Internal Solitary Waves in Lakes – a Closure Problem for Hydrostatic Models

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Abstract.
Laboratory experiments and field observations suggest that the degeneration of basin-scale internal standing waves into packets of solitary waves is an important mechanism for the transfer of energy within the internal wavefield in lakes. Since these shorter solitary waves break at the boundaries, this process provides a flux path from the wind to the turbulent benthic boundary layer. However, numerical models that make the hydrostatic approximation cannot simulate the evolution or propagation of internal solitary waves and therefore capture this important energy flux path. The feasibility of parameterising the important effects of solitary waves in hydrostatic models is discussed. Using a simple KdV model, some initial steps are suggested towards the development of such a solitary wave closure scheme.

Introduction

The internal wave field in a lake is energized by wind blowing over the surface, generating basin-scale waves or seiches. *Horn et al.* (1999) have shown that, for many lakes, an initial basin-scale seiche will steepen due to nonlinear effects until the steepening is balanced by dispersive effects, generating internal solitary waves. Packets of such solitary waves are frequently observed in lakes (*Thorpe et al.*, 1972; *Farmer,* 1978; *Hunkins and Fliegel,* 1973). The shoaling and breaking of these solitary waves is an important energy sink for the internal wave field and plays a major role in driving mixing in stratified lakes (*Imberger*, 1998). Importantly, the turbulent mixing events caused by the shoaling of internal solitary waves are local in both time and space. Unlike basin-scale waves, which gradually decay as they dissipate energy in the benthic boundary layer over the whole basin, internal solitary waves break on their first encounter with the sloping boundaries of a lake, dissipating most of their energy in a single breaking event (*Michallet and Ivey,* 1999).

Since the water quality and ecology of a lake are dependent on the vertical mixing of nutrients, oxygen and other biological agents, it is important that our models include this energy flux path: from the wind, to basin-scale internal waves, to shorter solitary waves, to turbulence in the benthic boundary layer (this is not to say that there are not other energy flux paths that result in mixing in the benthic boundary layer). Furthermore, it may not be sufficient to only reproduce the basin-wide rate of vertical mixing; it may be necessary to reproduce the spatial and temporal variability of mixing events, for it is these events that control many biogeochemical processes. It is likely that in many cases current hydrodynamic models of lakes do not correctly reproduce the cascade of energy from the basin-scale internal waves to solitary waves and hence do not reproduce the spacially and temporally local boundary mixing driven by the shoaling of these waves. The reasons for this shortcoming are twofold: (a) many hydrodynamic models of lakes make the hydrostatic assumption and therefore neglect the physics that allows the generation and propagation of solitary waves, and (b) in many cases the solitary waves would be sub-grid scale.

In this paper we address only the first of these shortcomings; we discuss the feasibility of parameterising the important effects of solitary waves in hydrostatic models. We make use of a simple inviscid two-layer Korteweg-de Vries model to describe the nonhydrostatic evolution of some simple initial conditions, comparing the evolution with that observed in an equivalent hydrostatic model. Some initial steps are suggested towards the development of a solitary wave closure scheme.

A KdV model of long internal waves

Hydrostatic models cannot reproduce the evolution and propagation of internal solitary waves because they neglect the vertical accelerations that generate these waves. The simplest model to include both weak nonlinearity and dispersion is the Korteweg-de Vries (KdV)

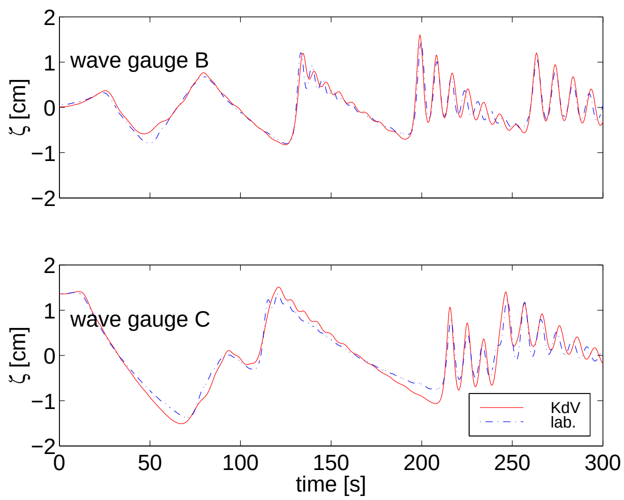

Figure 1. A comparison of the KdV model with laboratory experiment (from *Horn et al.*, 1999a).

equation (for a general review of the KdV equation see *Miles*, 1981). The KdV equation describes the evolution of weakly nonlinear long waves while assuming that nonlinear effects enter the equation at the same order as dispersive effects; solitary waves evolve when nonlinear effects are balanced by dispersive effects. In an inviscid two-layer system consisting of a fluid of depth h_1 and density ρ_1 overlying a fluid of depth h_2 and density ρ_2, and making the Boussinesq approximation, the KdV equation can be written as

$$\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0 \qquad (1)$$

where $\eta(x,t)$ is the interface displacement, $c_o = (g'\frac{h_1h_2}{h_1+h_2})^{1/2}$ is the linear long-wave speed, $\alpha = \frac{3}{2}c_o(h_1-h_2)/h_1h_2$ and $\beta = \frac{1}{6}c_oh_1h_2$. The solitary wave solution to (1) is given by

$$\eta(x,t) = a\,\mathrm{sech}^2\left(\frac{x-ct}{\lambda}\right) \qquad (2)$$

where a is the solitary wave amplitude, $\lambda = (\frac{12}{a}\frac{\beta}{\alpha})^{1/2}$ is the characteristic wavelength and $c = c_o + \frac{a\alpha}{3}$ is the solitary wave speed.

The KdV equation has been applied to observations of internal waves in lakes (e.g., *Hunkins and Fliegel*, 1973) and has been shown to agree well with experimental data (e.g., *Koop and Butler*, 1981; *Horn et al.*, 1999a). Figure 1 shows the very close agreement between the pseudo-spectral KdV model of *Horn et al.* (1999a) and a laboratory experiment designed to investigate the nonlinear steepening of standing internal waves. Given this close agreement between the KdV model and the laboratory experiments, we use the KdV model below to investigate the steepening of a number of different initial conditions. A useful feature of the KdV model is that by setting the dispersive term to zero ($\beta = 0$) in (1), the model becomes an equivalent hydrostatic model, allowing direct comparisons between the evolution under the hydrostatic and KdV equations.

Steepening time scale

To highlight the major differences between the behaviour of hydrostatic models and our KdV approximation of the real world, we first consider the evolution of a simple cosine initial condition. Figure 2 shows the evolution of such an initial condition in a frame of reference moving at the linear long wave phase speed. It can be seen that the initial condition evolves in a similar way in each model until a time $t = T_s$, by which time the initial cosine has steepened until the front face is almost vertical. T_s is the steepening time scale derived by *Horn et al.* (1999). The initial steepening of a long wave is described by the nonlinear hydrostatic wave equation (*Long*, 1972)

$$\eta_t + c_o\eta_x + \alpha\eta\eta_x = 0 \qquad (3)$$

where α is the nonlinear coefficient from (1). Equation (3) is just the KdV equation (1), neglecting the dispersive term. Balancing the unsteady and nonlinear terms, while moving in a frame of reference with the long-wave, the steepening time scale is defined as

$$T_s \sim \frac{\lambda}{\alpha\eta_o} \qquad (4)$$

where λ is some characteristic wavelength and η_o is the initial amplitude. In the absence of dispersion, T_s is the the time that it takes for a line of slope η_o/λ to steepen and become vertical. In a series of laboratory experiments, *Horn et al.* (1999) observed that solitary waves emerged from an intial condition at time T_s.

In both systems the horizontal length scale of the wave is initially long compared with the depth of the fluid, so the hydrostatic approximation is valid and the initial steepening is well described by (3). However, as the wave steepens, its horizontal length scale decreases until the vertical accelerations, and hence dispersive effects, become significant, eventually balancing the nonlinear steepening. The internal solitary waves subsequently evolve owing to this balance between nonlinear steepening and dispersion. The behaviour of the models diverge for $t > T_s$ because the hydrostatic model continues to neglect the vertical accelerations when they are clearly important.

In a hydrostatic model the wave would continue to steepen until breaking unless diffusion dissipates the

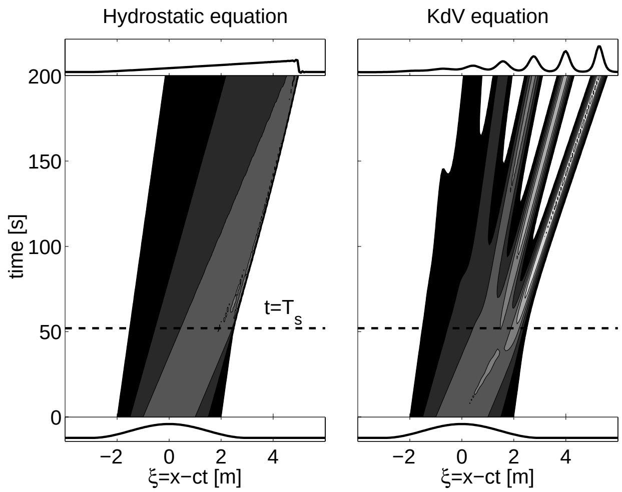

Figure 2. The evolution of a cosine initial condition under a hydrostatic equation and the KdV equation. The contours are of interface displacement. The initial and final interface profiles are shown below and above the X–T contour plots.

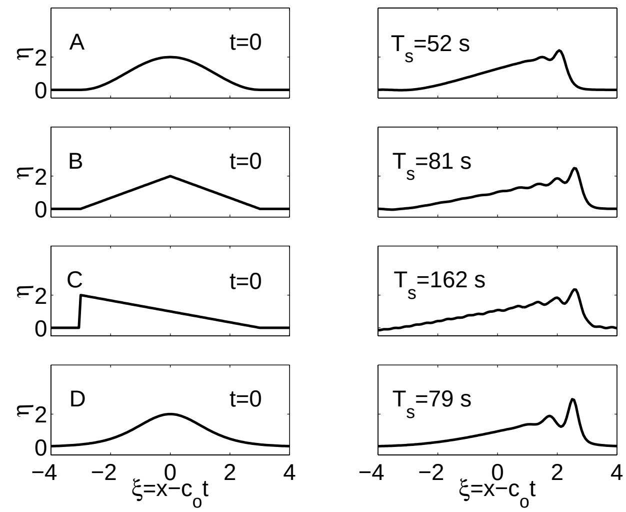

Figure 3. The evolution of four simple initial conditions under the KdV equation. The second column of panels represent the waves at $t = T_S$, where T_s has been calculated for each initial condition from (4).

shock. In the psuedo-spectral code used for these simulations, wave-breaking is prevented by a prescibed hyper-viscosity that dissipates the high wavenumber energy of the sharp front. In most finite-difference schemes numerical diffusion plays a similar role. The result is that in a hydrostatic model the wave continues to propagate as a steep fronted feature, dissipating energy as it travels, so that its amplitude quickly decays. In contrast, in the KdV model dispersive effects are retained and lead to the evolution of solitary waves at time $t = T_s$.

The propagation and the dissipation of internal wave energy are very different in the two models. In the hydrostatic model, energy is dissipated in the interior of the lake by diffusion across the sharp front. In the KdV model, the initial internal wave energy is re-packaged into coherent high energy-density solitons that propagate without loss until they encounter a boundary. In a real lake the propagation of solitons will be accompanied by some viscous losses and may even induce some local shear instabilities, but the fundamental difference remains that the hydrostatic model will always dissipate energy within the interior of the lake that would otherwise be dissipated at the boundaries.

To consider the effect of the shape and slope of an initial condition on the steepening process we let four simple initial shapes evolve under the KdV equation. Each of the shapes shown in Figure 3 was chosen to have the same vertical and horizontal length scales and the same area, but the front face of each shape has a different slope. The steepening time scale for each shape can then be defined as $T_s = 1/(\alpha S_o)$, where S_o is the

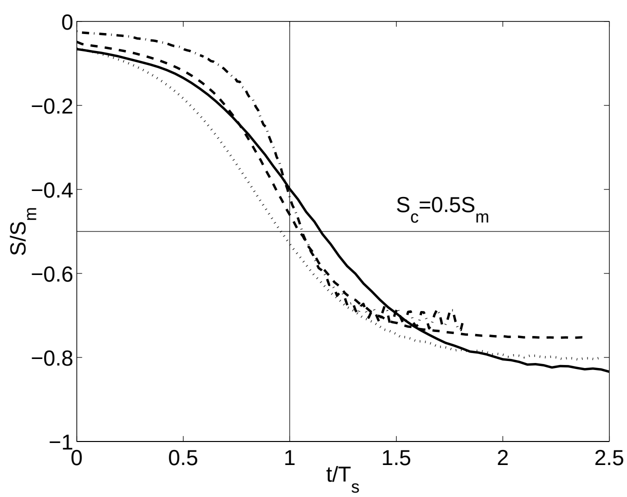

Figure 4. The maximum slope of the front face at any time t of each of the initial conditions in Figure 3 . Time is nondimensionalized by the steepening time scale given by (4) and the slope is nondimensionalized by the maximum possible slope given by (6).

maximum slope of the front face. Figure 3 shows that at $t = T_s$ each of the shapes has evolved into the same steep-fronted wave.

Figure 4 plots the maximum slope of the front face of each of the waves as they steepen. Each shape steepens in a similar way, achieving approximately the same slope at $t = T_s$ and asymptotically approaching the same maximum slope. It is this similar steepening that is important in the development of a solitary wave closure scheme. Although the steepening time scale is a useful parameter in determining *when* solitary waves will emerge, it requires some knowledge of the initial shape at $t = 0$ (specifically the initial slope S_o); information that is not available when implementing a closure scheme. However, since all of the shapes evolve to a common form with a similar frontal slope at $t = T_s$ it is possible to define the time and place at which solitary waves will emerge in terms of some critical slope, S_c. So what determines this critical slope?

Critical wave slope

Using the inverse scattering transform to solve the initial-value problem for the KdV equation, it is possible to determine the number and amplitude of the solitons that will emerge from some simple initial conditions (*Drazin and Johnson,* 1989). *Horn et al.* (1999b) used this method to investigate the solitons emerging from an idealised depression of the thermocline in a lake which they modelled as the triangular initial condition C shown in Figure 3. Figure 5 plots a series of curves representing the amplitudes of each of the solitons that emerge as a function of the size and shape of the initial triangle. It can be seen that the scaled amplitude parameter $K/\sqrt{u_m}$ of the leading soliton (given by the first curve) very quickly approaches a value of unity. Here K^2 represents an eigenvalue of the scattering problem and is related to the solitary wave amplitude and $u_m = a_0/h_1$ is the scaled amplitude of the initial condition. When the scaling is removed it can be shown that if $K/\sqrt{u_m} \sim 1$, the soliton amplitude is given by

$$a = 2K^2 h_1 \sim 2a_0 \qquad (5)$$

Since KdV solitary waves are one parameter waves described by (2), the amplitude of the leading soliton determines its shape and maximum interfacial slope

$$S_m = \frac{4\sqrt{2}}{9}\left(a_0^3 \frac{\alpha}{\beta}\right)^{1/2} \qquad (6)$$

This is the maximum slope that any initial condition can reach through nonlinear steepening in the presence of dispersion. The wave-slopes plotted in Figure 4 have been non-dimensionalized by this maximum slope, S_m, and the figure shows that at $t = T_s$ the front wave-slope is approximately $-0.5\,S_m$. We define the *critical slope* as $S_c = -0.5\,S_m$. If any wave steepens beyond this critical slope it will evolve into a packet of solitons. Any solitary wave closure scheme would be implemented when a slope exceeding S_c was detected in the model.

The appearance of the amplitude, a_0, of the initial condition in the right hand side of (6) raises a number

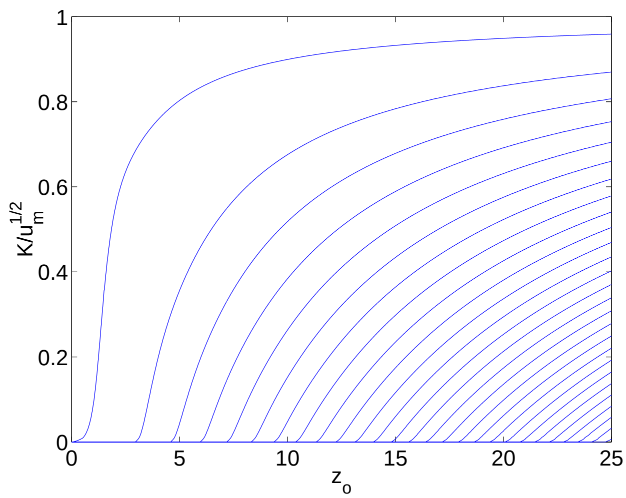

Figure 5. Curves representing the scaled amplitude of the solitons emerging from a triangular initial condition similar to C in Figure 3. z_o is a geometric parameter representing the area and shape of the triangle (from *Horn et al.*, 1999b).

of issues. Firstly, if the implementation of the closure scheme requires the detection of waves with some critical slope, S_c, this implies some measure of the amplitude of the initial condition. We have already acknowledged that any practical closure scheme will have no such knowledge of the initial conditions, even if these initial conditions could be defined in a developing system. The process of nonlinear steepening does not increase the amplitude of the initial condition until after the emergence of solitons at $t = T_s$. Therefore, the initial amplitude, a_0, can be assumed to be the amplitude of the wave at $t = T_s$, the time at which it will reach its critical slope. However, the calculation of the critical slope still requires knowledge of the initial amplitude of the wave. Any closure scheme must first define some characteristic amplitude for a feature and then use this amplitude to calculate the critical slope. If the front face of the feature exceeds the critical slope, then the closure scheme would be invoked.

The dependence of S_c on a_0 is not initially obvious. Equation (4) confirms that the steepening time scale depends only on the initial slope of the wave and is independent of the initial amplitude. However, (6) shows that the steepness of the front at $t = T_s$ is dependent on the initial amplitude of the wave and is independent of the initial slope. The consistancy of these statements can be confirmed by recalling that for KdV solitary waves the wavelength is inversely proportional to the square-root of the amplitude; large amplitude solitons are narrower and steeper than small amplitude solitons. Therefore, a small amplitude initial condition will evolve into small amplitude solitary waves that will

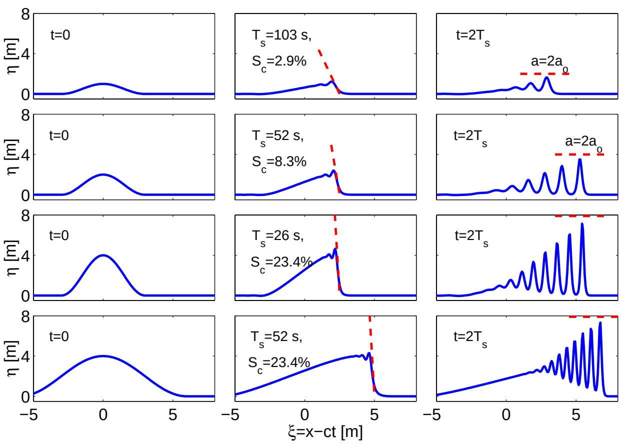

Figure 6. The evolution of four cosine initial conditions under the KdV equation. The second column of panels represent the waves at $t = T_S$, where T_s has been calculated for each initial condition from (4). The critical slope, $S_c = -0.5\,S_m$ is marked on each wave (where S_m is calculated from (6)). The third column of panels show the evolving packet of solitons at $t = 2T_s$. The dashed line in these panels represents the predicted amplitude $a = 2a_o$.

have relatively small slopes.

Figure 6 shows the evolution of four cosine initial conditions with different amplitudes and initial slopes. The figure confirms that at $t = T_s$ (where T_s is calculated for each initial condition) the wave has evolved into a steep front with a slope given by S_c (where S_c is calculated for each initial condition). S_c appears to be a good indicator of the time and place of the emergence of solitons.

Energy propagation

In most stratified lakes the thermocline is above mid-depth so any solitary waves are waves of depression. Although an initial disturbance can consist of both a depression and elevation of the thermocline (as in the case of a basin-scale internal seiche), it is the volume of the depression of the thermocline that determines the energy that is transferred from the initial condition to the emerging solitary waves (*Horn et al.*, 1999b). As a first approximation, all of the energy in that part of an initial condition that is comprised of a depression of the isopycnals will be transferred to the solitary waves. Assuming that there is an equipartition of potential and kinetic energy in the initial steep-fronted wave, the total energy per unit width of the initial wave can be estimated by

$$E_o = \Delta\rho g \int_{x_1}^{x_2} \eta_0^2(x)\,dx \tag{7}$$

where η_o is the interface displacement at the time when the wave-slope first exceeds S_c (where $S_c > 0$ for a wave of depression), and $\eta_o < 0$ over $[x_1, x_2]$ (for the case of a thin upper layer).

Given, from (5), the amplitude of the leading soliton that will emerge from the steep-fronted inital condition, and its shape from (2), we can determine the energy per unit width (to $O(a/H)$)

$$E = \frac{4}{3}\Delta\rho g a^2 \lambda \tag{8}$$

and its propagation speed

$$c = c_o + \frac{a\alpha}{3} \tag{9}$$

We have now built up a reasonably detailed picture of the formation of solitons from some simple initial conditions. We know when and where solitons will emerge from an initial conditions, we know the energy that will be transferred from the inital condition to the solitons, and we know the amplitude, shape, energy and speed of the leading soliton. Let us now consider a simple energy propagation model.

The motivation for developing a solitary wave closure scheme is to ensure that energy that is incorrectly dissipated in the interior of a lake by hydrostatic models is dissipated at the boundaries. This requires that, once invoked by the detection of a wave with a slope exceeding the critical slope, the closure scheme must (i) prevent any incorrect mixing in the interior of the lake, (ii) propagate the energy of the initial wave to the boundary at the correct speed, and (iii) dissipate the energy at the boundary, with the correct distribution between turbulent dissipation, mixing (and possibly reflection).

To prevent any further mixing in the interior, the closure scheme must effectively and rapidly reduce the diffusion caused by the steep front. The simplest method would be to remove the initial steep-fronted wave from the domain as quickly as possible. This could be achieved by directly resorting the vertical density profile over the extent of the wave, previously identified as the domain $[x_1, x_2]$. Alternatively, the initial wave could be rapidly damped with a locally enhanced *virtual* viscosity, provided that this enhanced viscosity was not associated with any diffusion of density. The rapid removal of the initial wave from the model ensures that the total energy of the system is conserved, since the energy of the initial wave has by then been transferred to the closure scheme for propagation to the boundary.

The propagation of the energy to the boundary can be achieved by advecting the energy of the solitary wave packet through the domain as a parameter. In the KdV model the packet of solitary waves gradually disperses, with the front of the packet propagating with the speed of the leading wave, c, given by (9), and the rear of the packet propagating at the long-wave speed, c_o. However, it unlikely that the length of the solitary wave packet will spread beyond two or three grid cells (1–2 km) in most models before encountering a boundary. The energy of the solitary wave packet can therefore be parameterized by a single value that is propagated through the domain at the speed of the leading soliton.

From a series of laboratory experiments investigating mixing due to internal solitary waves breaking on a slope, *Michallet and Ivey* (1999) determined the mixing efficiency and reflection coefficient as functions of the beach slope and the wave-slope. We have already determined a estimate of the wave-slope, S_m, for the leading wave. Although the wave-slopes of solitons towards the rear of the packet may not be as steep, S_m provides a reasonable approximation of a characteristic wave-slope for the packet that could then be used to estimate the mixing efficiency and reflection coefficient when the solitary wave packet encounters a sloping boundary.

The discussion so far has assumed that all of the energy transferred to the solitary waves is propagated to the boundary without loss. However, in a lake there a number of mechanisms that might lead to some internal dissipation and mixing as the solitary waves propagate. These include bottom boundary layer losses (*Leone et al.*, 1982), shear induced decay (*Bogucki and Garrett*, 1993) and wave–wave interactions. The first two of these are well understood and could be included in any solitary wave closure scheme by allowing the parameterized amplitude and energy to decay as they propagate through the lake.

Conclusions

Laboratory experiments and field observations suggest that the generation of internal solitary waves may be an important mechanism for the transfer of energy from basin-scale internal waves, energized by the wind, to turbulence in the benthic boundary layer in lakes. However, hydrostatic models, which are widely used to model the hydrodynamics of lakes, cannot simulate this important process because they neglect the vertical accelerations that are necessary to generate solitary waves. Furthermore, for many *engineering* models of lakes, internal solitary waves will not be resolved at practical grid-scales. If these models are to be used to simulate the spatial and temporal variablility of mixing events in lakes, it will be necessary to parameterize the effects of the evolution, propagation and dissipation of internal solitary waves. By considering the evolution of some simple initial conditions in a KdV model, we have suggested the first steps that could lead to the development of such a solitary wave closure scheme.

If such a solitary wave closure scheme is to be feasible, it must include some method of determining where and when in the model domain solitary waves will evolve. We have defined a critical wave slope, S_c, at which solitary waves emerge from an initial condition. This critical slope is dependent on the amplitude of the initial condition and on the background stratification. The proposed method would search the domain for a wave-front, the amplitude of the wave would be determined and the critical slope calculated. If the maximum slope of the wave-front exceeded the critical slope the closure scheme would be invoked.

Once invoked, the closure scheme must calculate the energy contained in the portion of the initial condition that contributes to solitary wave generation (that part of the wave in which isopycnals are below their equilibrium position for most cases) using (7). This energy would then be advected through the model as a parameter. The amplitude of the initial condition that invoked the closure scheme would be used to determine the amplitude of the leading soliton in the packet (5), and hence the speed at which the parameterized energy should be advected (9) and the reflection coefficient and mixing efficiency applied when the packet encountered a sloping boundary. The closure scheme could include the gradual decay of the solitons due to boundary layer losses and shear.

Although the principles of the proposed closure model have been applied to simple initial conditions propagating along an interface, the challenge is to apply these ideas to more complex initial conditions in a continuously stratified, three-dimensional model with complex topography.

Acknowledgments. The first author is grateful to Peter Müller for the opportunity to participate in the workshop. This research was supported by the Centre for Environmental Fluids and the Australian Research Council. This paper is Centre for Water Research reference ED 1477 DH.

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