Pathways for advective transport

A. D. Kirwan, Jr., M. Toner, and B. L. Lipphardt, Jr.

University of Delaware, Newark, Delaware-USA

Abstract. Advective transport in the ocean fundamentally is Lagrangian in nature. At model grid scales, this transport is parameterized by eddy or turbulent exchange coefficients in analogy to molecular diffusion. Nevertheless, much of the transport is by advection on scales larger than the grid cell yet smaller than basin scale. Moreover, flow at these scales tends to be nonstationary in time and irregularly distributed in space. Recently we have successfully used concepts from dynamical systems theory to analyze observations of such flows. The analysis is based on constructing, from synoptic observations of currents, material boundaries within the flow field that are not apparent in time varying Eulerian velocity fields. Here a short description of the approach is given and an application to the Gulf of Mexico is described where the analysis precisely identifies the boundaries of coherent vortical structures as well as pathways for advective transport.

Introduction

In a perceptive paper, *Eckart* (1948) developed a paradigm for ocean stirring and mixing in the ocean. He hypothesized this as occurring in three stages: the initial stage where coherent structures are formed, a stirring phase characterized by local strong gradients of fluid properties and localized but intense small scale advection, and a final stage in which small scale mixing obliterates the gradients and produces a uniform distribution of properties.

As Eckart's analysis is particularly pertinent to the results described here, it is appropriate to review his theory. The starting point is the diffusion equation for a substance C:

$$\partial C / \partial t + u_j \, \partial C / \partial x_j = K \, \partial^2 C / \partial x_j \partial x_j \qquad (1)$$

Here, the indicial notation in which repeated subscripts in a term are summed over the range of the spatial domain. The remaining terms are standard.

Taking the gradient of (1) yields

$$\partial \left(\partial C / \partial x_p \right) / \partial t + u_j \partial \left(\partial C / \partial x_p \right) / \partial x_j \\ + \partial u_j / \partial x_p \left(\partial C / \partial x_j \right) = K \partial^2 \left(\partial C / \partial x_p \right) / \partial x_j \partial x_j \qquad (2)$$

The last term on the LHS is sometimes referred to as "frozen-in." Frozen-in fields are not uncommon in hydrodynamics. The potential vorticity equation is one widely recognized example of a frozen-in field. Evolution equations for fields that include the frozen-in term change in time like a differential line element moving with the fluid. Moreover, these terms can be used to construct Lagrangian invariants for the basic field. Although frozen-in fields and associated Lagrangian invariants are of fundamental interest in the theory used below, they are not pursued here. Interested readers are referred to *Kuz'min* (1984) for details.

As is well known in theoretical hydrodynamics, the velocity gradient term in (1) can be divided into an isotropic tensor whose eigenvalue is the fluid divergence, a traceless and symmetric deviator (called the velocity strain rate or deformation), and a skew-symmetric tensor called the spin, which is one half the vorticity. These quantities are denoted respectively as $\mathbf{\Delta_{jp}}, \mathbf{\Phi_{jp}}, \mathbf{\Omega_{jp}}$. $\mathbf{\Delta_{ij}}$ accounts for the change in volume of a fluid parcel with no change in its shape or orientation, $\mathbf{\Phi_{jp}}$ describes the change of shape of the parcel without change in volume or orientation, and $\mathbf{\Omega_{jp}}$ quantifies the changes in orientation with no shape or volume changes.

What effect does the frozen-in term have on uncertainty in the gradients of C? To examine this, multiply (2) by $\partial C / \partial x_p$ and take the ensemble average. It is seen that the vorticity term sums to zero so that the evolution equation for the variance of the gradient C reduces to

$$\partial < \partial C/\partial x_p \; \partial C/\partial x_p > /\partial t +$$

$$u_j \partial \langle \; \partial C/\partial x_p \; \partial C/\partial x_p \rangle /\partial x_j +$$

$$2\Delta \langle \left(\partial C/\partial x_p \right)\left(\partial C/\partial x_p \right) \rangle +$$

$$2\Phi_{jp} \left\langle \left(\partial C/\partial x_p \right)\left(\partial C/\partial x_j \right) \right\rangle = \tag{3}$$

$$K\partial^2 \left\langle \partial C/\partial x_p \; \partial C/\partial x_p \right\rangle /\partial x_j \partial x_j \; -$$

$$2K \left\langle \left(\partial^2 C/\partial x_p \; \partial x_j \right)\left(\partial^2 C/\partial x_p \; \partial x_j \right) \right\rangle$$

In (3) Δ is 1/3 of the divergence of the velocity field and the angled brackets are the ensemble average operator. The noteworthy item about (3) is that the vorticity Ω does not effect the evolution of the variance of the gradient, even though it is the largest of the gradient terms. The physical explanation is that this merely reorients the variance but does not change its value. By far, the dominant frozen-in term in (3) involves the deformation Φ.

The heuristic picture that emerges from this is that the vorticity is the dominant frozen-in term in (2) while the deformation is critical in (3). Thus, in the Eckart picture, the vorticity is important in large-scale circulation effects on the gradient (stirring) while the fluid deformation is important in the final stage of mixing.

Here some ideas from dynamical systems theory are applied to quantify the Eckart paradigm. The next section reviews the requisite material from dynamical systems. This is applied to a reduced gravity primitive equation model in section 3 and then to interactions of rings and the Loop Current in the Gulf of Mexico.

Review of dynamical systems theory

The analysis focuses on those regions in the flow field that are dominated by fluid deformation and not vorticity. Generally these are small and ephemeral. As deformation dominates in these regions, one expects particle trajectories to move exponentially fast away from these regions or approach a critical point in the flow field. This is generally the case, but under appropriate conditions there is a special or distinguished trajectory that does neither. In the dynamical systems literature, such trajectories are called distinguished hyperbolic trajectories or DHTs. DHTs possess an important flow property, they are the seeds for the material surfaces, or manifolds in the dynamical systems literature, that define the barriers and pathways to advective transport.

Identifying the DHT and constructing the manifolds is still at a trial and error state. It is hoped that improvements in this phase of the analysis will soon improve this phase of the analysis so no attention is given here to this matter. Interested readers should consult *Guckenheimer and Holmes* (1983) or *Wiggins* (1990).

The essential ideas are contained in an example due to Ide (2000). Consider a flow field given by

$$dx/dt = \left(y - \sin t \right)/2$$
$$dy/dt = \left(x - 3\cos t \right)/2 \tag{4}$$

The particle paths resulting from this flow are

$$x = \left(X-1 \right)\cosh\left(t/2 \right) + Y\sinh\left(t/2 \right) + \cos t$$
$$y = \left(X-1 \right)\sinh\left(t/2 \right) + Y\cosh\left(t/2 \right) - \sin t \tag{5}$$

Here (X, Y) are the starting positions. Consider first the particle that starts at X=1, Y=0. The trajectory of this particle is a circle. This trajectory differs from all other particle trajectories, which exhibit exponential time behavior. For example, particles starting along the line y=1-x will flow towards the DHT as it executes its circular trajectory. All other particles will flow away from the DHT along the line y=x-1. The lines $y = \pm (x-1)$ define the inflowing and outflowing manifolds for this example.

It is important to note that particles cannot cross the quadrants delineated by the eigen-directions since they are asymptotes for the trajectories. Thus, the curves along the eigen-directions are the manifolds. The inflowing manifold is along the northwest-southeast direction and the outflowing manifold is along the other eigen-direction. Although the deformation is everywhere constant, locally the manifolds emanate from the DHT and so they move in space and time.

In natural settings one expects a multitude of DHT within a domain. Thus, one could expect a complicated web of intersecting inflowing and outflowing manifolds associated with different DHT. Such intersections are called principal intersection points or PIP. Particles flowing along an outflowing manifold will transfer to the inflowing manifold at a PIP. Examples of this are demonstrated below.

Manifolds in a reduced gravity primitive equation model

Here a few examples of DHT and manifolds are given for a reduced gravity primitive equation model. The model domain is 2000 km on a side and the forcing is by a meridianal wind stress. See *Poje* and *Haller* (1999) for details.

Figure 1 shows the flow field with several manifolds and DHT. Note that the two eddies north of the jet are connected by manifolds. This implies some mass exchange between these structures. Also note the manifolds bounding the jet. These indicate that there is no advective transport across the jet in those regions. In fact, the only communication between the waters on either side of the

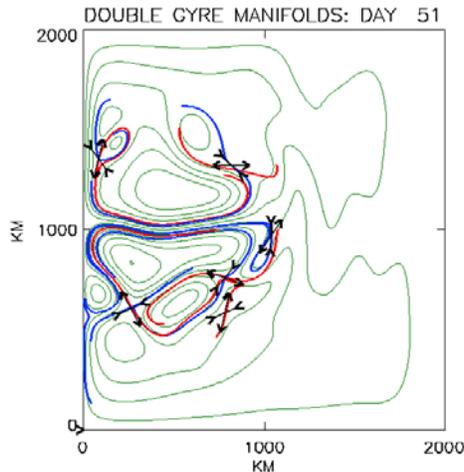jet are through manifolds that develop when eddies are shed.

Figure 1. Flow field with several manifolds and DHT.

Figure 2 depicts an example of the power of manifolds to channel advective stirring. Two blobs are started in about 50 kilometers apart. The one just to the east is initialized around a DHT so it is expected to be stretched along the outflowing manifold. The other blob is initialized near a principal intersection point or the intersection of an inflowing manifold of one DHT and the outflowing manifold of another DHT. It is attracted nearly equally to both and so the stretching effects are canceled. The result is that this blob is advected parallel to the outflowing manifold but with negligible stretching.
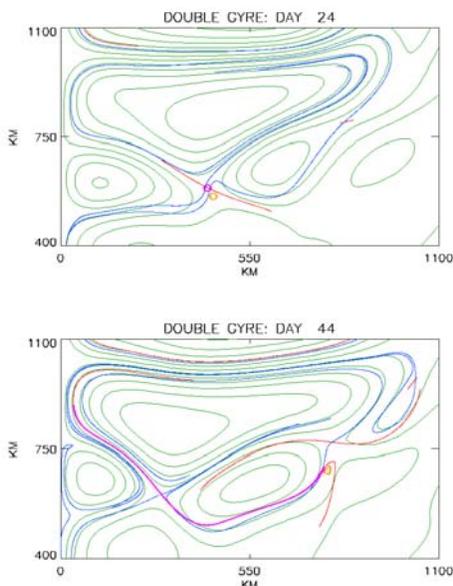

Figure 2. Manifolds that channel advective stirring.

Interactions of rings with the Loop Current

The purpose of this section is to apply the dynamical systems methods illustrated above to ocean data. To do this we use the University of Colorado model of the Gulf of Mexico. This is a 24-layer primitive equation model with 1/12 degree resolution, realistic bottom topography, and modified Mellor Yamada turbulence closure. It also is run in a predictive mode assimilating altimeter data, ABTs, and NOGAPS winds. For more details on the model and an assessment of its performance see *Toner et al* (2000).

Figure 3 shows a time slice of the model sea surface height and currents at 50 meters. Attention is directed to the eastern portion of the model domain where the Yucatan Current enters the Gulf and makes a large loop before egressing through the Straits of Florida. To the west of the Loop Current is a large anticyclonic ring that was shed several months before, and to the east is the Tortugas Eddy, a large semi-permanent cyclone sitting near the western most Florida Key. *Kutnetzkov et al* (2000) provide more details of the flow field.
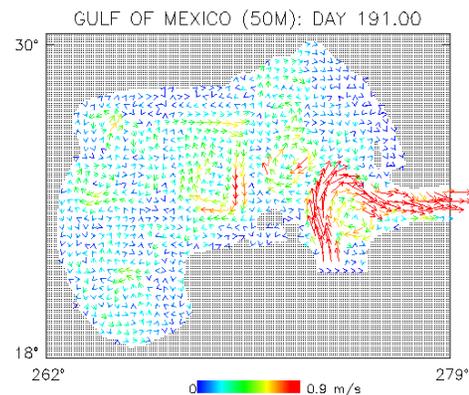

Figure 3. Time slice of the model sea surface height and currents at 50 m.

Four DHTs were located in this region and inflowing and outflowing manifolds were calculated for each. These are shown in Figures 4 and 5. It is important to note that the manifolds provide precise (but not necessarily accurate) delineations of the boundaries of the coherent structures noted above. Since they represent boundaries to advective transport the manifolds provide a quantitative measure of the size and shape of these structures.
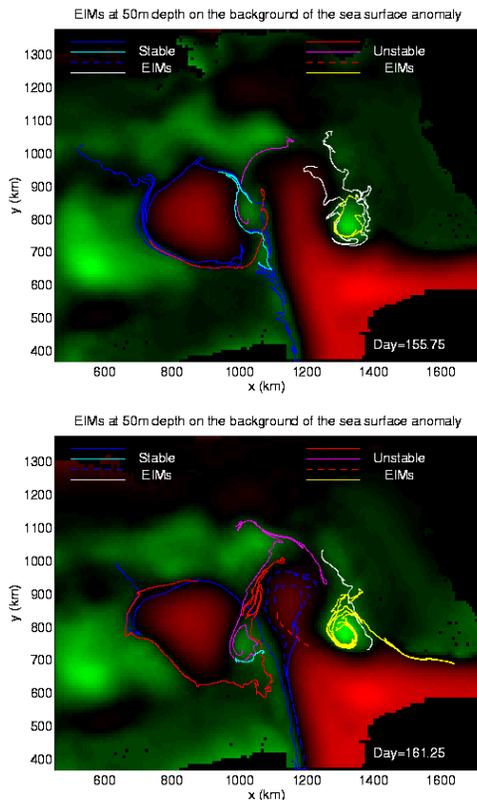
Figure 4. Inflowing and outflowing manifolds calculated for four DHTs.
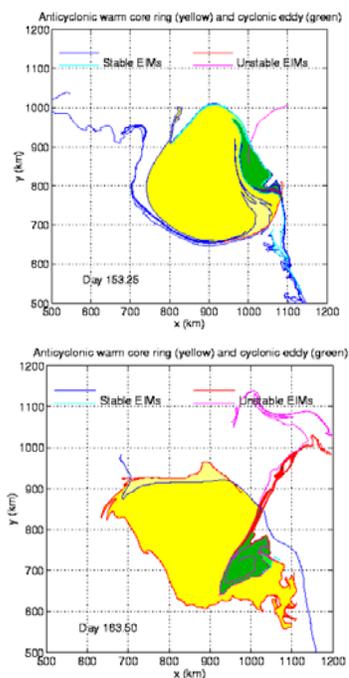
Figure 5. Evolution of a small cyclone originally to the northeast of the large anticyclonic ring.

Figure 5 shows the evolution of a small cyclone originally to the northeast of the large anticyclonic ring. As seen in this figure the cyclone moves anticyclonically around the ring and actually cuts off approximately 20%. This phenomenon was missed in the original examination of the Eulerian model velocities and height fields. The Lagrangian analysis used here highlighted this event.

Summary and discussion

This analysis is based on concepts from dynamical systems theory. The basic ingredients are hyperbolic points or saddle points and DHTs. The former is characterized by real eigenvalues of the velocity gradient at stagnation points. They have special significance in this approach in that they often signify a nearby DHT. These latter objects serve as seeds for calculating inflowing and outflowing manifolds or material boundaries that determine the advective transport pathways through a maze of coherent flow structures. In complex flows, such as the Loop Current region of the Gulf of Mexico, manifolds connect regions of intense mixing. In the language of *Eckart* (1948), the manifolds delineate stirring regimes and connect regions of intense mixing.

These pathways have now been found in simple time dependent QG and primitive equation models and in data assimilative GCMs (*Samelson*, 1992). Additionally, these techniques have been applied to merged drifters, moorings and MODAS data set on the LATEX Shelf (*Schulz*, 1999). It seems then they are telling us that stirring and mixing in the ocean may be highly time- and space-dependent and are governed by relatively simple processes determined by Lagrangian analysis. However, quantification of the importance of manifolds and DHTs to stirring and mixing, relative to traditional exchange processes, will have to await extension of these methods to three dimensions.

A comment on the stretching of blobs when they approach a DHT along an inflowing manifold is appropriate. The analysis indicates that a blob may be stretched hundreds of kilometers along the outflowing manifold, a purely advective process. We know of no observational evidence for this. Rather, it seems more likely that the blob undergoes intense small scale mixing and may lose its identity. But, both the reduced gravity primitive equation model and the assimilative model of the Gulf of Mexico include sophisticated and well-tuned mixing parameterizations. That the models are unable to predict the likely evolution of the blob in near a DHT suggests a deficiency in the present generation of mixing parameterizations.

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