Nonlinear four-wave interaction and freak waves

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Abstract. Four-wave interactions play an important role in the evolution of the spectrum of surface gravity waves. This follows from direct simulations of an ensemble of ocean waves using the Zakharov equation. The theory of homogeneous four-wave interactions, extended to include effects of nonresonant transfer, compares favourably with the ensemble averaged results of the Monte Carlo simulations. In particular, there is good agreement regarding spectral shape. Also, the kurtosis of the surface elevation probability distribution is well-determined by theory even for waves with a narrow spectrum and large steepness. These extreme conditions are favourable for the occurrence of freak waves and are well-described by the so-called Benjamin-Feir Index (BFI), which is essentially the ratio of the steepness of the waves and the width of the spectrum. In order to validate the theory, many occurrences of extreme wave events are required. Presently, this is only possible in the laboratory by performing wave tank experiments. Good agreement between observed and theoretical probability distribution functions is found. In October 2003 ECMWF introduced a new set of wave parameters, such as the BFI, which serve as an indicator for extreme events. Validation of such a Freak Wave Warning system is highly desirable.

Introduction

Recently, evidence for the existence of freak waves has been found. Since ships have not been designed to withstand these exceptional conditions, it is of the utmost importance to be able to predict the probability that freak waves occur.

For sure, in the context of modern wave forecasting systems it is not possible to predict individual wave events. We have to content ourselves with methods from probabilistic wave forecasting.

The programme of this paper is as follows.

- Wave prediction

  Wave prediction systems give the evolution of the average sea state in a box of the size of say 50×50 km. The sea state is characterized by the wave spectrum. In past decade we have seen a good improvement in our capability to predict parameters such as the significant wave height.

- Freak waves generation

  On the open ocean extreme waves are generated by *nonlinear focussing*, a process that also causes the Benjamin-Feir Instability.

- Freak waves prediction

  It is made plausible that, for given average sea state, the probability of extremes such as freak waves may indeed be obtained.

The main results of this work have been published elsewhere, see e.g. *Janssen* (2003, 2004).

Stochastic approach

For various reasons forecasting of individual ocean waves is not possible. One reason is that this requires knowledge of the initial phases of the waves, while another reason is that the deterministic equations exhibit chaotic behaviour, even when integrated over a relatively short time, of the order of 1000 periods. Hence we consider the evolution of average seastate: a spectral description!

Energy balance equation

It is therefore common practice in wave forecasting to concentrate on the prediction of the ensemble average of, for example, the action density spectrum $N(\vec{k}, t)$. Action plays the role of a number density and is defined in such a way that energy spectrum $F(\vec{k}, t)$ is given as

$$F(\vec{k}, t) = \omega(\vec{k}) \times N(\vec{k}, t)$$

which is the usual rule in wave mechanics. From first principles one finds the following evolution equation

$$\frac{\partial}{\partial t}N + \nabla_{\vec{x}} \cdot (\dot{\vec{x}}N) + \nabla_{\vec{k}} \cdot (\dot{\vec{k}}N) = S = S_{in} + S_{nl} + S_{ds}, \quad (1)$$

where $\dot{\vec{x}} = \partial\omega/\partial\vec{k}$, $\dot{\vec{k}} = -\partial\omega/\partial\vec{x}$, and the source functions S represent the physics of wind-wave generation, dissipation by wave breaking and nonlinear four-wave interactions.

Wave forecasting

The energy balance equation (including nonlinear transfer) is solved by modern wave prediction systems, where the forcing of the waves is provided by surface winds from weather prediction systems. For a large part ($\pm 80\%$) the quality of the wave forecast is determined by the accuracy of the surface wind field.

Operational ocean wave forecasting at ECMWF started in June 1992 and is based on WAM cy4. Discuss global implementation only.

The ECMWF global wave model (81 deg S to 81 deg N) is coupled to atmospheric model [two-way interaction with feedback of ocean waves on ocean surface roughness (since June 29, 1998) thus giving a sea-state dependent momentum and heat flux]. This setup is used in two applications:

- Deterministic forecasts:

  Here the spectrum $F(f, \theta)$ has 30 frequencies and 24 directions. The model is implemented on an irregular lat-lon grid, $\Delta x = 55$ km. In order to obtain an optimal initial condition, ENVISAT Altimeter wave heights and ERS-2 SAR spectra are assimilated. The wave model is forced by atmospheric winds every time step $\Delta t = 15$ min.

  Every day two 10–day forecasts from 00Z and 12 Z are issued.

- Probabilistic forecasts:

  In order to estimate forecast error an ensemble prediction approach is followed. It is well-known that the quality of the weather forecast is determined to a considerable extent by the accuracy of the initial conditions. A 50-member ensemble of low-resolution weather (and since the wave model is tightly coupled to the atmospheric model) and wave forecasts is generated by perturbing to the analysis with the most unstable perturbations. The current resolution of the ensemble prediction system is T_l255 for the atmosphere and $1°$ for the waves.

  The potential use of probablistic forecasting is in estimating the probability of high sea state and in ship routing. For example, error in forecast ship route may be obtained from the shiproutes generated by the winds and waves of the 50-member ensemble.

Wave forecast verification

Validation of the wind and wave forecast is of vital importance. Routinely analyses are compared with observations, whilst forecasts are validated against analysis and observations. For example, first-guess wave height and analyzed wind are compared with ERS2 and ENVISAT altimeter data, wave height and peak period are verified against buoy data, while forecast scores are obtained by verifying forecasts against the analysis.

From this validation effort we have found that the quality of the wave forecast is to a considerable extent determined by quality of the winds, but model improvements have contributed as well. Over the long term considerable improvements are seen. Apart from improved winds this is caused by new developments such as two-way interaction, introduced in June 1998.

This conclusion follows from all our verification studies, but I only show one example, namely the verification of forecast wave height against analyzed wave height data. Note the dramatic improvements from the middle of 1998 and onwards.

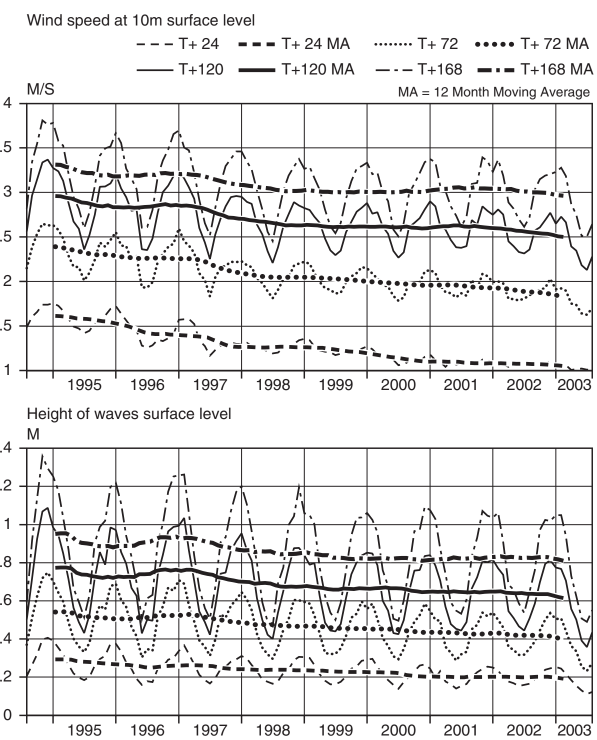

Figure 1. Forecast verification against analysis for wind and waves in the Northern Hemisphere.

Freak waves generation

In linear theory there is no interaction between ocean waves. Focussing of wave energy only occurs when the phases of the waves are favourable (*constructive interference*). Gives at best a doubling of wave height. Accordingly, the probability distribution function for the surface elevation η is given by the Gaussian distribution.

However, the situation for *nonlinear* waves is entirely different, because now there is the possibility of wave-wave interaction. Thus, a wave may borrow energy from its neighbours. Because of this extra focussing, wave height may become at most 3 times as large in 1D, while it 2 D it becomes 4.5-5 times as large as the average wave height.

As a consequence, for nonlinear waves the surface elevation distribution is rather different, in particular for the extremes.

Monte Carlo forecasting

I have shown this by performing Monte Carlo Forecasts with the 1D version of the Zakharov equation, which describes the evolution of the complex amplitude $a(\vec{k})$ of the free gravity waves:

$$\frac{\partial a_1}{\partial t} + i\omega_1 a_1 = -i \int \mathrm{d}\vec{k}_{2,3,4} T_{1,2,3,4} a_2^* a_3 a_4 \delta_{1+2-3-4}, \quad (2)$$

where $\vec{k}$ is the wave number and $\omega = \sqrt{gk}$. $T_{1,2,3,4}$ is a complicated function of frequency and wavenumber, and has a number of symmetries which garantee that the system is Hamiltonian.

From the numerical results of an ensemble of 500 members I obtained interesting quantities such as wave spectrum and the pdf of the surface elevation.

Evolution of ensemble mean

In 1D there are no resonant four-wave interactions that give rise to spectral change. Extended the well-known Hasselmann equation for the action density by including non-resonant four-wave interactions. These non-resonant interactions give rise to an irreversible change of the spectrum. In addition, the extended Boltzmann equation conserves the ensemble average of the Hamiltonian but, of course, not the linear energy.

For a *homogeneous* sea state the action density $N(\vec{k})$ is defined as

$$B_{i,j} =< a_i a_j^* >= N_i \delta(\vec{k}_i - \vec{k}_j), \quad (3)$$

and the task is to derive an evolution equation for N from the Zakharov equation. Because of nonlinearity, the equation for the second moment couples to the fourth moment, etc, resulting in an infinite hierarchy of equations, known as the BBGKY hierarchy. Closure is achieved by assuming that the waves, having a small steepness ϵ, are *weakly nonlinear* so that the pdf of the surface elevation is close to a Gaussian (Random-Phase Approximation (RPA)).

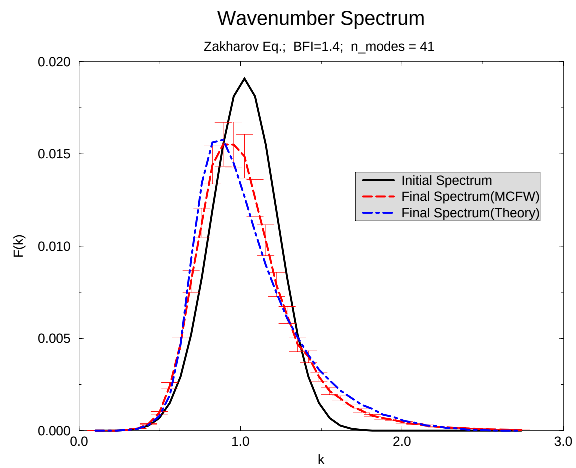

Figure 2. Initial and final time wave number spectrum according to Monte Carlo Forecasting of Waves (MCFW) using the Zakharov equation. Error bars give 95% confidence limits. Results from theory are shown as well.

For example, the fourth moment is

$$< a_j a_k a_l^* a_m^* > \;=\; B_{j,l}B_{k,m} + B_{j,m}B_{k,l} + D_{j,k,l,m},$$

where D is the so-called fourth cumulant, which vanishes for a Gaussian sea state. A similar relation applies for the 6th moment, and application of RPA closes the BBGKY hierarchy. As a consequence, the fourth cumulant D, subject to the initial value $D(t=0)=0$, becomes

$$\begin{aligned} D_{i,j,k,l} &= 2T_{i,j,k,l}\delta_{i+j-k-l}G(\Delta\omega, t) \\ &\quad [N_iN_j(N_k+N_l) - (N_i+N_j)N_kN_l] \quad (4) \end{aligned}$$

where $\Delta\omega = \omega_i + \omega_j - \omega_k - \omega_l$. This elegant result requires extensive use of the symmetries of T. In addition, the action density N is assumed to evolve on the slow time scale. The function G is defined as

$$G(\Delta\omega, t) = i\int_0^t \mathrm{d}\tau e^{i\Delta\omega(\tau - t)} = R_r(\Delta\omega, t) + iR_i(\Delta\omega, t).$$

Knowledge of the fourth cumulant is essential for

- evolution of N caused by *four-wave interactions*
- determination of deviations from *normality*.

Substitution of D in the equation for the second moment gives an evolution equation for the action density N

$$\begin{aligned} \frac{\partial}{\partial t}N_4 = 4\int \mathrm{d}\vec{k}_{1,2,3}T_{1,2,3,4}^2\delta(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)R_i(\Delta\omega,t) \\ \times[N_1N_2(N_3+N_4) - N_3N_4(N_1+N_2)], \quad (5) \end{aligned}$$

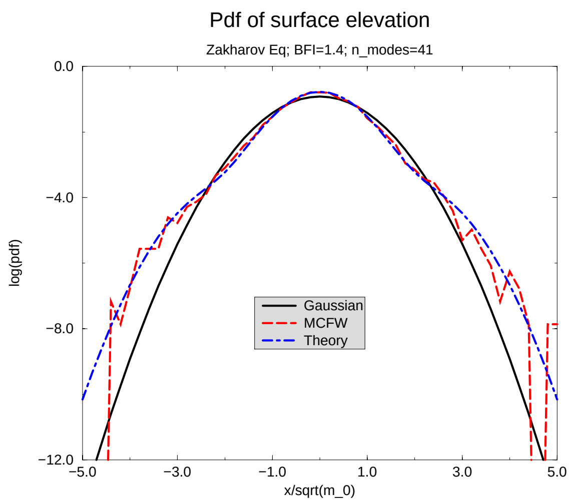

Figure 3. Log of PDF for surface elevation (BFI=1.4). For reference the Gaussian distribution is shown as well. Freak waves correspond to a normalized height of 4 or larger.

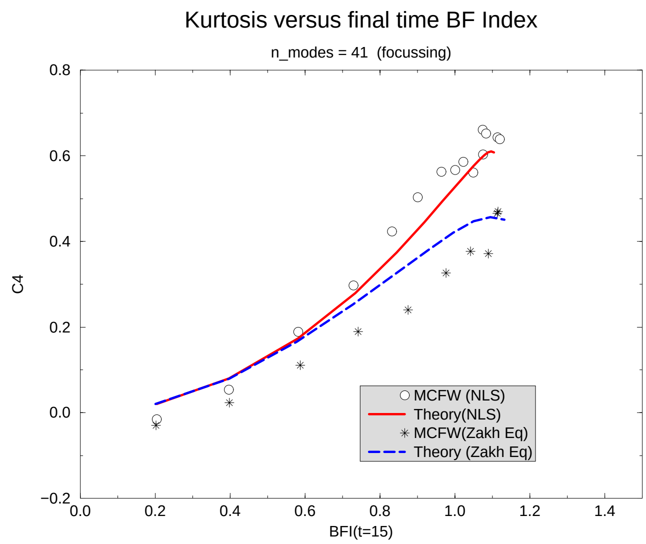

Figure 4. Normalized Kurtosis as function of the BF Index. Shown are results for focussing from simulations with NLS and with the Zakharov equation, and corresponding theoretical results.

This equation describes both the effects of resonant and non-resonant four-wave interactions, and therefore there are two timescales implied by $R_i(\Delta\omega, t) = \sin(\Delta\omega t)/\Delta\omega$: for short times $\lim_{t\to 0} R_i(\Delta\omega, t) = t$, hence $T_{NL} = O(1/\epsilon^2\omega_0)$, the Benjamin-Feir timescale, while for large times $\lim_{t\to\infty} R_i(\Delta\omega, t) = \pi\delta(\Delta\omega)$, corresponding to resonant wave-wave interactions, hence $T_{NL} = O(1/\epsilon^4\omega_0)$.

Deviations from Normality are most conveniently expressed by means of the kurtosis,

$$C_4 =< \eta^4 > /3 < \eta^2 >^2 -1,$$

Using D the kurtosis becomes

$$C_4 = \frac{4}{g^2 m_0^2} \int \mathrm{d}\vec{k}_{1,2,3,4} T_{1,2,3,4} \delta_{1+2-3-4} \left(\omega_1\omega_2\omega_3\omega_4\right)^{\frac{1}{2}} \times R_r(\Delta\omega, t) N_1 N_2 N_3, \qquad (6)$$

As $\lim_{t\to\infty} R_r(\Delta\omega, t) = P/\Delta\omega$, the kurtosis is determined by both resonant and non-resonant interactions.

An important recent advance is that kurtosis can be related to a spectral shape parameter, namely the Benjamin-Feir Index (BFI)

$$BFI = \epsilon\sqrt{2}/\sigma'_\omega, \qquad (7)$$

where $\sigma'_\omega = \sigma_\omega/\omega_0$ is the relative width of the frequency spectrum and $\epsilon = (k_0^2 < \eta^2 >)^{\frac{1}{2}}$ is an integral measure of wave steepness (with $< \eta^2 >$ the average surface elevation variance and k_0 the peak wave number).

Theoretically, the kurtosis is a very complicated expression in terms of the (action) wave spectrum N. For Gaussian-shaped spectra in the narrow band approximation the kurtosis shows a particularly simple form:

$$C_4 = \frac{\pi}{3\sqrt{3}} \times BFI^2, \qquad (8)$$

hence the kurtosis depends on the square of the BF index.

Results

The next figures show results for

- Spectral evolution
- PDF of surface elevation
- relation between Kurtosis and BFI

It is noted that, even for large values of BFI, there is a fair agreement between results from the theory of the ensemble mean on the one hand and the Monte Carlo simulations on the other hand.

Freak waves prediction

The theoretical picture has been confirmed by observations in a big wave tank in Trondheim (*Onorato et al.*, 2004). In particular, theory and Monte Carlo simulations seem to give a reasonable description of the dependence of kurtosis on fetch (cf. Fig. (5)).

An interesting quantity to obtain is the wave height distribution. Recently, for narrow spectra *Mori and Janssen* (2005)(see also *Mori and Yasuda*, 2002) found, starting from

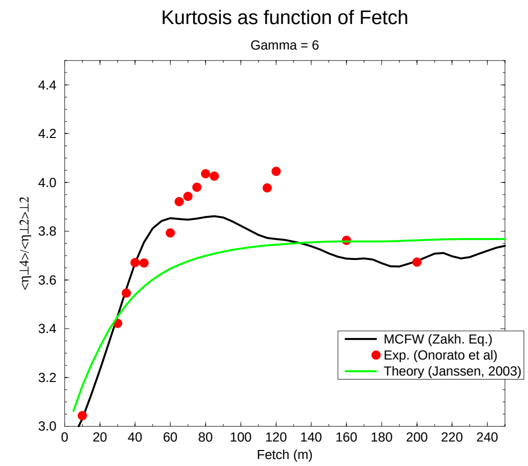

Figure 5. Kurtosis $\langle\eta^4\rangle/\langle\eta^2\rangle^2$ as function of fetch. Shown is a comparison between MCFW, Theory and experimental results from *Onorato et al.*, (2004).

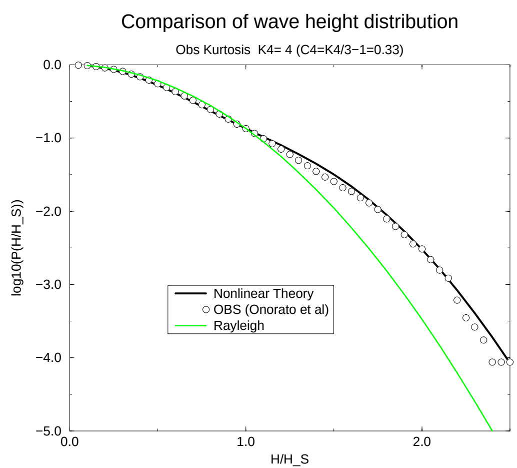

Figure 6. Comparison of theoretical and observed (*Onorato et al.*, 2004) wave height distribution. For reference, the linear Rayleigh result is shown as well.

a general expansion of the pdf in terms of cumulants (Edgeworth distribution) an extremely simple result. It was found that the probability that wave height exceeds $h \times H_S$ equals

$$P_H(h) = e^{-2h^2}\left[1 + C_4 B_H(h)\right], \tag{9}$$

where

$$B_H(h) = 2h^2\left(h^2 - 1\right). \tag{10}$$

This result verifies extremely well with observations, as can be seen from Figure 6.

In order to use a wave prediction system for extreme sea state prediction a reliable estimation of the BFI is required.

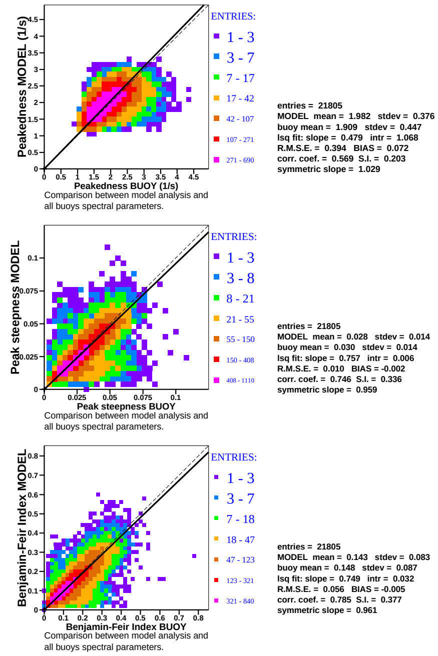

Figure 7. Comparison of modelled peakedness, integral steepness and Benjamin-Feir Index against buoy observations over the period October 2002 until April 2003.

This is possible by using a robust estimate of the width of observed spectrum (*Janssen and Bouws*, 1986). This is furnished by means of Goda's peakness factor Q_p defined as

$$Q_p = \frac{2}{m_0^2}\int d\omega\ \omega E^2(\omega) \tag{11}$$

As a consequence, the observed BFI becomes

$$BFI = k_0 m_0^{1/2} Q_p \sqrt{2\pi} \tag{12}$$

Figure 7 shows that the above procedure gives a robust estimate of the BFI. In addition, compared to buoy data, the ECMWF wave model shows good skill in estimating extreme sea states.

Operational implementation

At ECMWF we have implemented since October 2003 the following scheme:

- From the predicted wave spectrum we infer the B.F. Index.
- From the B.F Index we obtain the deviations from the Normal distribution, e.g. as measured by the kurtosis.
- Given the kurtosis and the significant wave height, we are able to answer question such as what is the enhanced probability on extreme events.

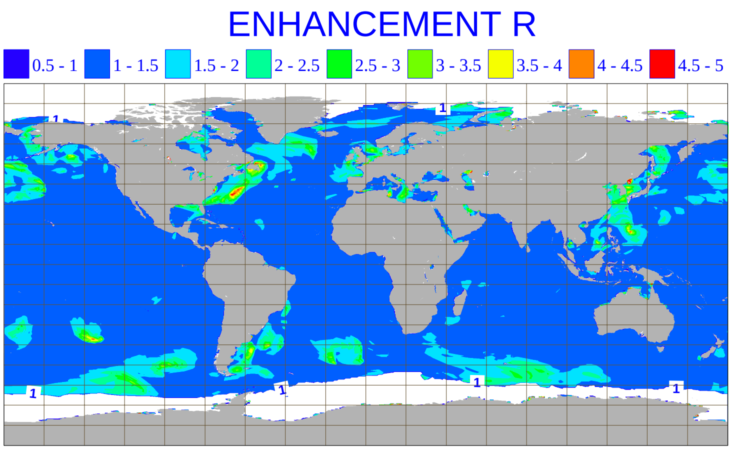

Figure 8. Enhancement of extreme events caused by nonlinear focussing for the one-day forecast from 21 October 2003.

> Probability that wave height exceeds $h \times H_S$ is then given by Eq. (9)

In Figure 8 we have given an example of output of a possible freak wave warning system. Here, we plotted the enhancement in the probability of occurrence of extreme sea states owing to nonlinear interactions compared to linear waves. Note that according to a normally distributed sea state every 3000 waves there is at least one wave that has a wave height larger than twice the significant wave height. For this example we see that in areas such as the East coast of the USA it is 5 times more likely that extreme events occur, signalling a warning.

Conclusions

From our work we have reached the following conclusions:

- Recent observations of freak waves have confirmed the theoretical picture of freak waves generation that actually already existed in the mid 1960's (*Benjamin and Feir*, 1967)!

- Using well-established methods one can, for given average sea state, obtain estimates of the enhanced probability of extreme events. The theoretical results are confirmed by laboratory results. Hence, in this sense, freak wave prediction is feasible.

- However, validation of all this in the field is of course desirable.

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