Exact Solitary Wave Solutions in Shallow Water

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Abstract. Long's equation describes stationary flows to all orders of nonlinearity and dispersion. Dissipation is neglected. In this paper, Long's equation is used to attempt to model the propagation of a solibore -- a train of internal waves in shallow water at the deepening phase of the internal tide.

1. The Solibore Phenomenon

The internal tide in shallow water often has a sawtooth shape rather than a sinusoidal shape. The point of the tooth is not a simple jump as in a turbulent surface tidal bore, but consists of a train of waves. This wave train is called a solibore. The individual waves are often referred to as solitons, because they were originally modeled as solitons in the Korteweg-de Vries (KdV) equation. In many situations, the KdV modeling is highly inaccurate, as in *Stanton and Ostrovsky*, 1998; often an equation called the extended KdV, eKdV, is used . An example of a solibore is shown in Figure 1.

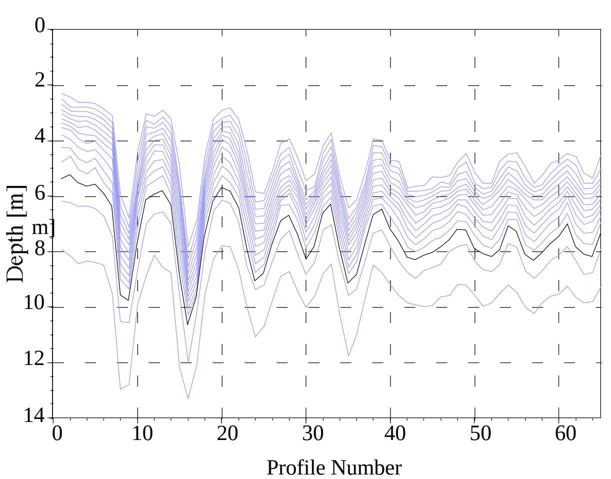

Figure 1. A solibore measured at Knight Inlet, British Columbia. Various isopycnals arwe shown. It can be seen that each isopycnal ends up lower than it started, forming the tooth of the sawtooth internal tide. The second wave is somewhat larger than the first, in contrast to the rank ordering predicted by the KdV theory. Data from E. D'Asaro.

Solibores are not all the same. Some have only a few waves in them, while others have many. Some are observed to have another solibore crossing them, while others seem to have two solibores merge into one, similar to the behavior of shocks and surface bores. They have a tendency to be very straight or segments of a circle, which is also the case with shocks and bores. This straightening out requires that dissipation (which may include radiation of smaller-scale waves) is important. This is because a straight line or circle has less entropy than the arbitrary shape that the waves might have been formed with. The only place for that entropy to go to is to smaller scales; small waves, turbulence, or heat. There is a remarkable case that appears to be this loss of entropy in the SAR images of *Fu and Holt* 1984, p 36. In this image, the waves near the source region are made out of many (relatively) small packets, going in various directions. The waves farther from the source region, presumably generated by the previous tide, have organized themselves into one set of waves made of straight lines and circular arcs.

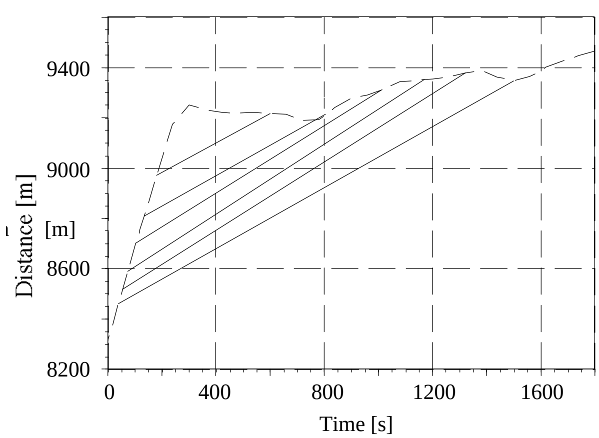

Figure 2 The trajectories of the wave crests of a Knight Inlet solibore. The ship's motion is shown as the dashed curve, and the line segments connect the points at which the ship encountered the crest of one of the waves. On the average, their is no dispersion of the waves, in contrast to KdV. The 3rd, 4th, and 5th waves are actually slightly faster than the others. Another feature of interest is the advection of the ship by the wave-caused surface current.

Another feature of solibores in that there is a tendency not to disperse, except when they are weak. A SARSEX image (*Liu*, 1988) shows four wave packets, generated by four successive tides. The youngest three appear to have the same wavelength; only the fourth one has clearly dispersed. One may object that these are different solibores, and may have been differently formed. During the 1995 Solibores and Sill flows experiment, Eric D'Asaro made multiple crosses of the same solibore. Figure 2 shows the space-time trajectories for one pair of passes; it can be seen that the velocities of all the waves are nearly the same. Such results contradict assertions commonly made in the literature, which seem to be based more on theoretical prejudice (KdV waves disperse) than on real evidence.

There is a phenomenon in lakes which is similar to the tidally generated solibores. The difference is that the waves in the lakes are generated by an internal seiche rather than an internal tide. The lake solibores warrant additional study (*M. E. Snyder*, private communication.)

The work discussed below started out as an attempt to model the shape of a solibore, but not its longer time dynamics.

2. Long's Equation

There are a number of assumptions made in the model. First are the safe assumptions, which are incompressibility and the Boussinesq approximation. Secondly are simplifying assumptions that are justified by the similarity of the solibore phenomenon in various conditions. Although actual conditions deviate a little from these assumptions, the nature of the solibore does not appear to be controlled by the amount of such deviations. The first such assumption is to neglect the Earth's rotation. This is justified by the absence of any clear dependence on latitude. The second simplifying assumption is that the solibore is two-dimensional. As mentioned above, this assumption is remarkably close to being true in many cases, and solibore properties do not seem to differ much when it is not quite so true. The last simplifying assumption is that there is no shear in the initial state. This assumption is observed to be approximately true in a number of cases, for no particular reason; if there was no net exchange transport over a tidal cycle, one would expect shears of opposite signs before and after the solibore passed. It appears however that the shear during and after the solibore often dominates.

Finally, there are the questionable assumptions that must be put into the category of hypotheses to be tested. There are two such assumptions. The first is to neglect dissipation. Although it is argued above that dissipation is important in the long-time dynamics, the shape of waves is usually determined on a time comparable to the wave period. On this short time, it is reasonable to neglect the dissipation. The second is to assume steady waves. The observed absence of dispersion argues that this assumption is reasonable. Many solibores, especially those with many waves, get longer by adding more waves on the back end of the wave train. This phenomenon is predicted by KdV, as pointed out by *Apel et al.*, 1997. In that case, the modeling should only apply to the front part of the wave train, which can be assumed steady.

The assumptions listed above lead to Long's equation (*Long*, 1953). Unlike the KdV or eKdV equations, there are no assumptions about the strength of the nonlinearity or the steepness of the waves. Long's equation is a nonlinear, inhomogeneous version of the Helmholtz equation. It has the form

$$b(\ {}^2 + K^2)\ \zeta = 0$$

In this equation ζ is the vertical displacement of isopycnals. An isopycnal has a constant value of $z - \zeta$. K contains the inhomogeneity and nonlinearity. It is a function of space which depends on the solution ζ. Setting $N(z)$ to be the buoyancy frequency in the initial state, and v to be the speed of the wave, K is given by

$$K = f(N(z\text{-}\zeta),v)$$

In many atmospheric applications, it is assumed that N is a constant, in which case Long's equation reduces to the Helmholtz equation. In the coastal ocean, however, constant N is usually a very poor description of the stratification. When solibores exist, the stratification is much more nearly a 2-layer type stratification. N is large near some depth, and small elsewhere.

The boundary conditions are $\zeta = 0$ at the top and bottom, and in the initial state. The final state ζ is discussed below. The work reported here is based on numerical solutions of Long's equation.

If the assumption of no shear in the initial state is relaxed, v , as well as N, is a function of $z\text{-}\zeta$. Now $v(z)$ is the speed of the wave relative to the water in the initial state, and we put v' = $f(dv,dz)$. Long's equation then generalizes to

$$v\ b\Big(v\ \ {}^2\zeta - v'\,(o(\mathbf{z},\hat{\ })-\ \ \zeta)^2 + v'\Big) = -N^2\,\zeta$$

where N, v, and v' are evaluated at $z - \zeta$.

3. Final State Conditions

One possible final state is $\zeta = 0$. This final state can represent a single solitary wave. Other final states that have $\zeta = 0$ at the top and bottom with $z - \zeta$ a monotonic function of z have solutions on the interior of the computational domain. However, in general, such solutions cannot be continued beyond the computational domain and maintain those properties; z – ζ will not correspond to any position in the water column. The way to allow the solution is to have the disturbance localized in the horizontal. Then the final state is independent of the horizontal coordinate x. In order for that to be possible, the final state must obey the ordinary differential equation obtained from Long's equation by dropping the x second derivative:

$$b(\__z^2 + K^2)\ \zeta = 0$$

This equation has solutions, one for each number of zero crossings (which are referred to as modes, generalizing the linear mode concept.). Usually, we want to determine the speed *v*, so for each mode there is a one-parameter family of solutions parameterized by *v*. For application to most solibores, we are only interested in the lowest mode, with no zero crossing in the interior.

It turns out, however, that there is another condition on the final state. This additional condition can be derived in a number of different ways. It prevents there from being a first *x* derivative of ζ at the final state. One way to derive the condition, which I have not seen previously in the literature, is by use of a variational principle. Long's equation is the Euler equation for the stationary points of the difference of kinetic energy and potential energy, both expressed in terms of ζ. If the initial and final states are *x*-independent, then a small horizontal translation δx changes the difference of kinetic energy and potential energy by the vertical integral of the difference of kinetic and potential energy densities in the final state multiplied by δx. (The energy is defined so these densities are zero in the initial state.) Since the solution must be a stationary point, this integral must be zero. By use of the other condition and integration by parts, this integral can be reexpressed in a particularly convenient form (see, e.g., *Lamb and Wan, 1998*) as

$$i(,,(\__z\ \zeta)^3)dz = 0$$

This additional condition picks out a single member of the one-parameter family; the lowest-mode solution of the two conditions is unique. This solution is called the "conjugate flow."

4. Solutions of Long's equation

The method used to find solutions was a relaxation technique. One guesses a configuration, and the computer code iterates toward a solution, starting from this guess. As a guess, I put in various configurations that resemble observed solibores. In every case, the relaxation technique converged to the same solution, a "kink" connecting $\zeta = 0$ and the conjugate flow, as shown in Figure 3. A set of waves that was present in the first guess would disappear during the relaxation. Assuming that this is the only solution, Long's equation cannot be used to model solibores. Even if there were a solution resembling a solibore, it is not reasonable to have a unique final state. The final state should be determined by the strength of the internal tide; generally this is weaker than the conjugate flow.

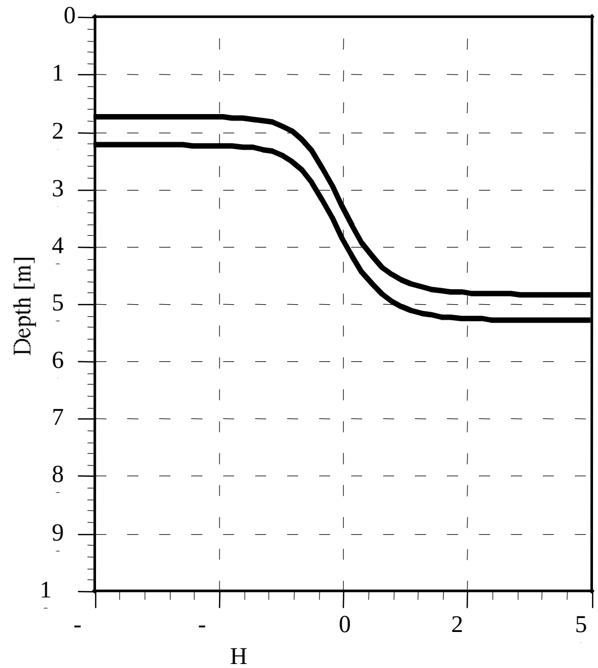

Figure 3. The kink solution of Long's equation. The depth of the water is 100 m, and the initial state (on the left) has a buoyancy profile, which is a Gaussian centered on 20 m depth and a width of 5 m. The isopycnals that start out 2.5 m on either side of the center of the Gaussian are shown.

Perhaps individual waves, rather than the entire solibore, should be modeled as solitary waves. In particular, the first wave in many solibores looks slightly separated from the next one. For individual waves separated from the rest, the appropriate final condition is $\zeta = 0$. However, in this case as well, the conjugate state plays an important role. (The theory of "exact" solitary waves is well known. See, for example, *Lamb and Wan, 1998*. There is a one-parameter family of lowest mode solitary solutions. The parameter can conveniently be taken as the energy of the solitary wave. As the energy varies from zero to infinity, the velocity of the wave varies from the linear wave speed to the kink speed; i. e. that of the conjugate flow. The amplitude varies from zero to that of the conjugate flow. Near the limit, the wave looks like a kink to the conjugate flow, followed after some distance (which asymptotically grows linearly with the energy) by an "antikink" which is the mirror image of the kink. Some of these solitary waves are shown in Figure 4.

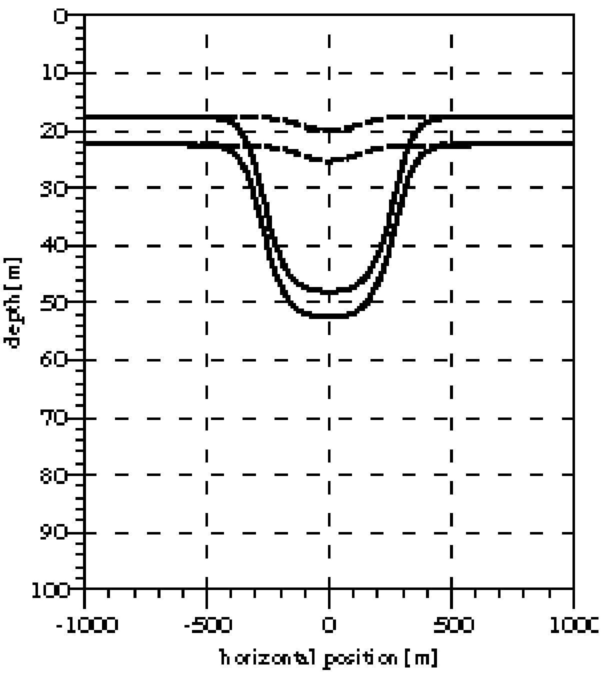

Figure 4. Two solitary waves are shown. The initial state and isopycnals shown are as in Figure 3. The smaller wave is near the limit of validty of KdV, while the larger is very close to the kink-antikink limit.

Thus to understand the family of solitary waves, one must understand the conjugate flow. For the two layer flow, K^2 is a delta function at the interface. The first condition on the final state then requires ζ to be linear in each layer, with a corner at the interface. The second condition reqires that the integrals of the cubes of these slopes be equal (up to the sign). The only way for this to happen is for the two layers to be exactly equal in thickness; the interface is half way through the water column. Calculations with various stratifications typical of shallow water describable as quasi-2 layer all show that this situation is not restricted to the true two-layer case. In all cases, it was found that the maximum stratification in the conjugate flow was extremely close to the mid-depth level. One need not calculate anything to compare to data. An example of such a comparison is with the data of *Podney and Sager, 1979*. The water in this experiment had a very good quasi-two-layer stratification, and the total water depth was 18 m. The first wave appears to be the very model of a solitary wave. The stratification returns closely to its initial configuration , and stays there for a while until the second wave comes along. The conjugate state has its maximum stratification at mid-depth, 9 m. However, the observed wave has its maximum stratification at 11m, and also looks a lot peakier than the kink-antikink type configuration. The second wave, not as well separated from what follows it, is even larger, its maximum is about 12.5 m deep, and it also is peaky. These waves are not at all the solitary waves they appear to be!

5. Comparison with KdV and eKdV

The limits of validity of the KdV equation have been examined by a number of investigators. Our calculations support the results others have obtained. The nondimensional parameter used to describe the validity of KdV in a quasi-two-layer stratification has been chosen as the ratio of the wave amplitude to the thickness of the thinner layer, usually the upper layer. For values of this parameter less than 0.1, KdV works fairly well. By the time the parameter reaches 0.2, the nonlinear part of KdV fails. In the COPE wave analyzed by *Stanton and Ostrovsky*, 1998, this parameter is about 4, and, as they show, KdV is extremely far off.

They claim, however, that it fits the eKdV model. Here we analyze that claim, using the formulas and values from their paper. The eKdV model also has a kink-antikink limiting wave, given by their equation 4. Unlike the full Long's equation , the position of the interface in the eKdV "conjugate flow" is not half way through the water column, but at a lesser amplitude. The curve for this limiting interface position, calculated from equations 2-4 of *Stanton and Ostrovsky* is shown in Figure 5. They quote the water depth as 147m. In their table 1 they state an initial interface depth as 7 m, while their figure 5 shows it at 6 m. The limiting position, with these values is 27.4m if the initial interface depth is 7 m, and 24.3 m if the initial interface depth is 6 m. Their wave has a maximum interface depth of 30.5 m, in excess of the limit for either choice of parameter. (Their theoretical curve in Figure 5 also exceeds this bound, so it doesn't agree with the 147 m depth.) Since the wave is so close to the eKdV limit, one expects it to be fatter in that theory than one expects in the true theory. Indeed, their theoretical curve is clearly fatter than the data. It would be interesting to compare their data to the Long's equation solution. Thus, eKdV also doesn't work for this wave.

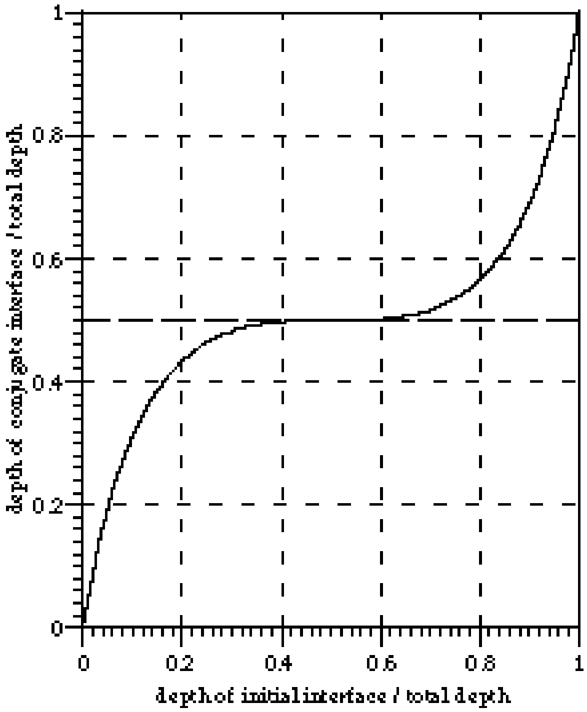

Figure 5. The limiting interface position of the “kink-antikink” wave for the 2-layer case, as a fraction of the total depth. The solid curve is for the eKdV equation, while the dashed line at half the total depth is the actual value given by Long’s equation. the * symbol marks the COPE wave values; it exceeds the eKdV limit but is only slightly in excess of 40% of the true limit.

6. Summary

Long's equation makes the assumptions of steadiness and lack of dissipation. With these assumptions it cannot describe the entire solibore wave train, nor, in at least one case one of the waves from a solibore that looks very much as if it should be a solitary wave.

The time-independent versions of KdV and eKdV, which are solved by solitons, make the same assumptions, but in addition, assume small amplitude. Therefore, comparison with Long's equation solutions can test if the amplitude is small enough. In many cases, the amplitudes are too large for either of these two equations.

The KdV equation predicts rank ordering and dispersion of the waves in the packet, both of which are often violated.

The conclusion that I draw from these considerations is that we do not have an adequate theoretical description of solibores; either dissipation or unsteadiness (or both) is important. My guess is that it is the dissipation that will prove to be the missing ingredient; data suggests it is present, but supports the assumption of steadiness.

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References

Apel, J. R., M. Badiey; C. S. Chiu, S. Finette, R. Headrick, J Kemp, J. F. Lynch, A. Newhall, M. H. Orr, B. H. Pasewark, and D Tielbuerger, An overview of the 1995 SWARM shallow-water internal wave acoustic scattering experiment, *IEEE J. Ocean. Eng., 22*, 465-500, 1997

D'Asaro, E., and R-C. Lien, Lagrangian Measurements of Waves and Turbulence in Stratified Flows, *J. Phys. Oceanogr*., in press (1999)

Fu, L-., and B. Holt, Seasat Views Oceans and Sea Ice With Synthetic Aperture Radar, *Jet Propulsion Laboratory Publ., 81-120*, 1982

Lamb, K. G., and B. Wan, Conjugate flows and flat solitary waves for a continuously stratified fluid, *Phys. Fluid*, *10*, 2061-2079, 1998

Liu, A. K., Analysis of Nonlinear Internal Waves in the New York Bight, *J. Geophys. Res., 93,* 12,317-12,329, 1988.

Long, R. R., Some aspects of the flow of stratified fluids. I. A theoretical investigation, *Tellus, 5,* 542-558, 1953.

Podney, W., and R. Sager, Measurement of fluctuating magnetic gradients originating from oceanic internal waves, *Science*, *205*, 1381-1382, 1979.

Stanton, T. P., and L. A. Ostrovsky, Observations of highly nonlinear internal solitons over the Continental Shelf, *Geophys. Res. Lett., 25,* 2695-2698, 1998