in the computations. Moreover, in the perturbation theory, any perturbation is treated as the integrated effect of the mass anomalies of a certain wave-length and the theory does not appear to offer the potentiality of yielding adequate information about the individual mass anomalies. Consequently, the results obtained from this theory give a rather smoothed picture of the gravity field.

## CHAPTER III

## DEVELOPMENT OF THE NEW THEORY FOR DETERMINATION OF GEOPOTENTIAL

### 1. Physical Basis of the New Theory

The gravity effect of the anomalous masses decreases with height and the rate of decrease is a function of the degree of harmonic term by which that specific gravity anomaly can be represented on the earth's surface. The shorter the wave length of the gravity anomaly, the faster the decrease of its effect with increasing height. Thus, a satellite will only sense those anomalies whose wave length is equal to or greater than the 'limiting wave length,' here defined as the shortest wave length which can be discerned by a satellite at its altitude. The limiting wave length is thus a function of satellite altitude. The gravity anomalies having wave lengths smaller than the limiting wave length will have no effect on satellite motion and hence information about them cannot be retrieved from the satellite data. The limiting wave length thus sets an upper limit on the degree of detail with which the geopotential can be derived from satellite motion. As pointed out earlier, the perturbation theory does not exploit this potentiality of satellite dynamics. It rather makes use of the orbital perturbation arising from the integrated effect of the mass anomalies of a specific wave length, and the geopotential coefficient thus determined reflects the cumulative effect of these mass anomalies rather than their individual contribution. Hence, when geopotential is recomputed from these harmonic coefficients, the contribution of each coefficient is to emphasize the mathematical pattern of its characteristic wave length and not the actual pattern of distribution of the individual mass anomalies of that wave length. This argument holds for any harmonic analysis carried to a finite number of terms.

As a satellite is more affected by a nearby mass than a remote mass of the same magnitude, its motion should yield information primarily weighted in favor of the area over which the satellite is passing, unless of course, the more distant masses outsize the nearer masses considerably and consequently have greater gravitational effect. Any method for the determination of geopotential from satellite motion which aims at giving the maximum degree of detail down to the limiting wave length, should therefore, exploit these facts. To illustrate, see Fig. 9. Let S be the position of the satellite, $R/4$ its height above the surface of the earth where R is the mean radius of the earth. The mass of the spherical part of the earth can be supposed to be concentrated at M and does not interest us in this investigation. We are looking for only the effects of the anomalous masses on the satellite. Consider anomalous masses of equal magnitude $\Delta M$, located at $P_i$ $(i=1,...4)$. Let the disturbing forces arising from these mass anomalies be $\Delta F_i$ $(i=1,...4)$. Then

$$\Delta F_1 = \frac{16G\Delta M}{R^2}$$

$$\Delta F_2 = \Delta F_3 = \frac{16G\Delta M}{41\ R^2}$$

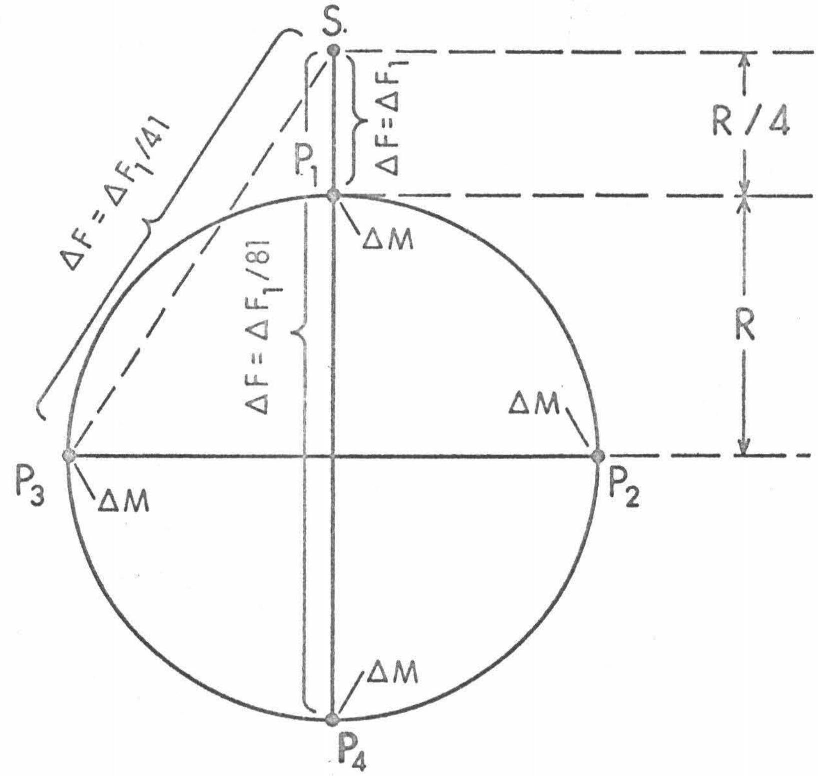


FIGURE 9. RELATIVE EFFECT ON THE SATELLITE S OF THE MASS ANOMALY ΔM PLACED AT DIFFERENT LOCATIONS

$$\Delta F_4 = \frac{16G\Delta M}{81\ R^2}$$

Notice that in case the mass anomalies at $P_1$, $P_2$, $P_3$ and $P_4$ are of equal magnitude, the effect of the mass anomalies at $P_2$ and $P_3$ on the satellites will be only about 2.5% of that of the anomaly at $P_1$ and that of the anomaly at $P_4$, about 1.25% of that of one at $P_1$. In order that the contribution $\Delta F_4$ be equal to $\Delta F_1$, the mass anomaly at $P_4$ has to be 81 times the one at $P_1$. For closer satellites, this percentage contribution of the distant mass anomalies becomes still smaller and their magnitude has to be still larger in order to have a contribution comparable to that of the mass anomaly located at $P_1$ (i.e., the point right underneath the satellite) or its neighborhood. For example, for the satellite whose height above the surface of the earth is equal to R/6, the contribution of the mass anomaly at $P_4$ is only about 0.5% and of those at $P_2$ and $P_3$ about 1.25% of the anomaly at $P_1$. In order to have the same contribution as that from $P_1$, the mass anomaly at $P_4$ has to be 169 times as large as the one at $P_1$. Thus barring a few exceptions when the mass anomalies located in the area over which the satellite is moving, are very much smaller in comparison to the distant anomalies, the anomalous masses in the neighborhood of points right below the satellite exert the controlling influence on the satellite. The instantaneous elements of a satellite can thus be exploited, in principle at least, to yield information concerning the nearer mass anomalies. The new theory presented here exploits this potentiality of satellite motion.

There exist a number of dynamical variables associated with the satellite motion which are approximately constant in time. One such variable is the Hamiltonian. In the ideal case which occurs when all the perturbations are ignored, it is possible to express the Hamiltonian as a function of the angular velocity of the earth as well as of the position and velocity of the satellite relative to the earth. With this as basis, however, it seems possible to take into consideration the effect of perturbations such as those arising from moon, air drag, radiation pressure, etc., or to modify the calculational procedures so as to reduce these effects to a minimum.

The potential function of the earth appears additively in the Hamiltonian. It is this function we seek to determine. If we expand this function in terms of spherical harmonics or the elements of the tensor of inertia, it is possible to determine the expansion coefficients which appear linearly as multiples of the spherical harmonics in the expansion. The determination of the expansion coefficients is made possible by the fact that they are assumed to remain constant over a short segment of the satellite orbit and are expressive of the mass distribution in the region right below the orbital segment under consideration. Minimally, there must be about as many observations as there are expansion coefficients. However, an abundance of measurements will be desirable to be able to apply the least squares method. In this way, this new theory enables us to obtain expressions for the gravitational potential expressive of regions immediately below the satellite trajectory. The region for which the information is primarily weighted has been termed as 'effective area' and can be defined by stipulating that it should extend only as far out as a point at which a certain mass anomaly would cease to have a certain minimum contribution. If this minimum limit is set at 50% of the contribution that the same mass anomaly would have if it were located right below the satellite, the area in question lies roughly within a circle whose center lies at the point immediately below the satellite and with a radius equal to the altitude of the satellite. If the minimum limit is 20%, the above-mentioned circle has a radius of twice the altitude of the satellite. If the satellite altitude is higher, the effective area will be greater, though the degree of detail with which the gravity field can be obtained will diminish with altitude. Theoretically, it is possible to cover the surface of the earth by such effective areas and thus to obtain a description of the earth's gravity field. In principle, the description obtained in this way should show at least some of the subtler features exhibited by terrestrial gravimetry.

In its initial form the new theory requires the measurement of the position vector and the relative velocity of the satellite at several points along a short segment of its trajectory. This could be done if Doppler and Interferometer measurements are simultaneously made at the same tracking station. The same could also hopefully be accomplished with a simultaneous use of laser and Baker Nunn camera tracking. The latter technique is already being experimented with by the Smithsonian Astrophysical Observatory, Cambridge, Massachusetts. However, the non-availability of the position vector and the relative velocity as directly measured quantities is no big hurdle in the implementation of the new theory. The position and velocity of a satellite at any time can be derived from its orbital elements which are available as the routine initial data and hence equations of condition can be set up in terms of the orbital elements. In this conversion we have to forego some of the ideal conditions which hold for the new theory in its original form but because of the mathematical approach adopted, it is believed that even with the same input data the new theory will yield a better degree of detail than the perturbation theory.

First, the ideal case ignoring all the perturbations is treated. A complete development of the equations of condition and the transformation of the position and relative velocity of the satellite to orbital elements is given. The 'extended theory,' taking into account the perturbing effects of the moon, the air drag, the radiation pressure, etc., is then developed and the equations of condition are derived for this case which reduce these perturbation effects to minimum. These are the equations which should be used for any practical applications.

## 2. Simplified Theory--The Ideal Case Ignoring All the Perturbations

Consider the ideal case of a satellite moving in a closed orbit around the earth, its motion being controlled only by the earth's gravity field. Let $\underline{V}$ denote the absolute velocity vector of the satellite in an earth-centered, space-fixed system of coordinates, $\underline{v}$, its relative velocity vector in an earth-centered, earth-fixed coordinate system, $\underline{w}$, the angular velocity of the earth and $\underline{r}$ the radius vector of the satellite from the origin of the coordinate system which is supposed to lie on the axis of rotation of the earth. Then we know (Wills, 1958,

p. 54) that

(1) $\underline{V} = \underline{v} + \underline{w} \times \underline{r}$

The Lagrangian L of the system is

(2) $L = \frac{1}{2} m \underline{V} \cdot \underline{V} - m U$

where

m = mass of the satellite

U = potential energy of the earth at the location of the satellite.

Substitution of (1) in (2) gives

(3) $L = \frac{1}{2} m (\underline{v} + \underline{w} \times \underline{r}) \cdot (\underline{v} + \underline{w} \times \underline{r}) - m U$

The canonical momentum $p_i$ is given by

$$p_i = \frac{\partial L}{\partial v_i} = m (\underline{v} + \underline{w} \times \underline{r})_i$$

where $v_i$ have been chosen as the generalized coordinates. In vectorial form the above equation is

(4) $\underline{p} = m (\underline{v} + \underline{w} \times \underline{r})$

The Hamiltonian H of the system is

(5) $H = \underline{p} \cdot \underline{v} - L$

Substitute (3) and (4) in (5) and get

$$H = m\,(\underline{v} + \underline{w} \times \underline{r}) \cdot \underline{v} - \tfrac{1}{2} m\,(\underline{v} + \underline{w} \times \underline{r}) \cdot (\underline{v} + \underline{w} \times \underline{r}) + m\,U$$

$$= m\,(\underline{v} + \underline{w} \times \underline{r}) \cdot [\underline{v} - \tfrac{1}{2}(\underline{v} + \underline{w} \times \underline{r})] + m\,U$$

$$= \tfrac{1}{2} m\,(\underline{v} + \underline{w} \times \underline{r}) \cdot (\underline{v} - \underline{w} \times \underline{r}) + m\,U$$

$$= \tfrac{1}{2} m\,[v^2 - (\underline{w} \times \underline{r}) \cdot (\underline{w} \times \underline{r})] + m\,U$$

Now

$$(\underline{w} \times \underline{r}) \cdot (\underline{w} \times \underline{r}) = w^2\, r^2 - (\underline{w} \cdot \underline{r})^2$$

Hence

$$H = \tfrac{1}{2} m\,[v^2 + (w \cdot r)^2 - w^2\, r^2] + m\,U \tag{6}$$

H is constant in time if the secular decceleration in the earth's rotation is ignored. U is regarded independent of time.

## 3. Geopotential

Consider a body M of finite dimensions and arbitrary mass distribution. Let $\underline{r}_c$ be the position vector of center of mass O of the body M, relative to a fixed coordinate system. Further, let (see Fig. 10):

$\underline{r} = x_1\,\underline{i} + x_2\,\underline{j} + x_3\,\underline{k}$ = position vector of a material particle at P with mass m.

$\underline{\rho} = x'_1\,\underline{i} + x'_2\,\underline{j} + x'_3\,\underline{k}$ = position vector of the mass element dM at point Q of the body M.

$\underline{\Delta} = \underline{r} - \underline{\rho}$

Then U, the potential of attraction of body M on the mass at P is

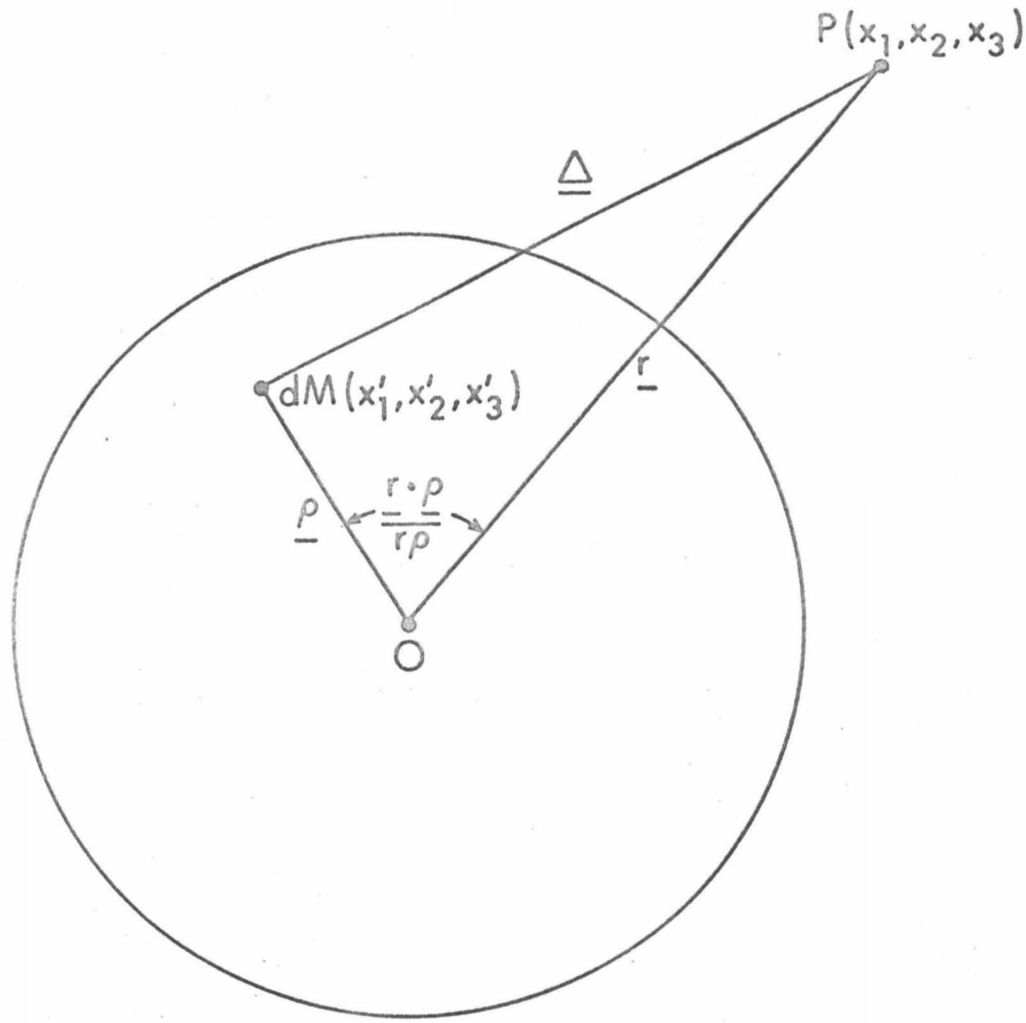


FIGURE 10. POSITION OF THE MASS ELEMENT dM, THE MATERIAL PARTICLE P AND THE VECTORS $\underline{r}$, $\underline{\Delta}$ and $\underline{\rho}$

given by

$$U = G \int_V \frac{D(\underline{\rho})}{\Delta} d^3 (\underline{\rho}) \quad .$$

where $D(\underline{\rho})$ is the density function and the symbol $\int_V$ indicates that integration is to be carried over the entire volume of the body.

The expression of $\Delta = |\underline{r} - \underline{\rho}|$ in terms of Legendre's polynomials which are function of the angle $\frac{\underline{r} \cdot \underline{\rho}}{r\rho}$ is well-known and is given in a slightly modified form in any good text on the subject (MacMillan, 1958; Kellog, 1953). Consequently, U can be expressed as

$$U = \frac{GM}{r} + \frac{G\underline{P} \cdot \underline{r}}{r^3} + \tfrac{1}{2} G \; {}_{i,j} \; Q_{ij} \frac{x_i x_j}{r^5} + \ldots \tag{7}$$

where

$$\underline{P} = \int \underline{\rho} \, D(\underline{\rho}) \, d^3 \, \underline{\rho} = M \, \underline{r}_c$$

$$Q_{ij} = \int (3 \, x'_i \, x'_j - \rho^2 \overset{\delta ?}{\rho_{ij}}) \, D(\underline{\rho}) \, d^3 \, \underline{\rho} \tag{8}$$

$\delta_{ij}$ is the Kronecher $\delta$ function, defined as

$$\delta_{ij} = \begin{matrix} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{matrix}$$

Note that the first term in equation (7) indicates the potential of the body if all its mass M were concentrated at its center.

The second term vanishes if the center of mass 0 of body M is taken as the origin of the coordinate system because in that case $\underline{r}_c = 0$

The moments of inertia A, B, C and the products of inertia D, E, F are usually defined by the integrals

$$
\begin{aligned}
A &= \int ({x'}_2^2 + {x'}_3^2)\, D(\underline{\rho})\, d^3\, \underline{\rho} \\
B &= \int ({x'}_3^2 + {x'}_1^2)\, D(\underline{\rho})\, d^3\, \underline{\rho} \\
C &= \int ({x'}_1^2 + {x'}_2^2)\, D(\underline{\rho})\, d^3\, \underline{\rho} \\
D &= \int x'_2\, x'_3\, D(\underline{\rho})\, d^3\, \underline{\rho} \\
E &= \int x'_3\, x'_1\, D(\underline{\rho})\, d^3\, \underline{\rho} \\
F &= \int x'_1\, x'_2\, D(\underline{\rho})\, d^3\, \underline{\rho}
\end{aligned} \tag{9}
$$

By comparing the above expressions with equation (8), we get

$$
\begin{aligned}
Q_{11} &= B + C - 2A \qquad & Q_{23}, Q_{32} &= 3D \\
Q_{22} &= C + A - 2B \qquad & Q_{13}, Q_{31} &= 3E \\
Q_{33} &= A + B - 2C \qquad & Q_{12}, Q_{21} &= 3F
\end{aligned} \tag{10}
$$

In the initial investigation we plan to determine only those quantities which are shown in equation (7).

One of the fundamental properties of the potential function U is that for all points not belonging to the mass M, it satisfies the Laplace's equation. A particular solution of this equation enables us to express U in terms of the spherical harmonics and can be written as (Byerly, 1959)

$$(11) \qquad U = \frac{GM}{r} \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm} (\sin \phi) \right]$$

If the center of mass of the body M is chosen as the origin of the coordinate system, the term for n = 1 vanishes and the equation (11) reduces to the form

$$(12) \qquad U = \frac{GM}{r} \left[ 1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm} (\sin \phi) \right]$$

where $P_{nm}$ (sin $\phi$) are called the Associated Legendre's functions. The first term in the expansion indicates the potential of the body M if all its mass were concentrated in its center.

A comparison of equation (11) with equation (7) gives some useful relations between the spherical harmonic coefficients $C_{nm}$, $S_{nm}$ and the physical constants of the earth. For the first few harmonics these relations are:

$$C_{00} = 1$$

$$C_{10} = \frac{\int x'_3 \, dm}{Ma_e} = \frac{x_{3c}}{a_e}$$

$$C_{11} = \frac{\int x'_1 \, dm}{Ma_e} = \frac{x_{1c}}{a_e} \qquad S_{11} = \frac{\int x'_2 \, dm}{Ma_e} = \frac{x_{2c}}{a_e}$$

$$C_{20} = -\frac{1}{Ma_e^2}\left[C - \tfrac{1}{2}(A + B)\right] \tag{13}$$

$$C_{21} = \frac{E}{Ma_e^2} \qquad S_{21} = \frac{D}{Ma_e^2}$$

$$C_{22} = \frac{B - A}{4Ma_e^2} \qquad S_{22} = \frac{F}{2Ma_e^2}$$

Note that in the foregoing expressions $\underline{r}_c = x_{1c}\,\underline{i} + x_{2c}\,\underline{j} + x_{3c}\,\underline{k}$ where $\underline{i}$, $\underline{j}$, $\underline{k}$ are unit vectors in the direction of $x_1$, $x_2$, $x_3$ respectively. In principle, there should be no difficulty in carrying out the development further to tie the higher harmonic coefficients to the physical constants of the earth (Jung, 1956).

Note that by a proper choice of the reference system, we can eliminate the coefficients $C_{10}$, $C_{11}$, $S_{11}$, $C_{21}$ and $S_{21}$.

## 4. Equations of Condition for the Simplified Theory

We can substitute the expansion of geopotential from equation (7) or equation (11) in equation (6) depending upon whether our most immediate purpose is to determine the spherical harmonic coefficients of geopotential or elements of the tensor of inertia. Substitution of equation (11) in equation (6) gives

$$\text{(14)} \qquad \frac{H}{m} = \tfrac{1}{2}\,[v^2 + (\underline{w} \cdot \underline{r})^2 - w^2\, r^2]$$

$$+ \frac{GM}{r}\,[\sum_{n=0}^{\infty} \sum_{m=0}^{n} (\frac{a_e}{r})^n\, (C_{nm} \cos m\lambda$$

$$+ S_{nm} \sin m\lambda)\, P_{nm}\, (\sin \phi)]$$

Let $\underline{r}_i$ and $v_i$ $(i = 1, 2, \ldots i)$ be the measured values of the position vector and the relative velocity of the satellite along a short segment of its orbit. Substitute these values in equation (14) and by successive subtractions get the equations of condition as follows:

$$\tfrac{1}{2}\,[(v_{i+1}^2 - v_i^2) + (\underline{w} \cdot \underline{r}_{i+1})^2 - (\underline{w} \cdot \underline{r}_i)^2 - w^2\, (r_{i+1}^2 - r_i^2)]$$

$$+ GM \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_e^n \left[ C_{nm} \left( \frac{\cos m\lambda_{i+1}\, P_{nm} \underset{(i+1)}{(\sin \phi)}}{r_{i+1}^{n+1}} \right.\right.$$

$$\left. - \frac{\cos m\lambda_i\, P_{nm} \underset{(i)}{(\sin \phi)}}{r_i^{n+1}} \right) + S_{nm} \left( \frac{\sin m\lambda_{i+1}\, P_{nm} \underset{(i+1)}{(\sin \phi)}}{r_{i+1}^{n+1}} \right.$$

$$\left.\left. - \frac{\sin m\lambda_i\, P_{nm} \underset{(i)}{(\sin \phi)}}{r_i^{n+1}} \right) \right] = 0 \qquad i = 1, 2, \ldots i$$

or

$$\text{(15)} \qquad \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(C_{nm}\, g_{\substack{nm\\(i)}} + S_{nm}\, h_{\substack{nm\\(i)}}\right)$$

$$= f_{i+1,i}\left(v_{i+1,i}, \underline{w}, \underline{r}_{i+1,i}\right) \qquad i = 1, 2, \ldots i.$$

where

$$g_{\substack{nm\\(i)}} = f\left[GM, a_e, r, \cos\lambda, P_{nm}(\sin\phi)\right]$$

$$= GM\, a_e^n \left( \frac{\cos m\lambda_{i+1}\, P_{\substack{nm\\(i+1)}}(\sin\phi)}{r_{i+1}^{n+1}} - \frac{\cos m\lambda_i\, P_{\substack{nm\\(i)}}(\sin\phi)}{r_i^{n+1}} \right) \qquad i = 1, 2, \ldots i.$$

(16)

$$h_{\substack{nm\\(i)}} = f\left[GM, a_e, r, \sin\lambda, P_{nm}(\sin\phi)\right]$$

$$= GM\, a_e^n \left( \frac{\sin m\lambda_{i+1}\, P_{\substack{nm\\(i+1)}}(\sin\phi)}{r_{i+1}^{n+1}} - \frac{\sin m\lambda_i\, P_{\substack{nm\\(i)}}(\sin\phi)}{r_i^{n+1}} \right) \qquad i = 1, 2, \ldots i.$$

and

$$f_{i+1,i}\ (v_{i+1,i},\ \underline{w},\ \underline{r}_{i+1,i}) = -\tfrac{1}{2}\,[(v^2_{i+1} - v^2_i) \tag{17}$$

$$+ (\underline{w}\cdot\underline{r}_{i+1})^2 - (\underline{w}\cdot\underline{r}_i)^2 - w^2\ (r^2_{i+1} - r^2_i)]$$

$$i = 1,\ 2, \ldots i.$$

Minimally $i-1 = (n+1)^2$ or $n = (i-1)^{\frac{1}{2}} - 1$. However, if it is desired to apply the least squares method of adjustment, $i >> (n+1)^2$ or $n << (i-1)^{\frac{1}{2}} - 1$.

## 5. Transformation of the Position Vector and the Relative Velocity of a Satellite into Orbital Elements

Consider a coordinate system $\overline{X}_i$ $(i = 1, 2, 3)$, the $\overline{X}_1\overline{X}_2$ plane of which coincides with the plane of the orbit and the $\overline{X}_1$ axis points toward perigee. In this coordinate system the coordinates of the satellite position are $(\overline{X}_1, \overline{X}_2, 0)$. The coordinate $\overline{X}_3 = 0$ but for purposes of generalization we will consider the case in which $\overline{X}_3$ is not zero and later make the substitution $\overline{X}_3 = 0$ for the derivation of our specific formulas. See Fig. 11 for illustration of this system.

If E be the eccentric anomaly, then it is obvious that

$$\begin{aligned} \overline{X}_1 &= a\ (\cos E - e) \\ \overline{X}_2 &= a\ (1-e^2)^{\frac{1}{2}} \sin E \end{aligned} \tag{18}$$

$\overline{X}_3 = 0$

Now consider another system of coordinates $X_i$ $(i = 1, 2, 3)$, the $X_1X_2$ plane of which coincides with the equatorial plane of the earth and the $X_1$ axis is directed towards the mean vernal equinox (see Fig. 11). This is the coordinate system to which the orbital elements of an artificial earth satellite are customarily referred. The transformation from the $\overline{X}_i$ system to the $X_i$ system $(i = 1, 2, 3)$, involves a rotation of coordinates which may be split up into the following component rotations (see Fig. 11 and Fig. 12):

1. A rotation $-w$ about the $\overline{X}_3$ axis given by the matrix $A(-w)$:

$$A(-w) = \begin{pmatrix} \cos w & \sin w & 0 \\ -\sin w & \cos w & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. A rotation $-i$ about the $X'_1$ axis (the transient position of the $X_1$ axis during the rotation from $\overline{X}_i$ system to the $X_i$ system) given by the matrix $B(-i)$:

$$B(-i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix}$$

and

3. A rotation $-\Omega$ about the $X_3$ axis given by the matrix $A(-\Omega)$:

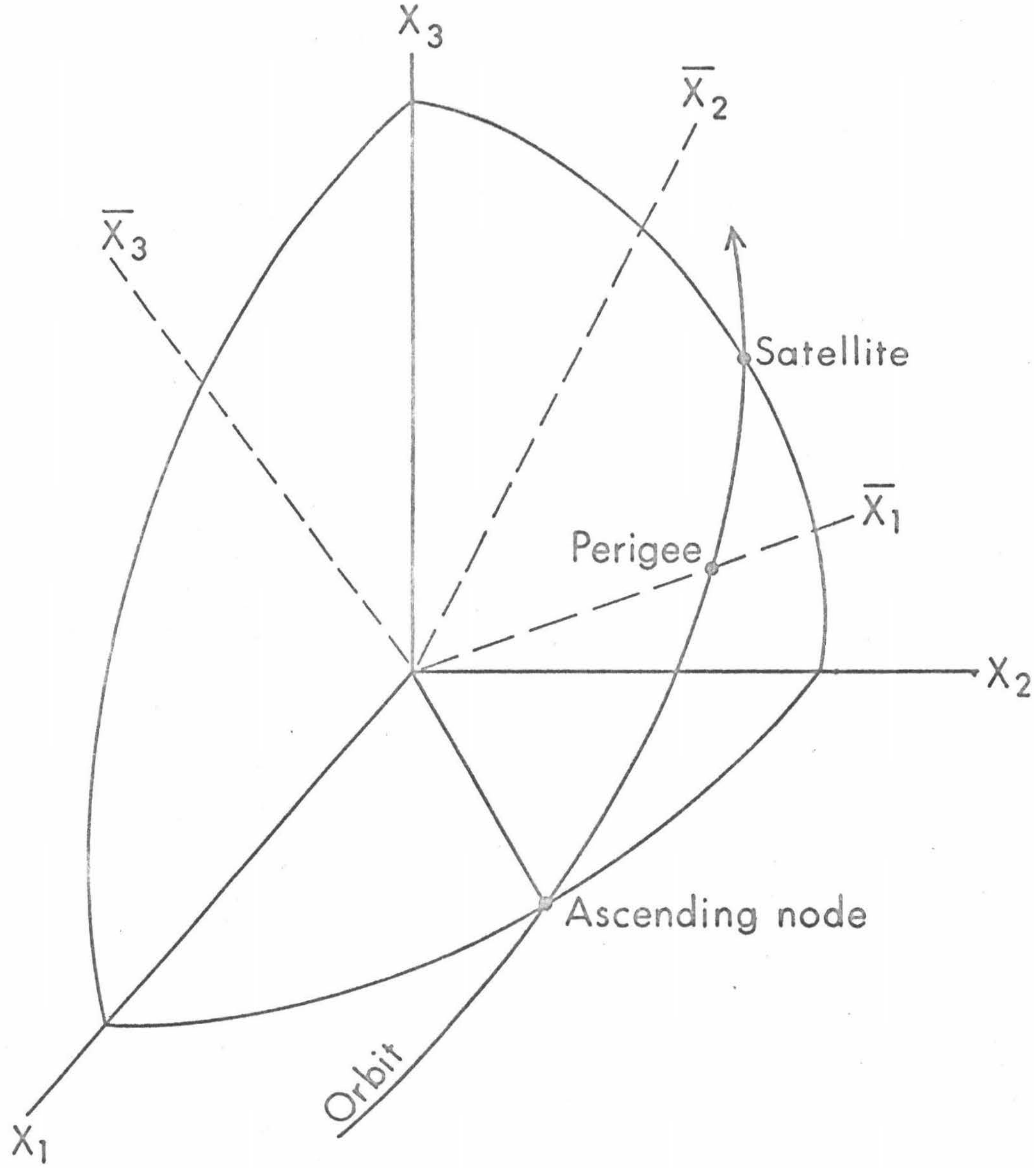


FIGURE 11. ORBITAL AND EQUATORIAL COORDINATE SYSTEM FOR AN ARTIFICIAL EARTH SATELLITE

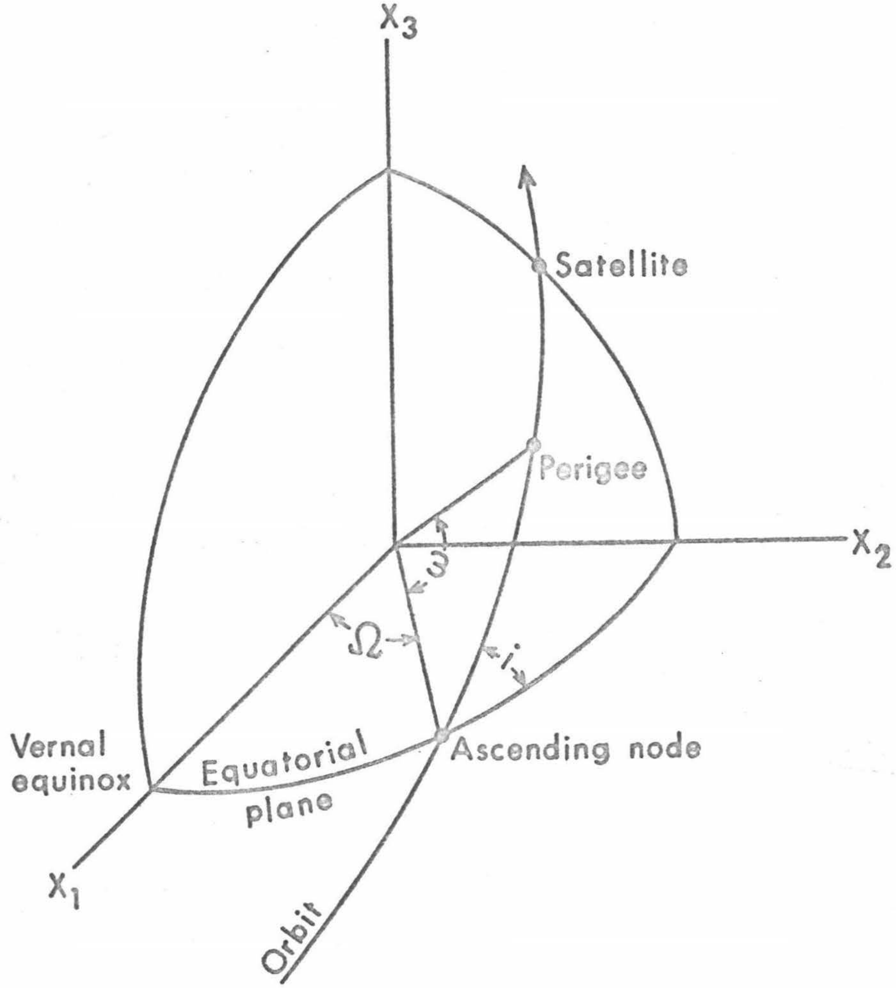


FIGURE 12. ORBITAL REPRESENTATION IN THE THREE DIMENSIONS

$$A\ (-\Omega) = \begin{pmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The complete transformation then is:

$$(X_1 X_2 X_3) = (\overline{X}_1 \overline{X}_2 \overline{X}_3)\ A\ (-w)\ B\ (-i)\ A\ (-\Omega)$$

In the numerical applications, the product A (-w) B (-i) A (-Ω) is obtained first. This product in the matrix form is:

$$A\ (-w)\ B\ (-i)\ A\ (-\Omega) = \begin{pmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{pmatrix}$$

so that the transformation finally becomes

$$(19) \qquad (X_1 X_2 X_3) = (\overline{X}_1 \overline{X}_2 \overline{X}_3) \begin{pmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{pmatrix}$$

$P_i$, $Q_i$ and $R_i$ (i = i, 2, 3) are called the vectorial orbital constants and are given by:

$$P_1 = \cos w \ \cos\Omega \ - \ \sin w \ \cos i \ \sin\Omega$$

$$Q_1 = -\sin w \cos \Omega - \cos w \cos i \sin \Omega$$

$$R_1 = \sin i \sin \Omega$$

$$P_2 = \cos w \sin \Omega + \sin w \cos i \cos \Omega$$

$$Q_2 = -\sin w \sin \Omega + \cos w \cos i \cos \Omega$$

(20)

$$R_2 = -\sin i \cos \Omega$$

$$P_3 = \sin w \sin i$$

$$Q_3 = \cos w \sin i$$

$$R_3 = \cos i$$

Note that the vectorial orbital constants are in reality the direction cosines of the axis $\overline{X}_i$ relative to the axis $X_i$ $(i = 1, 2, 3)$.

As is obvious from equation (18) and (19) the coordinates of the satellite position at any time are (Brouwer and Clemence, 1961):

$$X_1 = a (\cos E - e) P_1 + a (1-e^2)^{\frac{1}{2}} \sin E \, Q_1$$

(21) $$X_2 = a (\cos E - e) P_2 + a (1-e^2)^{\frac{1}{2}} \sin E \, Q_2$$

$$X_3 = a (\cos E - e) P_3 + a (1-e^2)^{\frac{1}{2}} \sin E \, Q_3$$

Kepler's equation can be written as

$$E - e \sin E = \overline{n} (t - T_0)$$

where

$T_0$ = time of perigee passage

$\bar{n}$ = mean motion of the satellite defined as $\bar{n} = \frac{2\pi}{P}$ and

P = the period of revolution of the satellite in its orbit.

Differentiating Kepler's equation with respect to time

$$\dot{E} = \frac{\bar{n}}{1 - e \cos E}$$

Differentiation with respect to time of equation (18) and the substitution of the above relation gives

$$\dot{\bar{X}}_1 = - \frac{a \bar{n} \sin E}{1 - e \cos E}$$

$$\dot{\bar{X}}_2 = \frac{a \bar{n} (1-e^2)^{\frac{1}{2}} \cos E}{1 - e \cos E}$$

Thus the absolute velocity components of the satellite are (Brouwer and Clemence, 1961):

$$\dot{X}_1 = \frac{\bar{n}}{1 - e \cos E} (- a \sin E \; P_1 + a (1-e^2)^{\frac{1}{2}} \cos E \; Q_1)$$

$$\text{(22)} \qquad \dot{X}_2 = \frac{\bar{n}}{1 - e \cos E} (- a \sin E \; P_2 + a (1-e^2)^{\frac{1}{2}} \cos E \; Q_2)$$

$$\dot{X}_3 = \frac{\bar{n}}{1 - e \cos E} (- a \sin E \; P_3 + a (1-e^2)^{\frac{1}{2}} \cos E \; Q_3)$$

Let $\rho$, $\phi$, $\lambda$ be the geocentric distance, latitude and longitude of the observer. Then the cartesian coordinates of the observer ($X_{10}$, $X_{20}$, $X_{30}$) are

$$X_{10} = \rho \cos\phi \cos(h_{Gr} + \lambda)$$

$$X_{20} = \rho \cos\phi \sin(h_{Gr} + \lambda) \tag{23}$$

$$X_{30} = \rho \sin\phi$$

where $h_{Gr}$ = hour angle of the vernal equinox with respect to the Greenwich meridian.

The absolute velocity components of the observer are obtained by differentiating equation (23), i. e.,

$$\dot{X}_{10} = -\rho\, \dot{h}_{Gr} \cos\phi \sin(h_{Gr} + \lambda)$$

$$\dot{X}_{20} = \rho\, \dot{h}_{Gr} \cos\phi \cos(h_{Gr} + \lambda) \tag{24}$$

$$\dot{X}_{30} = 0$$

Note that $\dot{h}_{Gr}$ is the rotation speed of the earth in radians per unit time used to express $\overline{n}$ and the other derivatives.

Equations (22) and (24) give the relative velocity components of the satellite ($\dot{X}_{1r}$, $\dot{X}_{2r}$, $\dot{X}_{3r}$), i. e.,

$$\dot{X}_{1r} = \dot{X}_1 - \dot{X}_{10}$$

$$\dot{X}_{2r} = \dot{X}_2 - \dot{X}_{20} \tag{25}$$

$$\dot{X}_{3r} = \dot{X}_3$$

To summarize, if the position vector and the relative velocity of the satellite appearing in the system of equations (15) are not available as directly measured data, the same can be obtained from the orbital elements of the satellite for a specific time with the help of equations (21) and (25). Consequently we can write the position vector $\underline{r}$ and the relative velocity v of the satellite as

$$v^2 = (\dot{X}_{1r})^2 + (\dot{X}_{2r})^2 + (\dot{X}_{3r})^2$$

$$r^2 = X_1^2 + X_2^2 + X_3^2 \tag{26}$$

$$\underline{r} = X_1\underline{i} + X_2\underline{j} + X_3\underline{k}$$

where $\underline{i}$, $\underline{j}$, $\underline{k}$ are the unit vectors in the direction of the axes $X_1$, $X_2$ and $X_3$, respectively. These substitutions in the system of equations (15) enable us to set up equations of condition when the initially available data are only in the form of the orbital elements. However, this process of conversion of the equations of condition from the form (15) to the form in which the position vector and the relative velocity are expressed in terms of the orbital elements, involves some of the assumptions of Keplerian motion which are used in the classical perturbation theory but which has been our basic concern to avoid in the development of this new theory.

## 6. Extended Theory--Inclusion of the Lunar and Air Drag Effects

Consider now the case of a satellite whose motion, though still primarily controlled by the earth's gravitational field, is being perturbed by the air resistance and the lunar attraction.

Let

$m_1$ = mass of the satellite

$m_2$ = mass of the moon

$M$ = mass of the earth

$\underline{v}_1$ = relative velocity vector of the satellite with respect to the earth

$\underline{v}_2$ = relative velocity vector of the moon with respect to the earth

$\underline{r}_1$ = the radius vector of the satellite from the origin of the coordinate system

$\underline{r}_2$ = the radius vector of the moon from the origin of the coordinate system

$\underline{w}$ = angular velocity of the earth

$L$ = the total Lagrangian of the system

$H$ = the total Hamiltonian function of the system

Then as before, the Lagrangian L of the system is

$$L = \tfrac{1}{2} m_1 (\underline{v}_1 + \underline{w} \times \underline{r}_1)^2 - m_1 U_1 + \tfrac{1}{2} m_2 (\underline{v}_2 + \underline{w} \times \underline{r}_2)^2 - m_2 U_2 - \frac{G m_1 m_2}{|\underline{r}_1 - \underline{r}_2|} \tag{27}$$

The canonical momenta $\underline{p}_1$, $\underline{p}_2$ are

$$\underline{p}_1 = \frac{\partial L}{\partial \underline{v}_1} = m_1 \ (\underline{v}_1 + \underline{w} \times \underline{r}_1)$$

(28)

$$\underline{p}_2 = \frac{\partial L}{\partial \underline{v}_2} = m_2 \ (\underline{v}_2 + \underline{w} \times \underline{r}_2)$$

$\underline{v}_1$ and $\underline{v}_2$ are thus given by

$$\underline{v}_1 = \frac{\underline{p}_1}{m_1} - \underline{w} \times \underline{r}_1$$

$$\underline{v}_2 = \frac{\underline{p}_2}{m_2} - \underline{w} \times \underline{r}_2$$

The Hamiltonian function H of the system is

$$H = \underline{p}_1 \cdot \underline{v}_1 + \underline{p}_2 \cdot \underline{v}_2 - L$$

$$= \underline{p}_1 \cdot (\frac{\underline{p}_1}{m_1} - \underline{w} \times \underline{r}_1) + \underline{p}_2 \cdot (\frac{\underline{p}_2}{m_2} - \underline{w} \times \underline{r}_2) - \tfrac{1}{2} \cdot \frac{p_1^2}{m_1}$$

$$- \tfrac{1}{2} \cdot \frac{p_2^2}{m_2} + m_1 \ U_1 + m_2 \ U_2 + \frac{G \ m_1 \ m_2}{|\underline{r}_1 - \underline{r}_2|}$$

or

$$H = \tfrac{1}{2} \frac{p_1^2}{m_1} + \tfrac{1}{2} \frac{p_2^2}{m_2} - \underline{p}_1 \cdot (\underline{w} \times \underline{r}_1) - \underline{p}_2 \cdot (\underline{w} \times \underline{r}_2) + m_1 \ U_1 \qquad (29)$$

$$+ m_2 U_2 + \frac{G m_1 m_2}{|\underline{r}_1 - \underline{r}_2|}$$

Partial differentiation of H with respect to $\underline{p}_1$, $\underline{p}_2$, $\underline{r}_1$ and $\underline{r}_2$ yields

$$\frac{\partial H}{\partial \underline{p}_1} = \frac{1}{m_1} \underline{p}_1 - \underline{w} \times \underline{r}_1$$

(30)

$$\frac{\partial H}{\partial \underline{p}_2} = \frac{1}{m_2} \underline{p}_2 - \underline{w} \times \underline{r}_2$$

$$\frac{\partial H}{\partial \underline{r}_1} = \underline{w} \times \underline{p}_1 - \underline{F}_1 - \underline{F}_{12}$$

$$\frac{\partial H}{\partial \underline{r}_2} = \underline{w} \times \underline{p}_2 - \underline{F}_2 + \underline{F}_{12}$$

where

$$\underline{F}_1 = - \frac{G m_1 M}{r_1^3} \underline{r}_1$$

$$\underline{F}_2 = - \frac{G m_2 M}{r_2^3} \underline{r}_2$$

and

$$\underline{F}_{12} = + \frac{G m_1 m_2}{|\underline{r}_1 - \underline{r}_2|^3} (\underline{r}_1 - \underline{r}_2)$$

Now define $H_1$ as

$$(31) \qquad H_1 = \underline{p}_1 \cdot \underline{v}_1 - \tfrac{1}{2} m_1 (\underline{v}_1 + \underline{w} \times \underline{r}_1)^2 + m_1 U_1$$

$$= \tfrac{1}{2} \frac{p_1^2}{m_1} - \underline{p}_1 \cdot (\underline{w} \times \underline{r}_1) + m_1 U_1$$

and similarly $H_2$ as

$$(32) \qquad H_2 = \underline{p}_2 \cdot \underline{v}_2 - \tfrac{1}{2} m_2 (\underline{v}_2 + \underline{w} \times \underline{r}_2)^2 + m_2 U_2$$

$$= \tfrac{1}{2} \frac{p_2^2}{m_2} - \underline{p}_2 \cdot (\underline{w} \times \underline{r}_2) + m_2 U_2$$

From equation (31) we obtain

$$(33) \qquad \frac{dH_1}{dt} = \frac{\partial H_1}{\partial \underline{p}_1} \cdot \frac{d\underline{p}_1}{dt} + \frac{\partial H_1}{\partial \underline{r}_1} \cdot \frac{d\underline{r}_1}{dt}$$

Now the Hamiltonian equations of motions are

$$\frac{dp_s}{dt} = - \frac{\partial H}{\partial r_s} \qquad \text{and} \qquad \frac{dr_s}{dt} = \frac{\partial H}{\partial p_s}$$

With the help of these equations, equation (33) reduces to the form

$$(34) \qquad \frac{dH_1}{dt} = - \frac{\partial H_1}{\partial \underline{p}_1} \; \frac{\partial H}{\partial \underline{r}_1} + \frac{\partial H_1}{\partial \underline{r}_1} \; \frac{\partial H}{\partial \underline{p}_1} = (H_1, H)$$

where $(H_1, H)$ is the Poisson's bracket expression and is defined as

$$(H_1, H) = \frac{\partial H_1}{\partial \underline{r}_1} \cdot \frac{\partial H}{\partial \underline{p}_1} - \frac{\partial H_1}{\partial \underline{p}_1} \cdot \frac{\partial H}{\partial \underline{r}_1}$$

From equation (31) we obtain

$$\frac{\partial H_1}{\partial \underline{p}_1} = \frac{1}{m_1} \underline{p}_1 - \underline{w} \times \underline{r}_1$$

so that the Poisson's bracket expression can be written as

$$(H_1, H) = \frac{dH_1}{dt} = \frac{\partial H}{\partial \underline{p}_1} \left(\frac{\partial H_1}{\partial \underline{r}_1} - \frac{\partial H}{\partial \underline{r}_1}\right) = \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial}{\partial \underline{r}_1} (H_1 - H)$$

$$(35) \qquad = \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial}{\partial \underline{r}_1} (- H_2 - H_{12})$$

$$= - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_2}{\partial \underline{r}_1} - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_{12}}{\partial \underline{r}_1}$$

where

$$H_{12} = \frac{G\, m_1\, m_2}{|\underline{r}_1 - \underline{r}_2|}$$

But $\frac{\partial H_2}{\partial \underline{r}_1} = 0$ which gives

$$(36) \qquad \frac{dH_1}{dt} = - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_{12}}{\partial \underline{r}_1}$$

Substitution of equation (30) in equation (36) gives

$$\frac{dH_1}{dt} = - \left(\frac{\underline{p}_1}{m_1} - \underline{w} \times \underline{r}_1\right) \cdot \frac{\partial H_{12}}{\partial \underline{r}_1}$$

Now

$$\frac{\partial H_{12}}{\partial \underline{r}_1} = \frac{\partial}{\partial \underline{r}_1} \left(\frac{G\ m_1\ m_2}{|\underline{r}_1 - \underline{r}_2|}\right) = - \underline{F}_{12}$$

and hence we finally get

$$\frac{dH_1}{dt} = \left[\frac{\underline{p}_1}{m_1} - \underline{w} \times \underline{r}_1\right] \cdot \underline{F}_{12}$$

(37)

$$= \underline{F}_{12} \cdot \underline{v}_1$$

If we now take into consideration the effect of air drag, the Lagrange's equations of motion become

(38) $$\frac{d}{dt} \cdot \frac{\partial L}{\partial \underline{v}_s} - \frac{\partial L}{\partial \underline{r}_s} = \underline{F}_{Ds} \qquad s = 1, 2$$

where $\underline{F}_D$ is the generalized force. In this particular case $\underline{F}_D$ has the dimensions of force because $\underline{r}_1$ has the dimensions of length. Note that in the general case of Lagrange's equations of motion, the generalized force appearing on the right hand side of the equation need not have dimensions of a force.

Since the canonical momentum

$$\frac{\partial L}{\partial \underline{v}_s} = \underline{p}_s \qquad s = 1, 2$$

The Lagrange's equations can be written as

$$\frac{d\underline{p}_s}{dt} - \frac{\partial L}{\partial \underline{r}_s} = \underline{F}_{Ds} \qquad s = 1,\ 2 \tag{39}$$

However, H was defined as

$$H = \underline{p}_1 \cdot \underline{v}_1 + \underline{p}_2 \cdot \underline{v}_2 - L$$

wherefrom

$$\frac{dH}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{v}_1 + \underline{p}_1 \cdot \frac{d\underline{v}_1}{dt} + \frac{d\underline{p}_2}{dt} \cdot \underline{v}_2 + \underline{p}_2 \cdot \frac{d\underline{v}_2}{dt} - \frac{\partial L}{\partial t}$$

$$- \frac{\partial L}{\partial \underline{v}_1} \cdot \frac{d\underline{v}_1}{dt} - \frac{\partial L}{\partial \underline{r}_1} \cdot \frac{d\underline{r}_1}{dt} - \frac{\partial L}{\partial \underline{v}_2} \cdot \frac{d\underline{v}_2}{dt} - \frac{\partial L}{\partial \underline{r}_2} \cdot \frac{d\underline{r}_2}{dt}$$

$$= \frac{d\underline{p}_1}{dt} \cdot \underline{v}_1 + \frac{d\underline{p}_2}{dt} \cdot \underline{v}_2 - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial \underline{r}_1} \cdot \frac{d\underline{r}_1}{dt} - \frac{\partial L}{\partial \underline{r}_2} \cdot \frac{d\underline{r}_2}{dt}$$

But from equation (39) we get

$$\frac{\partial L}{\partial \underline{r}_1} = \frac{d\underline{p}_1}{dt} - \underline{F}_{D1}$$

$$\frac{\partial L}{\partial \underline{r}_2} = \frac{d\underline{p}_2}{dt} - \underline{F}_{D2}$$

These substitutions in the expression for $\frac{dH}{dt}$ give

$$\frac{dH}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{v}_1 + \frac{d\underline{p}_2}{dt} \cdot \underline{v}_2 - \frac{\partial L}{\partial t} - \underline{v}_1 \cdot \left(\frac{d\underline{p}_1}{dt} - \underline{F}_{D1}\right)$$

$$(40) \qquad - \underline{v}_2 \left(\frac{d\underline{p}_2}{dt} - \underline{F}_{D2}\right) = - \frac{\partial L}{\partial t} + \underline{v}_1 \cdot \underline{F}_{D1} + \underline{v}_2 \cdot \underline{F}_{D2}$$

If L is independent of time, $\frac{dH}{dt}$ obviously becomes

$$(41) \qquad \frac{dH}{dt} = \underline{v}_1 \cdot \underline{F}_{D1} + \underline{v}_2 \cdot \underline{F}_{D2}$$

Following the same procedure for $H_1$, we get

$$(42) \qquad \frac{dH_1}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{v}_1 + \frac{d\underline{v}_1}{dt} \cdot \underline{p}_1 - \frac{dL_1}{dt}$$

where $L_1$ has been defined as

$$(43) \qquad L_1 = \tfrac{1}{2} m_1 (\underline{v}_1 - \underline{w} \times \underline{r}_1)^2 - m_1 U_1$$

But

$$\frac{dL_1}{dt} = \frac{\partial L_1}{\partial \underline{v}_1} \cdot \frac{d\underline{v}_1}{dt} + \frac{\partial L_1}{\partial \underline{r}_1} \cdot \frac{d\underline{r}_1}{dt}$$

$$= \underline{p}_1 \cdot \frac{d\underline{v}_1}{dt} + \frac{\partial L_1}{\partial \underline{r}_1} \cdot v_1$$

Substituting the above relation in Eq. (42), we finally get

$$\frac{dH_1}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{v}_1 + \frac{d\underline{v}_1}{dt} \cdot \underline{p}_1 - \underline{p}_1 \cdot \frac{d\underline{v}_1}{dt} - \frac{\partial L_1}{\partial \underline{r}_1} \cdot \underline{v}_1$$

$$(44) \qquad = \frac{d\underline{p}_1}{dt} \cdot \underline{v}_1 - \frac{\partial L_1}{\partial \underline{r}_1} \cdot \underline{v}_1$$

Now define a quantity $L_2$ such that

$$L_2 = \tfrac{1}{2} m_2 \; (\underline{v}_2 + \underline{w} \times \underline{r}_2)^2 - m_2 \, U_2$$

and another quantity $L_{12}$ as

$$L_{12} = - \frac{G \, m_1 \, m_2}{|\underline{r}_1 - \underline{r}_2|} = - H_{12}$$

Then

$$L = L_1 + L_2 + L_{12}$$

which gives

$$\frac{\partial L}{\partial \underline{r}_1} = \frac{\partial L_1}{\partial \underline{r}_1} + \frac{\partial L_2}{\partial \underline{r}_1} + \frac{\partial L_{12}}{\partial \underline{r}_1}$$

or

$$\frac{\partial L}{\partial \underline{r}_1} = \frac{\partial L_1}{\partial \underline{r}_1} - \frac{\partial H_{12}}{\partial \underline{r}_1}$$

Substituting the above in Lagrange's equation of motion we obtain

$$\frac{d\underline{p}_1}{dt} - \frac{\partial L_1}{\partial \underline{r}_1} + \frac{\partial H_{12}}{\partial \underline{r}_1} = \underline{F}_{D1}$$

or

$$\frac{\partial L_1}{\partial \underline{r}_1} = \frac{d\underline{p}_1}{dt} + \frac{\partial H_{12}}{\partial \underline{r}_1} - \underline{F}_{D1}$$

wherefrom

$$\frac{dH_1}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{v}_1 - \left(\frac{d\underline{p}_1}{dt} + \frac{\partial H_{12}}{\partial \underline{r}_1} - \underline{F}_{D1}\right) \cdot \underline{v}_1$$

$$= -\left(\frac{\partial H_{12}}{\partial \underline{r}_1} - \underline{F}_{D1}\right) \cdot \underline{v}_1$$

$$(45) \qquad = \underline{F}_{12} \cdot \underline{v}_1 + \underline{F}_{D1} \cdot \underline{v}_1$$

In view of Eq. (37) it is obvious that Eq. (45) takes into account the effect of both lunar attraction and air drag.

Integrate Eq. (45) between $t = t_i$ and $t = t_{i+1}$ and get

$$H_1 (t_{i+1}) - H_1 (t_i) = \int_{t_{i+1}}^{t_{i+2}} (\underline{F}_{12} + \underline{F}_{D1}) \cdot \underline{v}_1 \, dt$$

Now if the intervals $t_{i+1} - t_i$ and $t_{i+2} - t_{i+1}$ are not too large and $t_{i+2} - t_{i+1} = t_{i+1} - t_i$, the integrand does not vary too greatly in the interval $t_{i+2} - t_{i+1} = t_{i+1} - t_i$ and hence we obtain by subtraction

$$(46) \qquad 2 H_1 (t_{i+1}) - H_1 (t_i) - H_1 (t_{i+2}) = 0 \qquad i=1,2,\ldots i$$

## 7. Equations of Condition for the Extended Theory

Eq. (46) is the basic form of the observation equations. As before, let $\underline{r}_i$ and $\underline{v}_i$ be the observed values of the radius vector and the relative velocity of the satellite. Then substituting these values in the expression for the Hamiltonian function, we obtain the observation equations in the following form:

$$GM \sum_{n=0}^{\infty} \sum_{m=0}^{n} a_e^n \, [C_{nm} \, (\frac{2 \cos m \lambda_{i+1}}{r_{i+1}^{n+1}} P_{nm_{(i+1)}} - \frac{\cos m \lambda_i}{r_i^{n+1}} P_{nm_{(i)}}$$

$$- \frac{\cos m \lambda_{i+2}}{r_{i+2}^{n+1}} P_{nm_{(i+2)}} ) + S_{nm} \, (\frac{2 \sin m \lambda_{i+1}}{r_{i+1}^{n+1}} P_{nm_{(i+1)}} - \frac{\sin m \lambda_i}{r_i^{n+1}}$$

$$P_{nm_{(i)}} - \frac{\sin m \lambda_{i+2}}{r_{i+2}^{n+1}} P_{nm_{(i+2)}} )] + (v_{i+1}^2 - \tfrac{1}{2} v_i^2 - \tfrac{1}{2} v_{i+2}^2)$$

$$+ [(\underline{w} \cdot \underline{r}_{i+1})^2 - \tfrac{1}{2} (\underline{w} \cdot \underline{r}_i)^2 - \tfrac{1}{2} (\underline{w} \cdot \underline{r}_{i+2})^2]$$

$$- w^2 \, (r_{i+1}^2 - \tfrac{1}{2} r_i^2 - \tfrac{1}{2} r_{i+2}^2) = 0 \qquad i = 1, 2, \ldots i.$$

(47)
$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} [C_{nm} \, g_{nm_i} + S_{nm} \, h_{nm_i}] = f_i \qquad i = 1, 2, \ldots i.$$

where the functions $g_{nm_i}$, $h_{nm_i}$ and $f_i$ are defined as follows:

$$g_{nm_i} = GM \, a_e^n \, [\frac{2 \cos m \lambda_{i+1}}{r_{i+1}^{n+1}} P_{nm_{(i+1)}} - \frac{\cos m \lambda_i}{r_i^{n+1}} P_{nm_{(i)}}$$

$$- \frac{\cos m \lambda_{i+2}}{r_{i+2}^{n+1}} P_{nm_{(i+2)}} ]$$

(48)
$$h_{nm_i} = GM \, a_e^n \, [\frac{2 \sin m \lambda_{i+1}}{r_{i+1}^{n+1}} P_{nm_{(i+1)}} - \frac{\sin m \lambda_i}{r_i^{n+1}} P_{nm_{(i)}}$$

$$- \frac{\sin m \lambda_{i+2}}{r_{i+2}^{n+1}} P_{nm_{(i+2)}} \quad ]$$

and

$$f_i = [(v_{i+1}^2 - \tfrac{1}{2} v_i^2 - \tfrac{1}{2} v_{i+2}^2) + (\underline{w} \cdot \underline{r}_{i+1})^2 - \tfrac{1}{2} (\underline{w} \cdot \underline{r}_i)^2$$

$$- \tfrac{1}{2} (\underline{w} \cdot \underline{r}_{i+2})^2 - w^2 \; (r_{i+1}^2 - \tfrac{1}{2} r_i^2 - \tfrac{1}{2} r_{i+2}^2)]$$

Note that the functions $g_{nm_i}$, $h_{nm_i}$ and $f_i$ defined above are different from those used in Eq. (15). Also note that, in the above observation equations, $v_2$ denotes the relative velocity of the satellite at time $t_2$. This is not to be confused with the relative velocity of the moon for which the same symbol has been used in the theoretical development.

## 8. Extended Theory--Inclusion of the Effects of Lunar and Solar Attraction, Air Drag and Radiation Pressure

The equations of condition in the form given in Eq. (46) are derived on the assumption that the only disturbing forces (other than gravitational forces of the earth) acting on the satellite are air drag and the moon's pull, but further development of the theory brings out the interesting result that the equations of condition retain the same form when, in addition to the lunar and air drag effects, the effects of solar attraction and radiation pressure are taken into account.

In addition to the symbols of the previous section, denote the apparent velocity vector of the sun relative to the earth as $\underline{v}_3$ and its mass $m_3$. Also let L now denote the total Lagrangian of the new system in which the satellite is moving in the gravitational field of the earth under the perturbing influence of both the sun and the moon. Let H denote the Hamiltonian function of the new system. Then proceeding in the same way as in the previous section, we have

$$(49) \qquad L = \tfrac{1}{2} \sum_{i=1}^{3} m_i \left(\underline{v}_i + \underline{w} \times \underline{r}_i\right)^2 - \sum_{i=1}^{3} m_i U_i - \frac{G\, m_1\, m_2}{|\underline{r}_1 - \underline{r}_2|}$$

$$- \frac{G\, m_1\, m_3}{|\underline{r}_1 - \underline{r}_3|} - \frac{G\, m_2\, m_3}{|\underline{r}_2 - \underline{r}_3|}$$

Canonical momenta $\underline{p}_i$ are

$$\underline{p}_i = \frac{\partial L}{\partial \underline{v}_i} = m_i \left(\underline{v}_i + \underline{w} \times \underline{r}_i\right)$$

and the Hamiltonian function H is

$$H = \sum_{i=1}^{3} \underline{p}_i \cdot \underline{v}_i - L$$

$$(50) \qquad = \tfrac{1}{2} \sum_{i=1}^{3} \frac{p_i^2}{m_i} - \sum_{i=1}^{3} \underline{p}_i \cdot \left(\underline{w} \times \underline{r}_i\right) + \sum_{i=1}^{3} m_i U_i$$

$$+ H_{12} + H_{13} + H_{23}$$

where

$$H_{12} = \frac{G\, m_1\, m_2}{|\underline{r}_1 - \underline{r}_2|}$$

(51)
$$H_{13} = \frac{G\, m_1\, m_3}{|\underline{r}_1 - \underline{r}_3|}$$

and

$$H_{23} = \frac{G\, m_2\, m_3}{|\underline{r}_2 - \underline{r}_3|}$$

This gives

$$\frac{\partial H}{\partial \underline{p}_i} = \frac{1}{m_i}\, \underline{p}_i - \underline{w} \times \underline{r}_i \qquad i = 1, 2, 3.$$

$$\frac{\partial H}{\partial \underline{r}_1} = \underline{w} \times \underline{p}_1 - \underline{F}_1 - \underline{F}_{12} - \underline{F}_{13}$$

(52)
$$\frac{\partial H}{\partial \underline{r}_2} = \underline{w} \times \underline{p}_2 - \underline{F}_2 + \underline{F}_{12} - \underline{F}_{23}$$

where

$$\underline{F}_{12} = -\frac{\partial H_{12}}{\partial \underline{r}_1} = \frac{\partial H_{12}}{\partial \underline{r}_2}$$

(53)
$$\underline{F}_{13} = -\frac{\partial H_{13}}{\partial \underline{r}_1} = \frac{\partial H_{13}}{\partial \underline{r}_3}$$

$$\underline{F}_{23} = -\frac{\partial H_{23}}{\partial \underline{r}_1} = \frac{\partial H_{23}}{\partial \underline{r}_3}$$

Define $H_i$ as

$$H_i = \frac{1}{2}\,\frac{p_i^2}{m_i} - \underline{p}_i \cdot (\underline{w} \times \underline{r}_i) + m_i\, U_i$$

Differentiate this expression to obtain

$$\frac{dH_1}{dt} = \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial}{\partial \underline{r}_1} [ - H_2 - H_3 - H_{12} - H_{13} - H_{23}]$$

$$(54) \qquad = - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_{12}}{\partial \underline{r}_1} - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_{13}}{\partial \underline{r}_1}$$

$$= \underline{v}_1 \cdot [\underline{F}_{12} + \underline{F}_{13}]$$

Note that in the above expressions $\underline{F}_{12}$ and $\underline{F}_{13}$ are the forces acting on the satellite arising from the disturbing effects of the moon and the sun respectively.

Now let $\underline{F}_D$ and $\underline{F}_R$ be the generalised forces arising from the disturbing effect of the air drag and the radiation pressure, then the Lagrange's equations of motion expressed in terms of canonical momenta become

$$(55) \qquad \frac{d\underline{p}_i}{dt} - \frac{\partial L}{\partial \underline{r}_i} = \underline{F}_{Di} + \underline{F}_{Ri}$$

Note that in this particular case $\underline{F}_D$ and $\underline{F}_R$ have the dimensions of force, but they do not have to be essentially potential-derived.

If we denote $H_{12} = - L_{12}$ and $H_{13} = - L_{13}$ and define $L_1$ as in the previous section, the expression for $\frac{\partial L}{\partial \underline{r}_1}$ becomes

$$\frac{\partial L}{\partial \underline{r}_1} = \frac{\partial L_1}{\partial \underline{r}_1} + \frac{\partial L_{12}}{\partial \underline{r}_1} + \frac{\partial L_{13}}{\partial \underline{r}_1}$$

$$(56) \qquad = \frac{\partial L_1}{\partial \underline{r}_1} - \frac{\partial H_{12}}{\partial \underline{r}_1} - \frac{\partial H_{13}}{\partial \underline{r}_1}$$

which on substitution in equations of motion gives

$$\frac{d\underline{p}_1}{dt} - \frac{\partial L_1}{\partial \underline{r}_1} + \frac{\partial H_{12}}{\partial \underline{r}_1} + \frac{\partial H_{13}}{\partial \underline{r}_1} = \underline{F}_{D1} + \underline{F}_{R1}$$

wherefrom

$$(57) \qquad \frac{\partial L_1}{\partial \underline{r}_1} = \frac{d\underline{p}_1}{dt} + \frac{\partial H_{12}}{\partial \underline{r}_1} + \frac{\partial H_{13}}{\partial \underline{r}_1} - \underline{F}_{D1} - \underline{F}_{R1}$$

If this value of $\frac{\partial L_1}{\partial \underline{r}_1}$ is substituted in Eq. (44), we obtain

$$\frac{dH_1}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{v}_1 - \left(\frac{d\underline{p}_1}{dt} + \frac{\partial H_{12}}{\partial \underline{r}_1} + \frac{\partial H_{13}}{\partial \underline{r}_1} - \underline{F}_{D1} - \underline{F}_{R1}\right) \cdot \underline{v}_1$$

$$= - \underline{v}_1 \cdot \left(\frac{\partial H_{12}}{\partial \underline{r}_1} + \frac{\partial H_{13}}{\partial \underline{r}_1} - \underline{F}_{D1} - \underline{F}_{R1}\right)$$

$$(58) \qquad = \underline{F}_{12} \cdot \underline{v}_1 + \underline{F}_{13} \cdot \underline{v}_1 + \underline{F}_{D1} \cdot \underline{v}_1 + \underline{F}_{R1} \cdot \underline{v}_1$$

As may be noted, Eq. (58) allows for the forces arising from the four most important factors perturbing the satellite motion; i.e., lunar and solar attraction, air drag and radiation pressure.

If Eq. (58) is integrated between $t = t_i$ and $t = t_{i+1}$, we get

$$(59, a) \qquad H_1 (t_{i+1}) - H_1 (t_i) = \int_{t_i}^{t_{i+1}} [\underline{F}_{12} + \underline{F}_{13} + \underline{F}_{D1} + \underline{F}_{R1}] \cdot \underline{v}_1 \, dt$$

Similarly integration between $t = t_{i+1}$ and $t = t_{i+2}$ gives

$$(59, b) \qquad H_1 (t_{i+2}) - H_1 (t_{i+1}) = \int_{i+1}^{i+2} [\underline{F}_{12} + \underline{F}_{13} + \underline{F}_{D1} + \underline{F}_{R1}] \cdot \underline{v}_1 \, dt$$

Now if $t_{i+2} - t_{i+1} = t_{i+1} - t_i = \Delta t$ where $\Delta t$ is a small interval of time during which the integrand does not vary too greatly, we get

$$(60) \qquad 2 H_1 (t_{i+1}) - H_1 (t_i) - H_1 (t_{i+2}) = 0 \quad i = 1, 2 \ldots i.$$

Thus it may be seen that the equations of condition set up on the pattern of Eq. (46) should eliminate approximately the perturbing effects of air drag, radiation pressure and lunar and solar attractions, provided these effects can be considered constant or approximately so over short consecutive intervals of equal duration.

## 9. Applicability of the New Theory

The method of setting up the equations of condition in the new theory is primarily designed to cancel out (at least approximately) the effects of the disturbing forces, thus eliminating the necessity of computing the corrections arising from them. With our present knowledge of the atmosphere and radiation pressure at satellite altitudes, the computation of these corrections is an estimate at best and often involves some poorly-determined parameters. Thus the elimination of the necessity of computing these corrections enables us to avoid a potential source of error. Note however, that the equations of condition in the new theory are based on the assumption that the integrand of the disturbing forces remains constant over short consecutive intervals. This may not be true in cases when the satellite is entering from a rarified atmosphere at a high altitude to a relatively denser atmosphere near perigee, or when it is entering or leaving the 'shadow zone', in other words whenever its position is such that the effect of the air resistance and/or radiation pressure is likely to vary substantially over short intervals of time. However, by some selective process it should be possible to eliminate those data which were recorded when the satellite was in any of these 'critical transit positions'. However, if at some future time our knowledge of the atmospheric structure and solar radiation mechanism grows to a level where we could be assured of the necessary degree of accuracy required in the computation of the air drag and solar radiation corrections, it may be simpler and more convenient to use the equations of condition derived for the 'simplified theory' with due allowance for the luni-solar attraction.

Another major advantage of the new theory seems to be that it can be used for low altitude satellites because of the invariance of the equations of condition to the magnitude of the disturbing forces (i.e. the equations of condition would remain valid as long as the integrand of the disturbing forces is constant over the observation interval irrespective of their magnitude). This appears to offer the possibility for exploiting the 'short wave length sensing potentiality' of a low altitude satellite which cannot be done with advantage using perturbation theory. This factor coupled with the fact that the information yielded by the new theory reflects primarily the effect of the mass anomalies of the region immediately below the satellite, appears to provide a means for obtaining a representation of the earth's gravity field from satellite data which would show at least some of the subtler short wave length features exhibited in terrestrial measurements.

Since the limiting wave length of a satellite is a function of its altitude, the satellites at different altitudes will 'sense' the earth's gravity field to different degrees of detail and hence the geopotential coefficients descriptive of the gravity field of the region in the vicinity of the satellite projection on the ground will be a function of altitude also. Thus each set of coefficients will reflect the degree of detail of gravity field as 'sensed' by the satellite (from the measurements of which that particular set of coefficients has been obtained) at its particular height. This offers the possibility of studying the problem of the upward continuation of the earth's gravity field to greater heights.

The theory also appears to provide an application in applied geophysics. If surface gravity measurements in an area are available and if one is only interested in anomalous gravity composed of wave lengths shorter than a specific 'limiting wave length', one can obtain the desired part of the gravity field by simply subtracting the gravity field determined from a satellite with the above 'limiting wave length' from the observed gravity field. This seems to offer an effective way of removing the regional effect and to reduce the observed gravity to wave lengths of geological interest.

The time variant part of the gravity field can be determined if we can observe a satellite moving at a constant altitude over a long period of time. However, this phase of study may be handicapped by the fact that a satellite is continually changing its altitude in the successive transits over the same station. But if the variation of the gravity field could be accurately tied to altitude changes, we could study the time variation inspite of changes in satellite height. Applying the same principle, there is a possibility of studying any radial asymmetry in mass distribution of the core such as one caused by the convection currents. The theory could also be used in principle to detect any differential rotation between the core and the mantle provided an asymmetry of the type described above exists in the core. But any study of the above type will probably put very stringent restrictions on the accuracy of the observed data and also require a highly accurate knowledge of some of the hitherto poorly-determined or still unknown parameters.

As the 'limiting wave length' is a function of satellite height and since each set of measurements on a satellite at a certain altitude will yield a different set of geopotential coefficients for any specific region, a random combination of several sets of observations of satellites at different heights may not be possible in the same solution and this may limit the use of the new theory. This limitation can obviously be overcome by making a sufficient number of measurements on the same satellite and by combining data from satellites having the same height. However, this limitation may not prove to be as serious in practice as it appears to be in theory.

It is important to note that the above discussion outlines the theoretical development only and the theory has not been tested as yet.

Hence all the problems associated with the actual application of this theory cannot be foreseen at present. The applicability of the theory--its scope and limitations--will be defined with more confidence when it is applied in practice.

## APPENDIX

## ERROR ANALYSIS IN A TYPICAL CASE

In order to have an idea about the minimum limit of accuracy required in the measurement of position vector and the relative velocity of a satellite for obtaining meaningful second differences, consider a satellite moving in an elliptical orbit. Then under the ideal conditions of elliptical motion without any perturbation effects of any sort, assuming a sampling interval of one second, let us have

$$v = 10 \text{ km/sec}$$

$$r = 10{,}000 \text{ km}$$

$$\Delta v = 10 \text{ meters/sec}$$

$$\Delta r = 300 \text{ meters/sec}$$

Let $m_r$ and $m_v$ be the errors in the measurement of r and v. Then, assuming the simplified case in which $m_r$ and $m_v$ are independent, the error m in H is given by

$$m^2 = m_r^2 \left(\frac{\partial H}{\partial r}\right)^2 + m_v^2 \left(\frac{\partial H}{\partial v}\right)^2$$

If n be the number of H's involved in an expression, the final error $m_n$ for that expression will be

$$m_n = m(n)^{\frac{1}{2}}$$

with the assumption that the error m is the same for the quantities $H_1$, $H_2$,....$H_n$. For the second difference $d^2H$ therefore, the error $m_2$ will be

$$m_2 = 2m$$

and thus we should have

$$d^2H > 2m$$

Utilizing this restriction, it has been found for the case considered above, that the minimum accuracy required in the measurement of the position r is not critical and is well above the limit attainable at present. The error in v, however, appears to be critical for this case and the minimum accuracy required in the measurement of v comes out to be better than roughly a cm/sec or so. This minimum limit can be raised by increasing the sampling interval but that will reduce the number of observations available for any given orbital segment. In the actual application of the theory, therefore, a balance will have to be worked out in view of the accuracy of the available data, the magnitude of $\Delta r$ and $\Delta v$ and the wavelength of the gravity anomaly desired to be studied.

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