Renormalization Theories for Incompressible Flows with Gaussian Statistics

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Abstract. We review recent results on the large time / large scale properties of passive tracer transport by time dependent incompressible velocity fields with Gaussian statistics. We consider velocity fields with an infrared spectral singularity and we concentrate on the homogenization and renormalization procedures, one of our goals being to identify regimes of anomalous diffusion. We also discuss some mathematical conjectures motivated by the results of numerical simulations.

1. Introduction

We consider the problem of the asymptotic properties of the transport of passive tracers by an incompressible turbulent flow. This flow is given by a random velocity field $\{\vec{\mathbf{v}}(t,\mathbf{x})\}$ which is assumed to be

- mean zero (we are only interested in fluctuations around a mean deterministic motion) with Gaussian statistics,
- stationary in time,
- homogeneous in space and
- possesses a spectrum of Kolmogorov's type.

These assumptions are standard in statistical fluid mechanic. See for example *Yaglom* [1987] and *Monin and Yaglom* [1971]. We shall make even more restrictive assumptions on the velocity field. In particular we shall assume

- Incompressibility (i.e., it is divergence free). Throughout the paper we restrict ourselves to the 2-dimensional case. Because of the incompressibility assumption, this restriction implies that the velocity field can be derived from a *stream function* $\phi(t,\mathbf{x})$ via the formula $\vec{\mathbf{v}} = \nabla^{\perp}\phi$ or in other words:

$$\vec{\mathbf{v}}(t,\mathbf{x}) = \left[\frac{\partial\phi(t,\mathbf{x})}{\partial x_2}, \frac{-\partial\phi(t,\mathbf{x})}{\partial x_1}\right]^t.$$

- Space isotropy. In order to justify this assumption we note that, according to Kolmogorov's theory of fully developed turbulence, this property is universal (at least for large Reynolds numbers.)
- Markov property in time. This assumption is not as common as the other ones but it is also very natural.

These three conditions determine (up to a couple of parameters) the distribution of $\vec{\mathbf{v}}$. In order to make the notation simpler we describe the flow distribution in terms of the stream function ϕ. According to the above discussion we assume that the scalar random field $\{\phi(t,\mathbf{x});\ t \geq 0,\ \mathbf{x} \in \mathbb{R}^2\}$ is random, stationary in time and homogeneous in space. Since we also assume that it has Gaussian statistics, its distribution is entirely determined by its second order statistics. Their knowledge is equivalent to the knowledge of the spectral representation and in its general form it can be stated as

$$\phi(t,x) = \int\int e^{i(\omega t + k\cdot x)} E(\omega,k)^{1/2} W(d\omega, dk)$$

where $W(d\omega, d\mathbf{k})$ is a scalar valued orthogonal L^2 white noise measure. See, for example, *Adler* [1990] for the general theory of random fields. In fact the properties assumed in the bullets • above imply that the spectral density **must** have the specific form

$$E(\omega,\mathbf{k}) = \frac{\beta(|\mathbf{k}|)}{\omega^2 + \beta(|\mathbf{k}|)^2}\mathcal{E}(|\mathbf{k}|)$$

Notiçe that

◇ the function β determines the time (de-)correlation. Indeed a partial Fourier transform of the covariance function gives a superposition of terms of the form $e^{-\beta(|k|)|t|}$. This is an indication that the velocity field can be viewed as an Ornstein Ulhenbeck process parameterized by the time variable t with values in a (typically infinite dimensional) space of functions of the space variable $\mathbf{x}$. These processes were studied in *Antoniadis and Carmona* [1985], and this point of view was advocated in *Carmona* [1997].

◇ the function $\mathcal{E}$ controls the space correlations

For physical reasons it is desirable to impose scale invariance on the model. This last requirement forces the functions β and $\mathcal{E}$ to be power laws. In other words:

• $\beta(r) = r^z$

for some $z > 0$ (our first spectral parameter) and

• $\mathcal{E}(r) \approx r^{1-\epsilon}$ for some $\epsilon > 0$ (our second spectral parameter.)

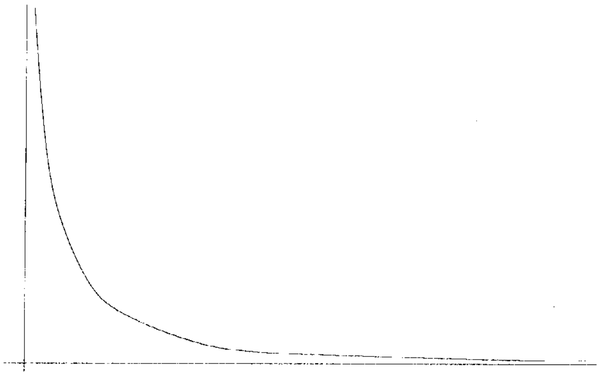

Figure 1. Power law spectral density $\mathcal{E}(r)$.

The fact that the spectral density is a power function implies that it **cannot** be integrable both at 0 and ∞! In other words,

- the velocity (random) field is singular (its realizations are Schwartz distributions at best)
- a spectral cut-off needs to be used to regularize the problem. See Figure 2 where the inertial range $[r_0, r_1]$ appears as a consequence of this cut-off.

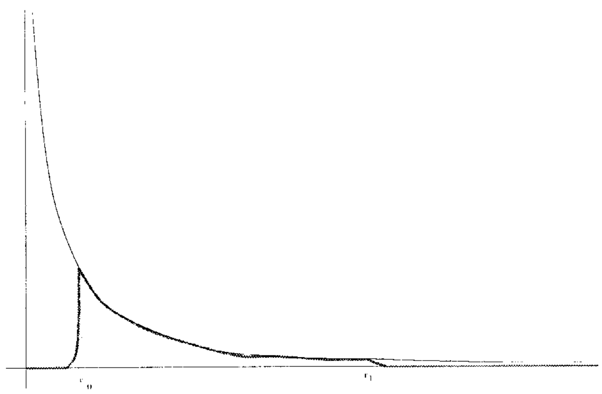

Figure 2. Cut-off spectral density $\mathcal{E}_c(r)$.

We say that the velocity field is regular when the spectral cut-off is used or more generally when the spectral density is integrable. In this case the realizations of the velocity field are bona fide functions $(t, \mathbf{x}) \hookrightarrow \phi(t, \mathbf{x})$. Notice that they are analytic functions of the $\mathbf{x}$ when the cut-off is used.

The analysis of the transport properties of the flow is performed by studying the characteristics of the solutions of the equation of motion:

$$dX_t = \vec{\mathbf{v}}(t, X_t)dt + \sqrt{2\kappa}dW_t \tag{1}$$

where $\mathbf{W} = \{W_t;\ t \geq 0\}$ is a Wiener process and $\kappa \geq 0$ is the diffusivity constant. Here X_t represents the position of a passive tracer. This equation of motion makes perfectly good sense when the velocity field is regular. It does not make sense when the spectral singularity forces the realizations of the velocity field to be Schwartz distributions which are not functions. So we shall only consider the equation of motion (1) when the velocity field is not singular !!

We close this introduction with the partial differential (PDE for short) formulation of the problem. Given a function $f(\mathbf{x})$ the quantity:

$$C(t, \mathbf{x}) = \mathbb{E}\{f(X_t)\}$$

(where we use the notation $\mathbb{E}\{\ \cdot\ \}$ to denote the expectation over the ensemble of velocity field realizations) can be interpreted as the concentration of tracers at time t if f is the concentration at time $t = 0$. This interpretation can be used to analyze quantities such as the temperature, the salinity, ... of the fluid in question. A simple application of Ito's calculus shows that this function is a solution of the equation

$$\begin{cases} \frac{\partial C(t,\mathbf{x})}{\partial t} = & \kappa \Delta C(t, \mathbf{x}) - (\vec{\mathbf{v}}(t, \mathbf{x}) \cdot \nabla) C(t, \mathbf{x}) \\ C(0, \mathbf{x}) = & f(\mathbf{x}), \end{cases} \tag{2}$$

which we shall call the concentration PDE. Note that since the velocity field is random, this equation is in fact a stochastic partial differential equation (SPDE for short). Also, since the velocity field is assumed to be mean zero Gaussian, the solutions of this equation keep the same statistical properties if we replace the minus sign by a plus, which we shall do in what follows.

2. First Simulations

Early simulations [*Carmona et al.,*, 1994] performed on a Massively Parallel machine (MASPAR II) gave a first taste of the theoretical results one could expect. These simulations were based on the *finite Fourier modes* model given in formula (10) below. We review the existing works on numerical simulations which we know of in Section 5 below.

Remark. Our first goal was to test the validity of the random velocity field model using real life drifter daily measurements. It is a well known fact that, if $\{X_t^{(1)};\ t \geq 0\}$ and $\{X_t^{(2)};\ t \geq 0\}$ are two solutions of the equation of motion (1) which correspond to two different initial conditions $X_0^{(1)} = \mathbf{x}^{(1)}$ and $X_0^{(1)} = \mathbf{x}^{(1)}$, then these two $\mathbb{R}^2$-valued processes are stationary in t and they have the same spectra. To use this theoretical fact to test our model we developed statistical test procedures to test if two given bivariate time series have the same spectra and we ran these tests on daily measurements of drifters at the surface of the ocean. See *Carmona and Wang* [1994]. The null hypothesis was rejected systematically: our model does not apply at the scale of the data at our disposal.

Our first simulations are at the origin of the theoretical results subsequently proven. For example, it was clear from the simulations that the longer the trajectory of a passive tracer. the more it looks like the trajectory of a Brownian particle. So our first attempt was to prove

- Homogenization in *Carmona and Xu* [1997] and
- Positivity of the (upper) Lyapunov exponent of the Jacobian flow in *Carmona et al.* [1997].

The positivity of the Lyapunov exponent is a very important property of the system. It holds in the case of Brownian flows because of the independence of the velocity increments (see *Le Jan* [1984] for a proof) but it does not hold in the case of a shear flow (see *Carmona* [1997]). It implies strong (exponential) ergodicity of the flow. This exponential mixing of the flow is at the origin of most of the numerical results presented in Section 5 but it is also a crucial ingredient in the mathematical proofs of the homogenization and renormalization results. As we see when discussing more general renormalization theories in Section 4 below and the conjectures of Section 5, the theoretical issues are far from being clear. There are still many theoretical challenges remaining.

3. Homogenization

Homogenization is a mathematical procedure to obtain effective equations when the phenomenon has two scales with an *infinite* separation between the scales. It has been used by analysts for equations with periodic coefficients and by probabilists for stochastic equations with a reasonable correlation structure. For our purpose, the names associated with this theory are Kozlov, Papanicolaou, Varadhan, Kipnis, ... The homogenization for the concentration PDE (2) can be formulated as follows. Given a regular (random) velocity field $\{\vec{\mathbf{v}}(t,\mathbf{x})\}$ we consider for each $\delta > 0$ the solution $C^{(\delta)}(t,\mathbf{x})$ of the rescaled (SPDE) equation

$$\left\{ \begin{array}{ll} \frac{\partial C^{(\delta)}(t,\mathbf{x})}{\partial t} = & \kappa \Delta C^{(\delta)}(t,\mathbf{x}) \\ & \quad + \frac{1}{\delta}(\vec{\mathbf{v}}(\delta^{-2}t, \delta\mathbf{x}) \cdot \nabla) C^{(\delta)}(t,\mathbf{x}) \\ C^{(\delta)}(0,\mathbf{x}) = & f(\mathbf{x}), \end{array} \right. \tag{3}$$

and we show that $C^{(\delta)}(t,\mathbf{x})$ converges (in probability for example) toward the solution $C(t,\mathbf{x})$ of a simpler equation of a similar form, say,

$$\left\{ \begin{array}{ll} \frac{\partial C(t,\mathbf{x})}{\partial t} = & \kappa^* \Delta C(t,\mathbf{x}) \\ C^{(\delta)}(0,\mathbf{x}) = & f(\mathbf{x}), \end{array} \right. \tag{4}$$

This equation is called the *effective equation* and κ^* is called the *effective diffusivity.*

3.1. The Probabilistic Approach

In terms of the probabilistic equation of motion (1) the homogenization problem can be reformulated in the following way. First we rescale space and time

$$\left\{ \begin{array}{lll} \mathbf{x} & \rightarrow & \delta\mathbf{x} \\ t & \rightarrow & \delta^2 t \end{array} \right. \tag{5}$$

and we considered the rescaled motion given by the process $\{Y_t^{(\delta)} : t \geq 0\}$ defined by

$$Y_t^{(\delta)} = \delta X_{t/\delta^2} \tag{6}$$

and we show that this process converges in distribution toward a process of Brownian motion with variance κ^{*2}. Notice that this formulation is stronger than the previous one since one can recover the PDE homogenization result from the convergence in distribution of the rescaled motion.

There are well established techniques to prove homogenization for random systems (see, for example, *Papanicolaou and Varadhan* [1981] or the more recent work [*Olla*, 1994]), but they do not apply to the present model so we had to design a new proof in *Carmona and Xu* [1997].

3.2. The Effective Diffusivity Constant κ^*

If we consider the case of a velocity field with finitely many Fourier modes, the large time / large scale behavior of passive tracers is diffusive, since it has been proved rigorously in *Carmona and Xu* [1997] that homogenization holds even in the purely advective case (i.e., when $\kappa = 0$.) The proof of this result was extended to include models with continuous spectrum and to allow for the possibility of a spectral singularity as long as the later is not too strong (R. Carmona and L. Xu, *Renormalization theory for passive tracers in time-dependent incompressible random flows*, ms. in preparation). One can also find in *Carmona and Xu* [1997] upper and lower bounds for the effective diffusivity constant κ^* but these bounds (and especially the upper bound) are too rough. The importance of good estimates on κ^* was stressed in the fundamental works [*Taylor*, 1953; *Kraichnan*, 1950, 1974].

The nature of the dependence of the effective diffusivity κ^* upon the characteristics of the flow (for example the Kubo number or the Peclet number, ... or the parameters κ, ϵ and z in the model considered in this paper) remains a very challenging open mathematical problem. *Isichenko* [1992] contains an extensive review of the numerical results published up until 1992, including results on the time-independent case.

4. Renormalization Theories

As in the case of homogenization, we use a small parameter δ to define the "large time / large scale " regime we want to find effective equations for. But the main difference is that, instead of having one single regular (random) velocity field $\{\vec{v}(t,\mathbf{x})\}$ we consider for each $\delta > 0$ a different (regular) velocity field $\{\vec{v}_\delta(t,\mathbf{x})\}$, the idea being that this velocity field is the result of a regularization procedure to remove a singularity like the infrared spectral singularity in the Kolmogorov spectrum introduced earlier. We now rescale time and space with the same small parameter δ controlling the spectral cut-off

$$\begin{cases} \mathbf{x} & \rightarrow & \delta\mathbf{x} \\ t & \rightarrow & \rho(\delta)^2 t \end{cases} \tag{7}$$

for some function ρ to be determined. At this stage we only need that $\rho(\delta) \to 0$ as $\delta \searrow 0$. As before we denote by $C^{(\delta)}(t,\mathbf{x})$ the solution of the rescaled concentration equation

$$\begin{cases} \frac{\partial C^{(\delta)}(t,\mathbf{x})}{\partial t} = & \kappa\Delta C^{(\delta)}(t,\mathbf{x}) \\ & \quad + \frac{1}{\delta}(\vec{v}(\rho(\delta)^{-2}t, \delta\mathbf{x}) \cdot \nabla) C^{(\delta)}(t,\mathbf{x}) \\ C^{(\delta)}(0,\mathbf{x}) = & f(\mathbf{x}), \end{cases} \tag{8}$$

and for each choice of the spectral parameter (ϵ, z) we try to find a rescaling function ρ, usually in the form of a power law in δ, for which that $C^{(\delta)}(t,\mathbf{x})$ converges (in probability for example) toward the solution $C(t,\mathbf{x})$ of an effective equation. Comparing the exponent in the definition of $\rho(\delta)$ and the form of the effective equation makes it possible to talk about *anomalous diffusion* whether it is because of a *sub-diffusive* or a *super diffusive* behavior of the system (as opposed to the *diffusive* regime when homogenization holds.)

4.1. The Probabilistic Approach

As in the case of the homogenization we consider the stronger form of the problem given by the probabilistic approach. In other words we try to find, for each choice of the spectral parameter (ϵ, z), a rescaling function ρ for which the rescaled motion process $\{Y_t^{(\delta)} : t \geq 0\}$ defined now by

$$Y_t^{(\delta)} = \delta X_{t/\rho(\delta)^2} \tag{9}$$

converges in distribution toward a process to be identified.

We recently succeeded in extending some of the results of Avellaneda and Majda to the general situation considered here. But since the results are rather technical (and still incomplete) we shall restrict ourselves to the case of shear flows (R. Carmona and L. Xu, *Renormalization theory for passive tracers in time-dependent incompressible random flows*, ms. in preparation). Shear flows are convenient models because explicit computations are possible and they give complete answers to all the question raised so far. But the main conceptual difficulty still remains: which procedure should we follow to regularize the spectral singularity? Moreover, one should wonder if the renormalized theories depend upon these choices or if they are universal.

We address these questions in the remaining part of this section.

4.2. A Toy Model

A naive approach consists in letting both endpoints r_0 and r_1 of the inertial range go to 0 linearly with the small parameter δ. More precisely we shall assume that $\mathcal{E}_\delta(r) = r^{1-\epsilon}\chi(r/\delta)$ where $\chi(r)$ is equal to 1 on an interval (r_0, r_1) and 0 otherwise. This choice is consistent with the idea of analyzing the contribution of the large scale singularity and this particular way of removing the infrared cut-off does just that. We shall say that the renormalization procedure follows a TM scheme as a reference to our toy model. The mathematical analysis gives a partition of the (ϵ, z)-plane given in Figure 3.1.

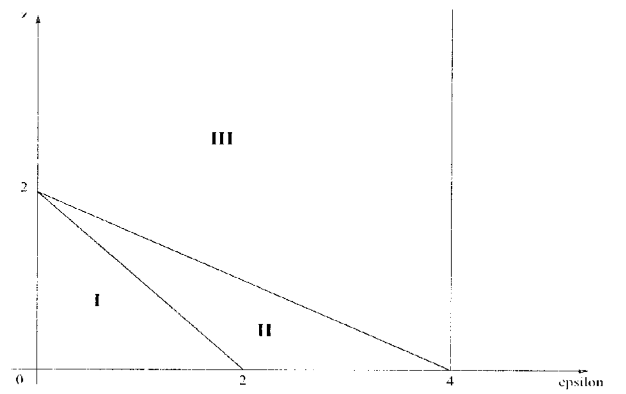

Figure 3.1. Partition of the (ϵ, z)-plane for the TM renormalization scheme

One can prove that

- if the spectral parameter (ϵ, z) is in region I and if we chose to rescale the time using $\rho(\delta) = \delta^{-1}$ then the rescaled motion converges in distribution toward a process of Brownian motion with variance $2\kappa t$
- if the spectral parameter (ϵ, z) is in region II and if we chose to rescale the time using $\rho(\delta) = \delta^{-2+(\epsilon+z)/2}$ then the rescaled motion converges in distribution toward a process of Brownian motion with variance $2\pi\left(\int_{r_0}^{r_1} |k|^{1-\epsilon-z} dr\right) t$
- if the spectral parameter (ϵ, z) is in region III and if we chose to rescale the time using $\rho(\delta) = \delta^{-1+\epsilon/4}$ then the rescaled motion converges in dis-

tribution toward a process of the form Rt where R is a mean zero Gaussian random variable with variance $\pi\left(\int_{r_0}^{r_1} |k|^{1-\epsilon} dr\right) t$

One sees that regimes I and II are similar to the homogenization since they are diffusive. Notice the difference in the time rescaling and most importantly in the reason for the diffusive behavior. Looking at the expression for the variance of the limiting Brownian motion we see that the original diffusion dominates in region I while the diffusion is created by the convection term and the singularity in region II. The asymptotic motion is ballistic in region III.

4.3. The Avellaneda-Majda Renormalization Theory

The second approach is due to Avellaneda and Majda. In a series of path-breaking papers starting with *Avellaneda and Majda* [1990a,b], they analyze the possible nontrivial limiting regimes when $r_1 = 1$ and $r_0 = \delta$. In each partition of the (ϵ, z)-plane given in Figure 3.2 they determine a spatial renormalization rate $\rho(\delta)$ and they identify the limit of the suitably rescaled solution of the concentration PDE (2).

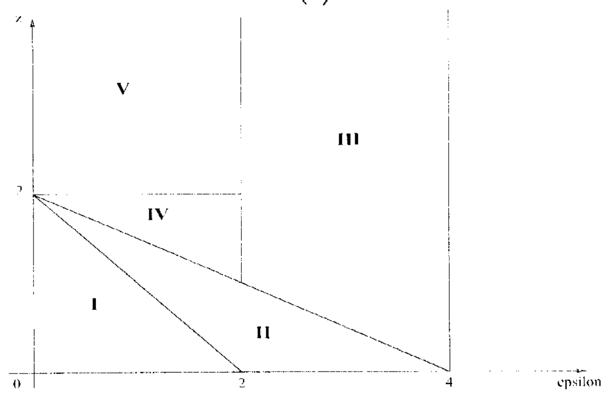

Figure 3.2. Partition of the (ϵ, z)-plane for the AM renormalization scheme

One can prove that

- if the spectral parameter (ϵ, z) is in region I and if we chose to rescale the time using $\rho(\delta) = \delta^{-1}$ then the rescaled motion converges in distribution toward a process of a Brownian motion with variance $\kappa + (4\pi/(2-\epsilon-z))t$
- if the spectral parameter (ϵ, z) is in region II and if we chose to rescale the time using $\rho(\delta) = \delta^{-2+(\epsilon+z)/2}$ then the rescaled motion converges in distribution toward a process of Brownian motion with variance $4\pi\left(\int_1^\infty |k|^{1-\epsilon-z} dr\right) t$
- if the spectral parameter (ϵ, z) is in region III and if we chose to rescale the time using $\rho(\delta) = \delta^{-1+\epsilon/4}$ then the rescaled motion converges in distribution toward a process of the form Rt where R is a mean zero Gaussian random variable with variance $2\pi\left(\int_1^\infty |k|^{1-\epsilon} dr\right) t$
- if the spectral parameter (ϵ, z) is in region IV and if we chose to rescale the time using $\rho(\delta) = \delta^{-z/(2(z-1)+\epsilon)}$ then the rescaled motion converges in distribution toward a fractional Brownian motion $B_f(t)$ with structure function:

$$\mathbb{E}\{[B_f(t) - B_f(0)]^2\} = \pi \int_{-\infty}^{\infty} |k|^{1-\epsilon-z} [1 - |k|^{-z}(1 - e^{-|k|^z})]\, dk\ t^{2-\frac{2-\epsilon}{z}}$$

- if the spectral parameter (ϵ, z) is in region V and if we chose to rescale the time using $\rho(\delta) = \delta^{-2/(2+\epsilon)}$ then the rescaled motion converges in distribution toward a process with the distribution of $\int_0^t M(\sqrt{2\kappa}Z(s))\, ds$ where $\{M(x);\ x \in \mathbb{R}\}$ is a mean zero Gaussian stationary random field with spectral density is $|k|^{1-\epsilon}$ and where $\{Z(s);\ s \geq 0\}$ is an independent standard Brownian motion

One sees that the first three regimes are the same as in the TM case. This means that in these regimes, the small scales (which were not included in the toy model) do not play a significant role after the rescaling.

4.4. Still Another Approach

Finally we discuss a possible third approach which we base on the following premise: why should we use an infrared cut-off when we do not need it. This point was advocated in *Glimm and Zhang* [1992], so we shall say that the renormalization procedure follows a GZ scheme. We then consider only the part of the (ϵ, z) plane where $\epsilon < 2$ and we do not use the cut-off. The net result is that, like in the case of homogenization, we use only one velocity field, it does not change with the small parameter δ. In this case one finds that the partition of the (ϵ, z)-plane becomes as shown here.

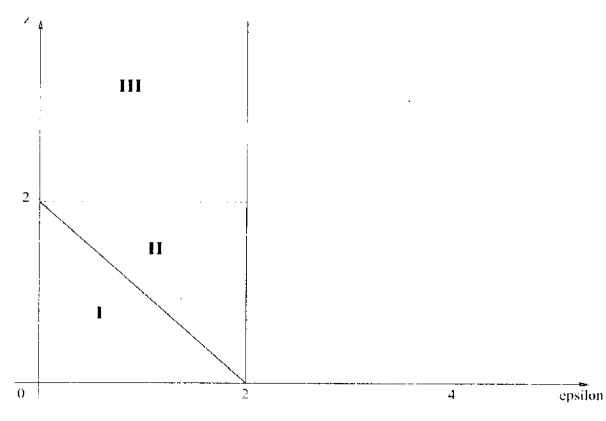

Figure 3.3. Partition of the (ϵ, z)-plane for the GZ renormalization scheme

One can prove that

- if the spectral parameter (ϵ, z) is in region I and if we chose to rescale the time using $\rho(\delta) = \delta^{-1}$ then the rescaled motion converges in distribution toward a process of a Brownian motion with variance $\kappa + (4\pi/(2-\epsilon-z))t$
- if the spectral parameter (ϵ, z) is in region II and if we chose to rescale the time using $\rho(\delta) = \delta^{-z/(2(z-1)+\epsilon)}$ then the rescaled motion converges in distribution toward a fractional Brownian motion $B_f(t)$ with structure function:

$$\mathbb{E}\{[B_f(t) - B_f(0)]^2\} = \pi \int_{-\infty}^{\infty} |k|^{1-\epsilon} [|k|^{-z} - |k|^{-2z}(1 - e^{-|k|^z})]\, dk) t^{2-\frac{2-\epsilon}{z}}$$

- if the spectral parameter (ϵ, z) is in region III and if we chose to rescale the time using $\rho(\delta) = \delta^{-2/(2+\epsilon)}$ then the rescaled motion converges in distribution toward a fractional Brownian motion with structure function:

$$\mathbb{E}\{[B_f(t) - B_f(0)]^2\} = \pi \int_{-\infty}^{\infty} |k|^{-\epsilon-1} [1 - |k|^{-2}(1 - e^{-|k|^2})]\, dk \;\; t^{2-\frac{2-\epsilon}{2}}$$

The first of these three regimes is identical to regime I in the AM theory but the other ones are different even though they bare some definite resemblance with the regimes IV and V respectively.

Remark. Notice that our point of view is different from the approach of *Bernard et al.* [1996] where the renormalization of the moments of the solutions of the concentration PDE is considered when the velocity field is a mean zero Gaussian field which is white in time and when a similar and statistically independent forcing term is added on the right hand side of (2).

5. Numerical Simulations and more Mathematical Conjectures

We now abandon the particular case of the shear flows and we come back to the general situation of the random velocity fields of the Ornstein-Ulhenbeck type (i.e., with a Kolmogorov spectrum.) The numerical simulations reported here have been performed by relying on a finite Fourier mode approximation of the spectral representation of the stream function (and consequently the velocity field.)

We choose a finite subset $\mathcal{K}$ (see Figure 4 for a typical choice of the set $\mathcal{K}$) of Fourier modes and we set

$$\phi(t,x) = \sum_{k\in\mathcal{K}} a_k(t)\cos(k\cdot x) + b_k(t)\sin(k\cdot x) \qquad (10)$$

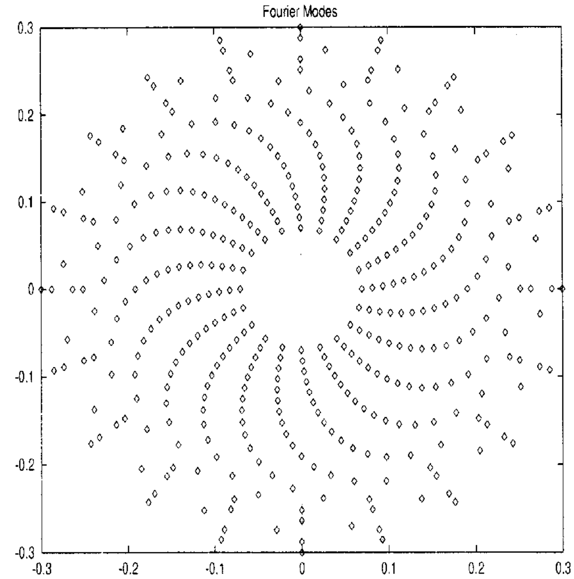

Figure 4. Typical choice for a set of Fourier modes

The reader can consult the Web page (http://www.princeton.edu) to see animations based on this model of the time evolutions of

- the velocity field intensity
- the velocity field direction
- the corresponding field of vorticity
- quivers in a small region

Other animations are also available at this URL. In particular, one of them shows the evolution of *drifters originally in a square.* Its goal is to illustrate the strong *mixing* of the flow which we attributed earlier to the positivity of the upper Lyapunov exponent of the Jacobian flow.

These animations were prepared to illustrate some of the points made in the review article of *Carmona and Cerou* [1997]. They show that the Gaussian models contain some of the vorticity features of the Poisson shot-noise models such as those discussed by E. Cinlar (unpublished information, 1994). These animations underline the time dependence of the velocity field. This time dependence can be found in some of the simulations reported in *Isichenko*[1992], but it is not present in the more recent works of Majda and his collaborators who implemented in the case of shear flows wavelet expansions as an alternative to the more traditional Fourier expansions. See *Elliott and Majda* [1994] and *Elliott et al.*[1997]. These expansions are desirable because of the presence of the spectral singularity, but unfortunately they are difficult to implement when it comes to Lagrangian simulations and their use seems to be restricted to the stationary (i.e., time independent) models. This point is argued in details in *Carmona and Cerou* [1997].

5.1. Interface Motions & Liquid Curves

We now switch to the discussion of the motion of interfaces and the so-called liquid curves. Let us consider a curve parameterized at time $t = 0$ by

$$[0,1] \ni \alpha \hookrightarrow \gamma_0(\alpha).$$

We choose a circle in the animations of *Carmona* [URL: http://www.princeton.edu/ rcarmona]. At each time t we consider the effect of the transport by the flow by considering the curve

$$[0,1] \ni \alpha \hookrightarrow \gamma_t(\alpha) = \varphi_{0,t}(\gamma_0(\alpha))$$

where we used the notation $\varphi_{0,t}$ for the (stochastic) motion flow. In other words, for each $\mathbf{x} \in \mathbb{R}$, $\varphi_{0,t}(\mathbf{x})$ is the solution X_t of the equation of motion (1) when the initial condition is $\mathbf{x}$, i.e., when $X_0 = \mathbf{x}$. Notice that at each time $t > 0$ the curve γ_t is C^∞. Nevertheless, Figure 5 (and most importantly the animations from *Carmona* [URL: http://www.princeton.edu/ rcarmona]) show the complexity of the shapes produced by the mixing of the flow and how intricate the curve γ_t (called *liquid curve*) can be, even when the initial curve γ_0 is as simple a curve as a circle. Notice that, when the curve is closed, the area inside the curve remains constant over time because of the incompressibility assumption.

5.2. Interface & Liquid Curve Lengths

It was shown in *Carmona* [1997] that the length of the liquid curves was growing at most linearly in the case of a shear flow. It was also argued that this length grows at least like $e^{\lambda t}$ where λ is the upper Lyapunov exponent of the Jacobian flow and it was conjectured on the basis of extensive numerical experiments that this exponent was strictly positive whenever the flow was not degenerate (i.e., not a shear flow.) This conjecture (which was known to be true in the case of the Brownian flows since *Le Jan* [1984] was proved (at least for the finite Fourier mode velocity fields) in *Carmona et al.* [1997] .

5.3. Fractal Dimension

Figure 5 (and the animations in *Carmona* [URL: http://www.princeton.edu/ rcarmona]) show that the liquid curves have a tendency to fill up the plane. This suggests that they should have a fractal dimension greater than 1. But this last statement has to be *taken with a grain of salt* since as we stated before, the liquid curves are C^∞ at all time if they are C^∞ at time $t = 0$. We try to measure how close to a fractal the liquid curves are by trying to approximate their Minkowski dimension. So for each $\epsilon > 0$ we consider the area $N_\epsilon(t)$ of the ϵ-sausage around the curve γ_t as shown in Figure 6.1.

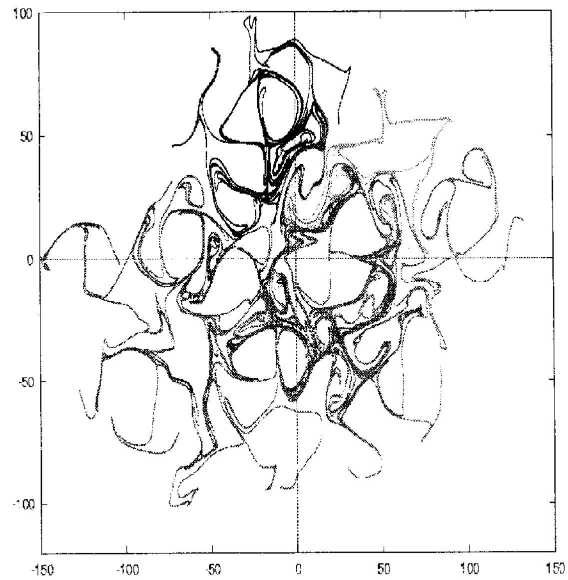

Figure 5. Liquid curve which started from a circle.

We expect that

- $N_\epsilon(t) \sim \epsilon$ as $\epsilon \searrow 0$ since the curve is C^∞ after all !!!
- obviously that $N_\epsilon(t) \sim \epsilon^2$ as ϵ is large since a single ball of radius ϵ can cover the whole γ_t when ϵ is large enough (compared to t) and we hope that:
- $N_\epsilon(t) \sim \epsilon^{2-d}$ in an intermediate regime for which $0 << \epsilon << t$

We shall say that the liquid curve or the interface has (Minkowski) dimension d if such an intermediate regime can be found.

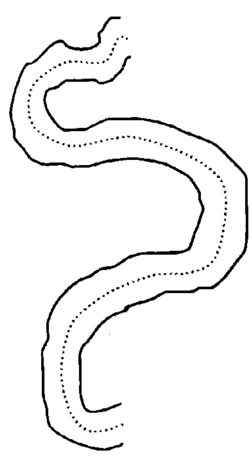

Figure 6.1. ϵ - sausage around the liquid curve γ_t.

Estimations

Figure 6.2 shows an attempt at identifying the intermediate regime and a reasonable candidate for the dimension. We see plots of estimates of the logarithms of $N_\epsilon(t)$ as a function of the logarithm of ϵ for different values of t ($t = 1$ is on top while $t = 0$ is at the bottom.) The estimate of the value of $N_\epsilon(t)$ was computed using a Monte Carlo method. One could argue for the existence of an intermediate regime and an effective dimension of $d \approx 1.5$ since the slope in the intermediate regime seems to be willing to be close to 0.5, but we have to admit that the results are not very convincing.

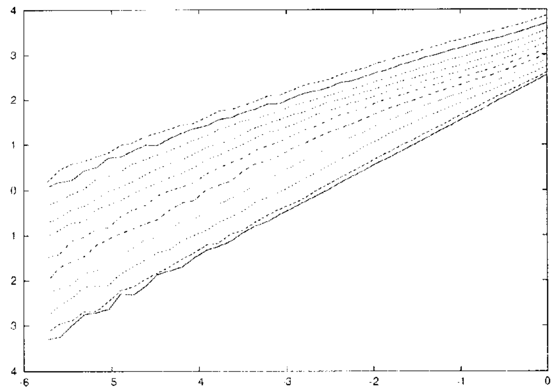

Figure 6.2. Logarithm of $N_\epsilon(t)$ versus $\log \epsilon$.

5.4. Diameter Asymptotics

We now consider the diameter of the region enclosed by the liquid curve, namely the quantity

$$d_t = \text{diam}(\gamma_t)$$

and we ask the question of the almost sure behavior of d_t as $t \to \infty$. The simpler problem of the time evolution of the center of mass and of the moment of inertia of initially well localized mass distributions has been considered in *Cinlar and Zirbel* [1996] in the case of Brownian flows. In fact, it is easy to see (at least for a Brownian flow for which the random velocity is white in time)

$$\liminf_{t\to\infty} \frac{d_t}{\sqrt{2t \log\log t}} = \infty \qquad \text{a.s.}$$

and people have conjectured that the asymptotic behavior of the diameter of the liquid curves should be

$$d_t \approx t \qquad \text{a.s.}$$

i.e., the large time behavior of the diameter should be **almost surely linear**!! See, for example, *Isichenko* [1992]. We heard independently this conjecture from Ya. Sinai (private communication, 1997). In any case, this is a very challenging open mathematical problem, even in the case of Brownian flows the answer is not known.

Percentiles

In order to get a feel for the conjecture, we plotted the percentiles of the distance to the origin (or any fixed point for that matter). One can see in Figure 7 that the diameter (or equivalently the maximum of the moduli) increases significantly with the number of points while the percentiles behave differently: this indicates (as confirmed by the animation) that the maximum is produced by thin long fingers !

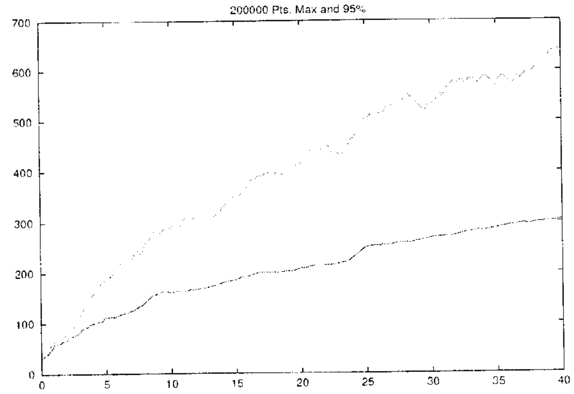

Figure 7. Maximum and 95%-tile of the moduli of 200,000 points initially on the unit circle

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