

# Vertical shear plus horizontal stretching as a route to mixing

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**Abstract.** The combined effect of vertical shear and horizontal stretching leads to thin, sloping structures in tracer fields, whose vertical length scale is much smaller than their horizontal length scale. These structures are then vulnerable to vertical mixing processes. This effect needs to be taken into account when interpreting the horizontal structure of oceanic (and atmospheric) tracers and may explain recent oceanographic observations.

## 1. Introduction

In both atmosphere and ocean it is useful to divide velocity fields into isentropic or isopycnal parts, which move fluid parcels along potential temperature or potential density surfaces, and diabatic or diapycnal parts, which move fluid parcels across such surfaces. This is because diabatic or diapycnal motion can be accomplished only by non-conservative physical processes such as radiative heating (in the atmosphere) and molecular diffusion of temperature enhanced by three-dimensional turbulence (in atmosphere and ocean). In large parts of the atmosphere and ocean, such processes are weak in some average sense, so that diabatic or diapycnal velocities must be much smaller than isentropic or isopycnal velocities.

The process of stirring may therefore be regarded as layerwise two-dimensional, in the sense that differential advection by the quasi-horizontal isentropic/isopycnal velocity field acts independently on each isentropic or isopycnal surface to distort tracer fields into complex geometrical configurations. If atmospheric and oceanic flows were purely two-dimensional, similar to many numerical simulations or laboratory experiments, then stirring would simply strengthen horizontal spatial gradients of tracer fields until horizontal diffusion became competitive with advection and mixing occurred. But in layerwise two-dimensional atmospheric and oceanic flows, horizontal advection that varies in the vertical may also act to increase vertical gradients, thereby enhancing the effects of vertical diffusion, or more complicated vertical mixing processes. The fact that vertical scales are almost always observed to be considerably smaller than horizontal scales suggests that mixing diffusion rather than horizontal mixing is the dominant process.

Most previous theoretical investigations of stirring

and mixing of tracers have focussed on models of quasi-isotropic flows, either in two or three dimensions. The considerations above suggest that the most relevant model for the atmosphere and the ocean is an anisotropic model in which the vertical velocity is zero, but the horizontal velocities depend on the vertical coordinate. *Haynes and Anglade* (1997, hereafter HA97) used such an anisotropic model to study the enhancement of vertical gradients by horizontal stirring plus vertical shear. This is a relatively simple extension to previous work, but essential if theoretical results are to be relevant to real flows. The HA97 work was motivated primarily by the stratosphere. This article reviews and expands on some of the arguments in that paper, focussing in particular on the implications for stirring and mixing in the ocean.

The model to be presented is based on the paradigm of ‘chaotic advection’, that flows that are smoothly varying in space and time may lead to patterns of advected tracer that are spatially highly complex. One of the requirements on a chaotic advection flow is that there is a useful division of scales between the smallest active scale in the advecting velocity field and the diffusive scale (or more generally scale at which genuine mixing processes act). The case of three-dimensional turbulence, for example, does not fall into this category, since the velocity field has complex spatial structure at scales down to the diffusive scale and it may be argued that the stirring of tracer features at any scale larger than the diffusive scale is dominated by the flow at the same scale (rather than at the large scale). In fact the chaotic advection paradigm is relevant in three-dimensionally turbulent flows for tracers with diffusivity much less than the momentum diffusivity (large Schmidt number). The tracer field may then have non-trivial structure on a spatial scale smaller than the Kolmogorov scale, with the flow at the Kolmogorov

scale providing the ‘large-scale’ stirring. This is often called the Batchelor regime of turbulence and has been studied as an important special case in recent theoretical work on tracer fields in turbulent flows (e.g. see *Shraiman and Siggia, 2000* and references therein).

In the stratosphere it seems to be useful to consider tracer fields to be determined by the stirring effect of the large-scale flow (varying on spatial scales of hundreds to thousands of kilometers) and by the mixing effect of localised patches of three-dimensional turbulence, with vertical scales of a hundred meters or so and horizontal scales of a few kilometers. There is no strong evidence of active stirring by eddies on scales from a few hundred down to tens of kilometers, and in this range of scales the chaotic advection paradigm is therefore relevant.

Similarly in the ocean one might argue that mesoscale eddies with scales of tens of kilometers are the dominant part of the flow in stirring the tracer field at scales from a few tens of kilometers down to the scale of small-scale mixing events.

The structure of the paper is as follows. Section 2 describes a simple model problem of steady flow including both horizontal stretching and vertical shear, focussing on the spreading of a tracer from a point release in such a flow. Section 3 discusses some aspects of tracer evolution in more general random flows. Section 4 considers application of these ideas to the observations of *Ledwell et al. (1993)* (hereafter L93). Section 5 gives a brief discussion of the possible competing role of mixing by double-diffusive intrusions. Section 6 describes some possible future lines of research.

## 2. Steady flow model

The simplification allowed by the chaotic advection paradigm is that there is a finite scale below which the flow may be considered as a linear function of the space coordinates, i.e. the velocity field may be expanded as a (multidimensional) Taylor series about some reference location and only the linear term in the series retained. The effects of small-scale mixing events may be represented by a diffusion term (but more sophisticated representations are also possible).

We follow HA97 in considering first a steady linear flow that has the two ingredients of horizontal strain,  $\Gamma$ , plus vertical shear,  $\Lambda$ . In much of the atmosphere and ocean it is relevant to consider the parameter regime  $\Lambda \gg \Gamma$ , i.e. vertical shear much larger than horizontal strain. This may be argued from the observed fact that horizontal length scales tend to be larger than vertical scales or by appealing to theoretical arguments that the ratio of horizontal to vertical scales is of order  $f/N$  (Prandtl’s ratio), where  $f$  is the Coriolis parameter and

$N$  the buoyancy frequency.  $f/N$  is  $O(100)$  in the atmosphere and in the oceanic thermocline and perhaps reduces to  $O(10)$  in the deep ocean.

The flow is taken to have components  $(\Gamma x, -\Gamma y + \Lambda z, 0)$  in the  $x$ ,  $y$  and  $z$  directions respectively, where  $(x, y, z)$  are Cartesian coordinates, with  $z$  vertical. The steadiness assumption implies that the horizontal strain  $\Gamma$  and the vertical shear  $\Lambda$  are constants. The equation for the evolution in this flow of a tracer with concentration  $\chi(x, y, z, t)$  is therefore

$$\frac{\partial \chi}{\partial t} + \Gamma x \frac{\partial \chi}{\partial x} + (\Lambda z - \Gamma y) \frac{\partial \chi}{\partial y} = \kappa \left( \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} + \frac{\partial^2 \chi}{\partial z^2} \right), \quad (1)$$

where  $\kappa$  is the diffusivity, assumed constant.

HA97 considered sinusoidal solutions of this equation. Another useful solution of this equation, and indeed of the equation for tracer concentration in any flow where velocity components are linear functions of space coordinates (including time-dependent flows), is an ellipsoidal Gaussian solution, i.e. an exponential function of a negative definite quadratic function of  $x$ ,  $y$  and  $z$ . This is conveniently obtained by considering the second-order spatial moments of the solution, such as  $m_{xx} = \int x^2 \chi dV / \int \chi dV$ ,  $m_{xy}$  etc. The six moments are sufficient to determine the quadratic function and the constraint that the total amount of tracer remains constant then determines the coefficient of the exponential function. In the above flow, the equations for the moments may be shown to be

$$\frac{d}{dt} m_{xx} = 2\Gamma m_{xx} + \kappa \quad (2)$$

$$\frac{d}{dt} m_{xy} = \Lambda m_{xz} \quad (3)$$

$$\frac{d}{dt} m_{xz} = \Gamma m_{xz} \quad (4)$$

$$\frac{d}{dt} m_{yy} = -2\Gamma m_{yy} + 2\Lambda m_{yz} + 2\kappa \quad (5)$$

$$\frac{d}{dt} m_{yz} = -\Gamma m_{yz} + \Lambda m_{zz} \quad (6)$$

$$\frac{d}{dt} m_{zz} = \kappa \quad (7)$$

These equations may be solved straightforwardly. For brevity we consider the case where all moments tend to zero as  $t \rightarrow 0^+$ , corresponding to a point release. Then the solutions are

$$m_{xx} = \frac{\kappa}{\Gamma} e^{2\Gamma t} \quad (8)$$

$$m_{xy} = m_{xz} = 0 \quad (9)$$

$$m_{yy} = \frac{\kappa\Lambda^2}{\Gamma^3} \{2\Gamma t + 4e^{-\Gamma t} - e^{-2\Gamma t} - 3\} + \frac{\kappa}{\Gamma} \{1 - e^{-2\Gamma t}\} \quad (10)$$

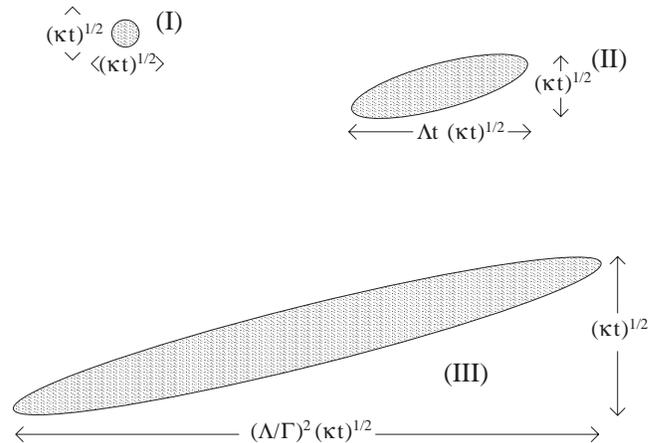
$$m_{yz} = 2\frac{\kappa\Lambda}{\Gamma^2} \{\Gamma t - 1 + e^{-\Gamma t}\} \quad (11)$$

$$m_{zz} = 2\kappa t \quad (12)$$

First consider the case where  $\Lambda = 0$ , i.e. the case of purely two-dimensional flow. Then  $m_{xx} = \kappa(e^{2\Gamma t} - 1)/\Gamma$  and  $m_{yy} = \kappa(1 - e^{-2\Gamma t})/\Gamma$ . For small times,  $t \ll \Gamma^{-1}$ , diffusion dominates and both  $m_{xx}$  and  $m_{yy}$  increase linearly with time. When  $t \sim \Gamma^{-1}$  advection by the horizontal strain flow becomes competitive with diffusion. The tracer elongates in the direction of the stretching axis, so that for  $t \gg \Gamma^{-1}$ ,  $m_{xx} \sim \kappa e^{2\Gamma t}/\Gamma$ , whilst in the direction of the compression axis a steady-state balance between strain and diffusion is achieved, with  $m_{yy} \sim \kappa/\Gamma$ . At all times the only vertical transport is through diffusion, so  $m_{zz}$  increases linearly with  $t$  at all times.

What is the effect of adding vertical shear, i.e. taking  $\Lambda \neq 0$  (with  $\Lambda \gg \Gamma$ )? The elongation of the tracer along the stretching axis of the horizontal strain axis is unchanged. However, the spreading of the tracer in the direction of the compression axis is rather different. Examination of the expression for  $m_{yy}$  above shows that there are three regimes. For  $t \lesssim \Lambda^{-1}$  (regime I) there is purely diffusive spreading as above. For  $\Lambda^{-1} \lesssim t \lesssim \Gamma^{-1}$  (regime II) the diffusive spreading of the tracer in the vertical allows the vertical shear to augment the spreading in the horizontal, so that  $m_{yy} \sim \kappa\Lambda^2 t^3$ . Finally when  $t \sim \Gamma^{-1}$  horizontal advection begins to inhibit the horizontal spreading, so that for  $t \gtrsim \Gamma^{-1}$  (regime III),  $m_{yy} \sim \kappa t \Lambda^2 / \Gamma^2$ . These three regimes are depicted graphically in Figure 1. Note that  $m_{yy}$  continues to increase with time, in contrast to the case with  $\Lambda = 0$ . In regime III  $m_{yy} \sim \Lambda m_{yz} / \Gamma \sim \Lambda^2 m_{zz} / \Gamma^2$  suggesting that the  $y$  and  $z$  scales of the tracer patch are in the ratio  $\Lambda/\Gamma$ . Indeed what happens is that the tracer patch forms a sheet that slopes at angle  $\Gamma/\Lambda$  (from the horizontal) in the  $(y, z)$  plane. The extent of the sheet in the vertical direction is  $(\kappa t)^{1/2}$ , therefore the extent of the sheet in the horizontal ( $y$ ) direction is  $\Lambda(\kappa t)^{1/2} / \Gamma$ . This explains the dominant behaviour of  $m_{yy}$ ,  $m_{yz}$  and  $m_{zz}$ . In order to deduce the spread of the tracer on a given horizontal surface it is useful to consider  $m_{yy} - 2\Lambda m_{yz} / \Gamma + m_{zz} \Lambda^2 / \Gamma^2$ , i.e. the moment of  $(y - \Lambda z / \Gamma)^2$ . It is straightforward to show that

$$m_{yy} - 2\Lambda m_{yz} / \Gamma + m_{zz} \Lambda^2 / \Gamma^2 = \frac{\kappa}{\Gamma} \left(1 + \frac{\Lambda^2}{\Gamma^2}\right) (1 - e^{-2\Gamma t}) \simeq \frac{\kappa}{\Gamma} \frac{\Lambda^2}{\Gamma^2} (1 - e^{-2\Gamma t}). \quad (13)$$



**Figure 1.** The spread of tracer in the  $(y, z)$  in each of regimes I ( $t \lesssim \Lambda^{-1}$ ), regime II ( $\Lambda^{-1} \lesssim t \lesssim \Gamma^{-1}$ ) and regime III ( $\Gamma^{-1} \lesssim t$ ).

Thus the spread of the tracer on a given horizontal surface is  $(\kappa/\Gamma)^{1/2} \Lambda/\Gamma$ . The effect of the vertical shear is therefore to increase the equilibrium width of the tracer patch by a factor  $\Lambda/\Gamma$ .

In summary, the effect of the vertical shear is to lead to sloping structures in the tracer field with aspect ratio (ratio of horizontal length scale to vertical length scale)  $\alpha = \Lambda/\Gamma$ . The equilibrium width of these structures is larger by a factor  $\alpha$  than would be the case without vertical shear. One might say that it is *as if* a horizontal diffusivity  $\kappa\alpha^2$  were acting.

### 3. Time-dependent flow models

In a flow that varies in space and time, the velocity gradient encountered by a fluid parcel changes with time as the particle moves through the flow. The evolution of small-scale tracer features is therefore governed by the equation for tracer evolution in a time-dependent linear flow. In the study of turbulence there is a long tradition of considering the effect of a linear flow that varies randomly in time (using so-called ‘random-straining’ models). This approach has been used most recently by various authors to give a rather complete description of the statistics of tracer fields in Batchelor-regime turbulence. [See *Balkovsky and Fouxon*, 1999 and *Falkovich et al.*, 2001 for further details of this work.]

The random-straining model most relevant to the atmosphere and ocean is non-standard in that the statistics of the velocity gradient tensor is not isotropic (since the vertical velocity is zero, but the vertical gradient of the horizontal velocity is not). HA97 considered such random-straining models and showed that the important predictions of the steady flow model presented in

Section 2 carried over, in the sense that the tracer field was predicted to form sloping sheets, with an aspect ratio  $\alpha$  that depended on the statistics of the horizontal strain and vertical shear fields.

We review some of the HA97 results below, emphasising the dependence of  $\alpha$  on the different flow parameters. One particular goal is to identify conditions under which  $\alpha$  is not simply equal to the aspect ratio  $\Lambda/\Gamma$  of the flow itself.

It is convenient to consider the normalised gradient of the tracer field, defined by  $\mathbf{k}(t) = \chi^{-1}\nabla\chi$ , evaluated at the position  $\mathbf{x} = \mathbf{X}(t)$  following a fluid parcel.  $\mathbf{k}$  might be considered as a local wavenumber of the tracer field in the case where the tracer field varies rapidly in space compared to the velocity field. It may be shown from the gradient of the tracer advection equation that

$$\frac{d\mathbf{k}}{dt} = -(\nabla\mathbf{u})\cdot\mathbf{k} \quad (14)$$

where the scalar product on the right-hand side applies to the second index of the tensor  $\nabla\mathbf{u}$ . The evolution of  $\mathbf{k}$  therefore depends on the time series of  $\nabla\mathbf{u}$  (encountered following the point  $\mathbf{X}(t)$ ). In a random-straining model the effect of turbulent flow on gradients of tracer (or line elements) is considered as a kinematical problem, in which the tensor  $\nabla\mathbf{u}$  is taken to be given by a random time series, without addressing questions of the flow dynamics.

For the layerwise two-dimensional flows of interest here, the matrix  $W$  of components of the velocity gradient tensor, with  $W_{ij} = \partial u_i/\partial x_j$ , has  $W_{31} = W_{32} = W_{33} = 0$ . (The index 3 is taken to correspond to the  $z$  coordinate.) It is useful to write  $\mathbf{k} = (\mathbf{k}^{(h)}, m)$ , with  $\mathbf{k}^{(h)}$  the horizontal wavenumber and  $m$  the vertical wavenumber, and define  $\mathbf{W}^{(h)}$  to be the 2nd rank tensor with components  $W_{11}, W_{21}, W_{12}$  and  $W_{22}$ , and  $\mathbf{\Lambda}$  to be the vector with components  $(W_{13}, W_{23})$ . It then follows that

$$\frac{dk_i^{(h)}}{dt} = -W_{ji}^{(h)}k_j^{(h)} \quad (15)$$

and

$$\frac{dm}{dt} = -W_{j3}k_j^{(h)} = -\Lambda_j k_j^{(h)} \quad (16)$$

In what follows the summation convention is used for repeated suffices and the suffices run through the values  $\{1, 2\}$ . Equation (15) is just that for the tracer wavenumber in two-dimensional (horizontal) flow. Note that it may be solved independently of (16). Thus variation in the  $z$ -direction makes no difference to the evolution of the horizontal wavenumber vector. However (16) shows that the vertical wavenumber evolves through the vertical shear (in the horizontal flow) acting on the horizontal wavenumber vector.

To formulate a suitable random-straining model we assume that each of the six non-zero components of the matrix  $W$  is a realisation of a random function of time, with average (over all realisations) zero. We also assume that the wavenumber at  $t = 0$  is randomly chosen and is statistically independent of the  $W_{ij}$ . Equation (16) may be integrated to give

$$m(t) = m(0) + \int_0^t \Lambda_j(t')k_j^{(h)}(t')dt' \quad (17)$$

and then squaring, and taking the ensemble average, it follows that

$$\langle m(t)^2 \rangle = \langle m(0)^2 \rangle + \int_0^t dt' \int_0^t dt'' \langle \Lambda_j(t')k_j^{(h)}(t')\Lambda_k(t'')k_k^{(h)}(t'') \rangle, \quad (18)$$

where  $\langle \cdot \rangle$  denotes the ensemble average, over all realisations of the flow field and all realisations of the initial wavenumber vector.

Useful quantitative estimates are possible if a number of further assumptions are made concerning the statistics of the velocity gradient tensor. The first is that the components of  $\mathbf{\Lambda}$  are independent of all components of  $\mathbf{W}^{(h)}$ , i.e. that the vertical shear is statistically independent of the horizontal deformation. It follows that  $\mathbf{\Lambda}$  is also independent of  $\mathbf{k}^{(h)}$  so that averages of terms involving  $\mathbf{\Lambda}$  and terms involving  $\mathbf{k}^{(h)}$  may be taken separately in (18). The second important assumption is stationarity, from which it follows  $|\langle \Lambda_j(t')\Lambda_k(t'') \rangle| \sim \Lambda^2 g(\sigma_\Lambda |t' - t''|)$ , for some function  $g$ , where  $\sigma_\Lambda$  is an inverse correlation time for the vertical shear and  $\Lambda^2 = \langle \mathbf{\Lambda}^2 \rangle$ .

In order to estimate the integral appearing in (18) it is also necessary to have information on the evolution of the horizontal wavenumber. This follows exactly as in the random straining models of isotropic two-dimensional flow considered by *Kraichnan* (1974) and others. The basic predictions of these theories are that the horizontal wavenumber increases exponentially in time, at a rate,  $S$  say, governed by the statistics of the horizontal strain field and depending in particular on the root mean square rate of strain,  $\Gamma$  say, and on the inverse correlation time  $\sigma_h$  for the horizontal strain field. Detailed analysis of explicit models (*Kraichnan*, 1974; *Chertkov et al.*, 1995; HA97) suggests that  $S \sim \Gamma \min(\Gamma/\sigma_h, 1)$ .

Again, making suitable stationarity assumptions, this suggests that  $|\langle k_i^{(h)}(t')k_j^{(h)}(t'') \rangle| \sim k_0^2 e^{S(t'+t'')} h(\mu|t' - t''|)$ , where  $k_0^2 = \langle \mathbf{k}^{(h)}(0)^2 \rangle$ . The inverse time scale  $\mu$  and the precise form of the function  $h(\cdot)$ , depend on the statistics of the horizontal strain field, but explicit calculation in various models suggests that  $\mu \sim S$ .

Substituting these estimates into (18) it follows that, for large times, when the second term on the right-hand side dominates the first,

$$\langle m(t)^2 \rangle \sim \frac{\Lambda^2 k_0^2 e^{2St}}{S^2} \min\left\{1, \frac{S}{\sigma_\Lambda}\right\}. \quad (19)$$

Note that the dominant contribution to the first integral, over  $t''$ , performed in (18) comes from a neighbourhood of  $t'' = t$  of size  $\min\{S^{-1}, \sigma_\Lambda^{-1}\}$  and that to the second integral, over  $t'$ , comes from a neighbourhood of  $t' = t$  of size  $S^{-1}$ .

The ratio  $\alpha$  of horizontal length scale to vertical length scale, or, equivalently, the ratio of vertical wavenumber to horizontal wavenumber may therefore be estimated as

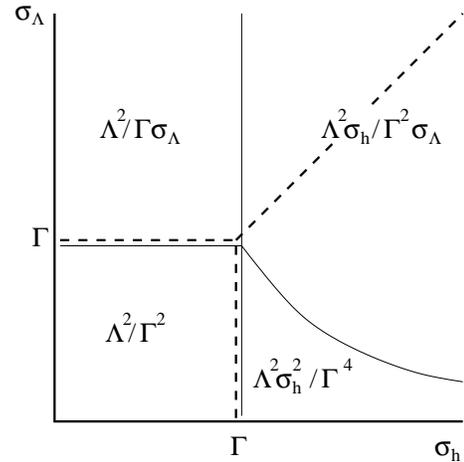
$$\begin{aligned} \alpha^2 &\sim \frac{\langle m(t)^2 \rangle}{\langle \mathbf{k}^{(h)}(t)^2 \rangle} \sim \frac{\Lambda^2}{S^2} \min\left\{1, \frac{S}{\sigma_\Lambda}\right\} \\ &\sim \frac{\Lambda^2}{\Gamma^2} \max\left(1, \frac{\sigma_h^2}{\Gamma^2}\right) \min\left\{1, \frac{\Gamma}{\sigma_\Lambda} \min\left(1, \frac{\Gamma}{\sigma_h}\right)\right\}. \end{aligned} \quad (20)$$

with the last estimate following from the estimate for  $S$ .

This semi-quantitative analysis suggests that the vertical wavenumber increases exponentially at the same rate as the horizontal wavenumber, and that the ratio  $\alpha$  is equal to the ratio of vertical shear to horizontal strain,  $\Lambda/\Gamma$ , multiplied by a number that depends on  $\sigma_h/\Gamma$  and  $\sigma_\Lambda/\Gamma$ . These predictions agree with explicit calculations based on suitably formulated random-straining models. (See HA97 for more details.)

Figure 2 summarises the variation of the aspect ratio  $\alpha$  with the parameters  $\sigma_h$  and  $\sigma_\Lambda$ . Note that  $\alpha$  is anomalously large (i.e. greater than  $\Lambda/\Gamma$ ) when the horizontal strain varies on a timescale that is shorter than both the inverse horizontal strain rate and the timescale of variation of the vertical shear. On the other hand,  $\alpha$  is anomalously small (i.e. less than  $\Lambda/\Gamma$ ), when the vertical shear varies on a timescale that is shorter than both the inverse horizontal strain rate and the timescale of variation of the horizontal strain. As might be expected, in the limit where  $\sigma_h \ll \Gamma$  and  $\sigma_\Lambda \ll \Gamma$  the estimate for  $\alpha$  agrees with that obtained for the steady flow considered in the previous section.

Note that in this section we have not considered the effects of diffusion explicitly. Nonetheless, it is plausible that diffusion acts in a similar way to that deduced from the steady flow model in Section 2, i.e. diffusion  $\kappa$  acts on structures in the tracer field that slope with aspect ratio  $\alpha$  and the result, e.g. in achieving a balance between diffusion and horizontal straining, is that it is as if there is a horizontal diffusivity with magnitude  $\kappa\alpha^2$ . This is supported (with certain limitations) by explicit calculations in *Vanneste and Haynes* (2001).



**Figure 2.** Scaling of  $\alpha^2$  (where  $\alpha$  is aspect ratio of horizontal to vertical scale) with parameters  $\Gamma$  (horizontal strain rate),  $\Lambda$  (vertical shear),  $\sigma_h$  (inverse correlation time for horizontal strain) and  $\sigma_\Lambda$  (inverse correlation time for vertical shear). The thin dashed lines delimit the regions where the different estimates hold. The thick dashed lines delimit the regions where  $\alpha$  scales as the aspect ratio of the velocity field ( $\Lambda/\Gamma$ ) (lower left-hand region), is larger than that aspect ratio (right-hand region) and is smaller than that aspect ratio (upper region).

#### 4. Application to the Ledwell et al. (1993) observations

In the tracer release experiment reported by L93 the dispersion of a tracer in the ocean thermocline was followed over a period of several months. The vertical (cross-isopycnal) diffusion could be estimated directly and a value of vertical diffusivity  $\kappa_v$  of about  $10^{-5} \text{m}^2 \text{s}^{-1}$  was inferred. In the horizontal the tracer was, in the later stages of the experiment, observed to be confined to thin streaks, whose width apparently reached an equilibrium value of about 3 km. This was interpreted as the stretching out of the tracer patch by the mesoscale eddy field, resulting in filaments of tracer whose width was the equilibrium value determined by a balance between horizontal stretching (which tends to reduce the width of the filament) and horizontal dispersive effects, perhaps associated with small-scale mixing processes (which tend to increase the width of the filament). From the observed length of the tracer filament it was estimated that the average stretching rate experienced was about  $3 \times 10^{-7} \text{s}^{-1}$ . It followed that if the horizontal dispersive effects could be represented by an effective horizontal diffusivity  $\kappa_h$ , it must have a value of about  $3 \text{m}^2 \text{s}^{-1}$  ( $\simeq (3 \text{ km})^2 \times 3 \times 10^{-7} \text{s}^{-1}$ ). Thus  $\kappa_h/\kappa_v \simeq 3 \times 10^5$ . Previous work by *Young et al.* (1982), taking account of the combined effect of vertical mixing and horizontal advection by inertio-gravity waves had

suggested  $\kappa_h/\kappa_v \simeq (N/f)^2 \simeq 1500$ , i.e. 200 times too small to account for the L93 observations.

HA97 noted that the combined effect of horizontal stretching and vertical shear could account for the value of  $\kappa_h/\kappa_v$  if the aspect ratio  $\alpha$  was about 500. According to the analysis presented in Section 3 this requires  $\Lambda \sim 1.5 \times 10^{-4} \text{s}^{-1}$ , based on the estimate  $\alpha \sim \Lambda/S$  (which holds if  $\sigma_\Lambda \lesssim S$ , i.e. the correlation time for the vertical shear must be larger than  $S^{-1}$ ). This value of vertical shear (equivalent to  $15 \text{ cm s}^{-1}$  per km) does not seem indefensible as realistic.

If the models described in this article are to be relevant to the L93 observations, one might ask further whether the observed morphology of the tracer is consistent with the model predictions. First it is clear that in a flow with finite length scales the exponential stretching in a single direction predicted by the model in Section 2 must break down when the largest length scale of the tracer patch (or, at this stage, tracer filaments) becomes comparable to the length scale of the flow. What happens is simply that the filaments meander on the length scale of the flow (as observed in countless laboratory and numerical experiments). Perhaps more crucial is the shape of tracer filaments in cross-section. The model of Section 2 predicts that this will ultimately be highly elongated, so that the tracer is actually concentrated within sloping sheets. However, this applies on time scales much greater than  $\Gamma^{-1}$ , where  $\Gamma$  is the strain rate and therefore, in the steady model, the stretching rate. For times comparable to  $\Gamma^{-1}$  (see Figure 1), the tracer distribution is not sheet-like, but it is simply the case that the filament cross section has much larger horizontal extent than vertical extent (by a factor  $\alpha$ ). The observing period reported by L93 is such that  $\Gamma t \lesssim 6$ , so that sheet-like features are not necessarily expected. [Note that the extent of the sheets is proportional to  $(\Gamma t)^{1/2}$ .]

## 5. Possible effects of double diffusion

*Garrett* (1982) (hereafter G82) noted that the stirring by mesoscale eddies of temperature and salinity along isopycnal surfaces would lead to large local gradients in these quantities manifested as thermohaline fronts. These in turn might lead to double-diffusive intrusions. G82 argued that the sharpening of gradients of temperature and salinity by stirring would be halted by the mixing effects of the intrusions when the growth rate of the intrusions was equal to the convergence acting on the fronts, i.e. to the stretching rate associated with the stirring process. He gave the formula (based on previous theoretical and experimental work)

$$\lambda_{max} = 0.075 \frac{g\beta S_x}{N} \quad (21)$$

for the maximum growth rate  $\lambda_{max}$  of the intrusions, where  $g$  is the gravitational acceleration,  $\beta = \rho^{-1}(\partial\rho/\partial S)_T$  is the rate of change, at constant temperature, of density with salinity, and  $S_x$  is the salinity gradient in the neighbourhood of the front. He argued that if the front had width  $L_f$  then the approximate value of  $S_x$  would be  $S_x = \tilde{S}_x L_{eddy}/L_f$  where  $\tilde{S}_x$  was the large-scale salinity gradient and  $L_{eddy}$  was the typical size of the mesoscale eddies. If  $s$  is the stretching rate then it follows that  $L_f$  is given by the formula

$$L_f = 0.075 \frac{g\beta S_x}{N} \frac{L_{eddy}}{s}. \quad (22)$$

For the values of  $s$  and  $L_f$  observed by L93 (taking  $L_f$  to be the width of the filaments) and assuming  $L_{eddy} \sim 100 \text{ km}$ , this would require  $g\beta S_x/N \simeq 10^{-7} \text{s}^{-1}$ . This is, in fact, considerably weaker than the values suggested by G82 as examples. This therefore appears to leave open the possibility that it is the mixing effects, along isopycnal surfaces, of double diffusive intrusions, rather than any process directly related to ‘background’ vertical mixing, that determines the width of the filaments observed by L93. On the other hand it is not at all clear that the filaments of a tracer injected at an arbitrary location will match the regions of enhanced gradients in temperature and salinity whose distribution is set on the large scale and therefore feel the same mixing effects.

## 6. Discussion

The work reported in HA97 may be extended in various ways, some of which are relevant to oceanographic considerations. For example, *Vanneste and Haynes* (2001) have considered the effect of vertical shear on the horizontal wavenumber spectrum of passive tracers, in particular on the range of scales where the effects of diffusive mixing become important. In particular this work highlights the limitations of estimating the effective horizontal diffusivity as  $\kappa\alpha^2$ . The recent theoretical work on Batchelor-regime turbulence has made explicit predictions about probability density functions for tracer concentrations, tracer concentration differences (over a finite distance) and tracer concentration gradients and this work needs to be extended to the layerwise two-dimensional case if it is to be applicable to real atmospheric and oceanic flows.

Much insight into atmospheric tracer distributions has been obtained by using models driven by observed velocity fields, or velocity fields extracted from quasi-realistic global models. Approaches have included full numerical integrations of the tracer evolution equation (e.g. on an isentropic surface), non-diffusive reconstructions of tracer fields by following back trajectories to an

initial condition and calculations of the statistical properties of large numbers of trajectories, e.g. to give the distribution of finite-time stretching rates. See, for example, the papers by *Schoeberl and Newman* (1995), *Ngan and Shepherd* (1999) and *Hu and Pierrehumbert* (2001). All of these methods have substantial savings over integrations of a full model including tracer equations and dynamical equations and some may be worth applying in the oceanic context. Velocity fields might be extracted from eddy-resolving models for this purpose, but as a first step it might be worth considering ‘synthetic’ velocity fields that are generated quite artificially to have a plausibly realistic spatial and temporal structure in space and time.

**Acknowledgments.** I am grateful to Chris Garrett for bringing possible double-diffusive effects to my attention.

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This preprint was prepared with AGU’s L<sup>A</sup>T<sub>E</sub>X macros v4, with the extension package ‘AGU++’ by P. W. Daly, version 1.6a from 1999/05/21.