

A Quick Introduction to Path Integrals

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Abstract The meaning and derivation of path integrals, as used in ocean acoustics, is briefly reviewed. A comparison to path integrals used for diffusion problems is made. The version of the path integral useful for wave propagation in random media is mentioned.

Introduction

Feynman developed path integrals for quantum mechanics by 1948 [see *Feynman and Hibbs*, 1965]. When it was discovered by *Leontovich and Fock*, [1946] that the Schrödinger equation is useful for electromagnetic and acoustic wave propagation, and then used it in ocean acoustics [see *Tappert*, 1977], it must have been obvious to many people that techniques from quantum mechanics could be used essentially unchanged. Path integrals are one such technique; those who knew of Leontovich and Fock's work would immediately know that Feynman's ideas could be applied to such problems. Path integrals have been used by many people in ocean acoustics, probably most extensively by Dashen and coworkers [see *Flatté et al.*, 1981]. Good general references on path integrals are *Feynman and Hibbs* [1965], *Schulman* [1981], and *Gutzwiller* [1990].

Path Integrals

The first step in constructing the path integral for ocean acoustics is to obtain a Schrödinger equation. This is done by first selecting a single frequency, neglecting the time dependence of the ocean medium through which the sound travels, obtaining the Helmholtz equation, and then making a small angle approximation. The derivation of the Schrödinger equation from the Helmholtz equation is exactly the same as its derivation in quantum mechanics from the Klein-Gordon equation. In the ocean case, it is an approximation that the angle of propagation relative to an axial direction is small: $d\mathbf{z} \ll dx$, while in the quantum mechanics case, it is a small velocity approximation: $d\mathbf{x} \ll c dt$. A sign difference between the Klein-Gordon (hyperbolic) and Helmholtz (elliptic) equations disappears when these approximations are made. In ocean acoustics, the Schrödinger equation is often referred to as the "parabolic equation" (sometimes "narrow-angle parabolic equation"). This terminology is misleading to those who think of the classification of second-order real PDEs. They think of parabolic equations as representing diffusion, whereas the Schrödinger equation represents wave propagation; it is

complex, with an i multiplying the first derivative term. The Schrödinger equation for ocean acoustics is very much less accurate than that for quantum theory, since $d\mathbf{z}/dx$ (acoustics) $\gg d\mathbf{x}/c dt$ (quantum) for most problems in either subject. The inaccuracy of the Schrödinger equation in ocean acoustics must be considered in applications. The axial direction is taken to be in the horizontal for essentially every ocean acoustics problem of interest. The two-dimensional transverse vector \mathbf{z} is usually replaced by the one-dimensional z , since the transverse horizontal is rather unimportant.

The potential in the Schrödinger equation for ocean acoustics is related to the index of refraction n by

$$V = k_0 \frac{n^2 - 1}{2} \quad (1)$$

where k_0 is the reference wavenumber, i.e., the frequency divided by the speed c_0 used in defining the index of refraction as $n = c_0/c$. The acoustics Schrödinger equation is

$$i \frac{\partial \Psi}{\partial x} = -\frac{1}{2k_0} \nabla^2 \Psi + V \Psi \quad (2)$$

This can be compared with the truly parabolic equation

$$-\frac{\partial \rho}{\partial x} = -\frac{1}{2k_0} \nabla^2 \rho + V \rho \quad (3)$$

representing diffusion with a potential.

Any equation of the form $\partial \Psi / \partial x = \Omega \Psi$, where Ω is any operator that has the semigroup property, that the Green's function for a total x -extent from 0 to X , considered as a function of z , is the convolution of the Green's function for the extent 0 to X_1 with that for X_1 to X , where X_1 is any value between 0 and X . That result is established by considering $\Phi = \Psi \theta(x - X_1)$, where θ is the unit step function. Then Φ obeys the equation $\partial \Psi / \partial x = \Omega \Phi + \Psi \delta(x - X_1)$, i.e., the same equation as Ψ except for an initial source (at X_1) given by Ψ . We now

take Ψ to be the original Green's function (as a function of both x and z), and the result follows.

Given a subdivision of the interval into two subintervals, we can further subdivide those subintervals, and continue the process as long as we like, making each subinterval as small as we want. The Green's function is then the multifold convolution of the Green's functions for each of the subintervals. Explicitly

$$G(X_0, z_0, X_F, z_F) = \int dz_1 dz_2 \cdots dz_n \times G(X_0, z_0, X_1, z_1) G(X_1, z_1, X_2, z_2) \cdots G(X_n, z_n, X_F, z_F) \quad (4)$$

One can think of the points $(X_0, z_0), (X_1, z_1), \dots, (X_n, z_n), (X_F, z_F)$ as representing a path, sampled at $x = X_j$. The integral is then an integral over the coordinates of that path, i.e., a path integral. That part of the G that don't depend on z can be included as part of the metric for the paths, and the remainder is the integrand. If a good approximation of the Green's function for small x intervals can be found, then the expression in Eq. (4) may converge to the exact answer as all the interval sizes approach zero. For Eqs. (2) and (3), such approximations exist.

An approximation that works for this purpose is the "phase screen" approach. The potential V is treated as if it were concentrated (as delta functions) at the planes $x = X_1, x = X_2$, etc. Then the Green's function for the interval is the known Green's function for the "free" problem with $V=0$, multiplied by $\exp(-iV\delta x)$ or $\exp(-V\delta x)$ for Eq. (2) or (3), respectively. The free Green's function is

$$G_0 = \frac{\exp\left(i \frac{k_0 \delta z^2}{2\delta x}\right)}{\delta x} \quad (5)$$

or

$$G_0 = \frac{\exp\left(\frac{k_0 \delta z^2}{2\delta x}\right)}{\delta x} \quad (6)$$

for the two cases. In each case the important paths are those for which the integrand is not too large, which means that $\delta z \sim \sqrt{\delta x}$. For Eq. (6), G_0 is real and positive, and can be included in the integration measure, so paths more wild than $\delta z \sim \sqrt{\delta x}$ have no measure in the limit $\delta x \rightarrow 0$. For Eq. (5), however, the absolute value of G_0 is $1/\delta x$, which does not provide any cutoff on wild paths if included in the measure. The restriction in the wave problem to $\delta z \sim \sqrt{\delta x}$ relies on phase cancellations in the integrand. The inclusion of arbitrarily wild paths in the wave case is what makes mathematicians say that the path integral is not defined as an integral over paths in the limit $\delta x \rightarrow 0$, but of course there is no difficulty at finite

δx . For the diffusion case, the restriction of paths makes the path integral well-defined in the limit. In the diffusive case, a path can be thought of as a Brownian motion path taken by a random walker.

If one thinks of the Helmholtz equation rather than the parabolic equation, the free Green's function is

$$\frac{\exp\left(i k_0 \sqrt{dx^2 + dz^2}\right)}{\sqrt{dx^2 + dz^2}} \quad (7)$$

for which all paths are important; only the slow $1/\delta z$ reduces the integrand at large δz . For the Helmholtz equation, or wide-angle parabolic equation with this same feature, the path integral doesn't make sense.

Paths with $\delta z \sim \sqrt{\delta x}$ are not differentiable (more precisely, differentiable paths have zero measure). Nevertheless, one writes $\delta z = \frac{dz}{dx} \delta x$. With this notation, the Green's function for one step is

$$G(\delta x, \delta z) = \frac{\exp(i L dx)}{dx} \quad (8)$$

where

$$L = \frac{k_0}{2} \left(\frac{dz}{dx}\right)^2 - V \quad (9)$$

The denominators of Eq. (8) are absorbed into the integration measure of Eq. (4). The integrand is then the product of the exponentials, i.e., $\exp(\sum L \delta x)$. With the obvious change of notation from sum to integral for small δx , Eq. (4) becomes

$$G = \int_{\text{paths}} Dz[x] \exp(i \int L dx) \quad (10)$$

In this expression $Dz[x]$ means $dz_1 dz_2 \dots dz_n$, multiplied by other factors absorbed into the measure. Equation 10 is the final path integral for deterministic problems, i.e., when V is specified. (Sometimes a "normalization factor" is written in front of the integral to indicate that the measure has been changed.)

Random Media

For wave propagation in random media, however, one wishes to treat V as a sum of a specified part and a stochastic part (in space and time), and one wishes to obtain expectation values of expressions that can be constructed from G . In particular, one can consider moments, which are expressions made from products of some number called the order of the moment of G . In the products, the k_0 might be different for different factors, and/or the endpoints of the paths might be different, and/or the time at which the V is evaluated might be different. One multiplies the expressions of Eq. (10)

together, and then takes the expectation value. In applications, it has always been assumed that the stochastic part of V comprises a Gaussian process. In that case, one evaluates the expectation value of the exponent of the part involving the random piece of V as the exponential of half the variance.

The path integral for a moment has an multiple integral over a number of paths equal to the order of the moment.

The path integral is a formal expression for any desired moment. Some evaluation method must be used; in general the method is only approximate, and one ought to be in control of how good the approximation is. One standard method, the Gaussian integral approximation, is to treat the exponent as if it is quadratic in z ; in that case the path integral can be carried out analytically, giving the integrand at the saddle point divided by the square root of a determinant made from the coefficients of the quadratic. One should imagine doing the integrals at finite δx , and then taking the limit of the answer as $\delta x \rightarrow 0$. When we evaluate the integrand at the saddle point and the determinant numerically, we omit the $\delta x \rightarrow 0$ step, by choosing δx small enough for acceptable accuracy. The saddle point in the deterministic problem is the Fermat path, i.e., the ray of ray tracing.

References

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