A Simple Model of Abyssal Flow

Rick Salmon

Scripps Institution of Oceanography, University of California at San Diego, La Jolla Ca 92093-0225

Abstract. The planetary geostrophic equations (PGE) have special properties that greatly facilitate analytical and numerical solution. In particular, when the potential vorticity is assumed to be an arbitrarily prescribed function of the buoyancy, then the ideal three-dimensional PGE exactly reduce to a pair of coupled equations in two space dimensions. As an example of this method of reduction. I offer a simple model of abyssal flow in the southwestern Pacific.

I, among many others, have advocated the use of ocean models based upon the planetary geostrophic equations (hereafter PGE),

$$
\begin{aligned}
-fv &= -\frac{1}{r\cos\theta}\frac{\partial p}{\partial\lambda} - \varepsilon u \\
fu &= -\frac{1}{r}\frac{\partial p}{\partial\theta} - \varepsilon v \\
0 &= -\frac{\partial p}{\partial z} + T \\
&\frac{\partial}{\partial\lambda}\left(\frac{u}{r\cos\theta}\right) + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{v}{r}\right) + \frac{\partial w}{\partial z} = 0
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
\frac{\partial T}{\partial t} + \frac{u}{r\cos\theta}\frac{\partial T}{\partial\lambda} + \frac{v}{r}\frac{\partial T}{\partial\theta} + w\frac{\partial T}{\partial z} = \\
\nabla\cdot(\kappa\nabla T) + \frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right)
\end{aligned}
\tag{2}
$$

Here, θ is the latitude, λ the longitude, z is the vertical distance, r the radius of the Earth, T is the buoyancy (which I will call temperature), and the other symbols have their conventional meanings. Wind- and thermal forcing terms can also be appended.

The defining characteristic of the PGE (1-2) is their complete neglect of inertia, leading to *linear* equations of motion *except* for the advection of temperature in (2). However, this single nonlinearity is enough to make the PGE dynamics both challenging and very rich. Nevertheless, because of their relative simplicity (compared to say, the primitive equations) the PGE have a number of important advantages.

First, numerical solutions of the PGE (with steady forcing) seem always to approach a steady state. This effectively reduces the number of independent variables by one, and it means that if only the final steady state is of interest, it can usually be found using numerical relaxation methods that are much more efficient than time-stepping.

Second, since the PGE omit the advection of momentum, they do not require a diffusive (i.e., Laplacian) friction. In fact, the simpler Rayleigh friction in (1) is sufficient to meet boundary conditions of *no-normal-flow* at rigid boundaries *provided* that the ocean depth vanishes smoothly at the coastline (Salmon, 1986, 1992). (However, if any part of the boundary is vertical, then the vertical momentum equation (1c) must also contain a Rayleigh friction term.) The simpler Rayleigh friction leads to a much simpler boundary- and internal-layer structure and greatly facilitates analytical and numerical solution.

Third, analytical and numerical solutions of the PGE contain internal boundary layers of thickness $\kappa^{1/2}$, corresponding to the ocean's mean thermocline (Salmon 1990, Salmon and Hollerbach 1991) and leading to a picture of the subtropical ocean as two *inhomogeneous* layers in which temperature diffusion is unimportant, separated by a thin region in which T changes rapidly and diffusion is important *no matter how small the value of* κ. This result calls into question the many attempts to explain the structure of the main thermocline on the basis of the ideal (κ=0) equations and to justify such explanation by appeal to the smallness of measured values of κ.

Finally, on account of their simplicity, the ideal PGE admit an exact reduction from three to two space dimensions. This reduction, which leads to equations that generalize the conventional two-layer PGE equations, further facilitates analytical and numerical solution. In this brief report on work in progress, we show how the reduction principle can be used to obtain a simple equation governing the flow of a dense layer of fluid along the ocean bottom. The simplicity of the dynamics offsets the difficulty of incorporating real bathymetry and makes the results easier to understand.

The basic idea of reduction goes back to Welander (1971) and Needler (1971). In the ideal-fluid limit (ε=κ=0), the PGE conserve the temperature and potential vorticity on fluid particles,

$$\frac{DT}{Dt}=0, \quad \frac{D}{Dt}\left(f\frac{\partial T}{\partial z}\right)=0. \tag{3}$$

Therefore, the ansatz,

$$f\frac{\partial T}{\partial z}=G(T), \tag{4}$$

where G( ) is an arbitrary function is *consistent*, in that sense that if (4) holds at some initial time, it then holds at all future times. But (4) integrates immediately to

$$T=F''\left(\frac{z}{f}+S(\lambda,\theta,t)\right), \tag{5}$$

where $F''()$ is another arbitrary function, related to G, and the primes, which denote differentiation, are introduced for later convenience. $S(\lambda,\theta,t)$ is a function of integration, *independent of* z, which must be determined by substituting (5) back into (1-2). The result (still assuming $\varepsilon=\kappa=0$) is

$$r^2 f\frac{\partial S}{\partial t}+\frac{1}{\cos\theta}\frac{\partial(P,S)}{\partial(\lambda,\theta)}+\frac{1}{f\sin\theta}\frac{\partial D}{\partial\lambda}=0, \tag{6}$$

an evolution equation for $S(\lambda,\theta,t)$ in which the vertical coordinate z does not appear.

The S-equation (6) contains two additional dependent variables, $P(\lambda,\theta,t)$ and $D(\lambda,\theta,t)$, which are determined by boundary conditions at the top and bottom of the ocean. If these boundary conditions are taken to be no-normal-flow at the ocean surface and bottom, then D is easily determined, and P (or alternatively ψ, the streamfunction for the vertically integrated horizontal velocity) is determined by a second equation, also containing S. The dynamics then reduce to a pair of coupled equations in $S(\lambda,\theta,t)$ and $\psi(\lambda,\theta,t)$, which together determine the whole flow. Salmon (1994) called these the *generalized two-layer equations* (GTLE), because they reduce to the conventional, two-homogeneous-layer analogue of (1-2) when the arbitrary function $F''()$ in (5) is chosen to be a Heaviside function. However, other choices of $F''()$ were found to be both more realistic and numerically convenient. In particular, the conventional two-layer model is an inconvenient basis for numerical modeling because of the difficulty in following the *outcropping line* at which the meniscus between the layers intersects the ocean surface or bottom. However, if two nearly homogeneous layers are really wanted, then $F''()$ can be chosen to be a function that changes rapidly but *continuously* between temperature values corresponding to the two layers. The outcropping lines are then regions of rapid but continuous temperature change, which need not be explicitly followed.

Of course, full-basin solutions require wind- and thermal forcing, and the PGE can satisfy *coastal* boundary conditions of no net transport across coastlines only if $\varepsilon \neq 0$. Moreover, if the model ocean spans the equator, then the ansatz (5) itself contains a singularity at $f=0$. To avoid this singularity, Salmon (1994) generalized (5) to

$$T=F''\left(\frac{z}{\sqrt{f^2+f_0^2}}+S(\lambda,\theta,t)\right), \tag{7}$$

where f_o is a small constant. With T of the form (7), the linear equations (1) and no-normal-flow boundary conditions can still be completely satisfied; they serve to determine the velocity field (u,v,w) in terms of T. However, substitution of this velocity field and (7) back into (2) no longer yields a z-independent equation. This is because of the modification of (5) to (7) and because forcing, friction, and diffusion anyway destroy the conservation properties (3) on which (5) relies. Salmon (1994) therefore replaced (2) by its vertical average. The resulting equation, which is essentially (6) with forcing and dissipation terms appended, has "three-dimensional accuracy" except at very low latitude, where the f_o term in (7) is significant, and the GTLE have the character of a Galerkin approximation. For many further details, refer to Salmon (1994).

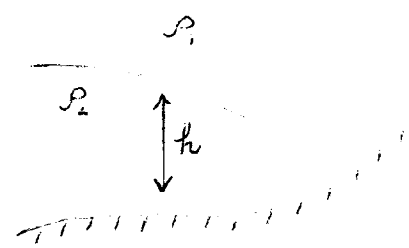

Figure 1. A simple model of the flow of dense water along a bumpy ocean bottom—the so-called one-and-one-half layer model. The water above the moving layer is assumed to be at rest.

In this note, we consider an ansatz of the general form (5) or (7) that leads to dynamics even simpler than the generalized two-layer equation. This new model, which could accurately be described as a *generalized one-and-one-half layer model*, bears the same relation to the simple model of abyssal flow shown in Figure 1, as the GTLE bear to the conventional two-layer model. In Figure 1, the upper fluid layer is assumed to be infinitely deep and at rest, so that (neglecting the inertia, the lower-layer dynamics are governed by

$$-fv=-\frac{g'}{r\cos\theta}\frac{\partial\eta}{\partial\lambda}-\varepsilon u$$

$$+fu = \frac{g'}{r}\frac{\partial\eta}{\partial\theta} - \varepsilon v$$
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial\lambda}\left(\frac{hu}{r\cos\theta}\right) + \frac{1}{r\cos\theta}\frac{\partial}{\partial\theta}(hv\cos\theta) = 0 \tag{8}$$

where h is the thickness of the moving layer, $\eta = h - H$ is the height of the interface between the layers, and g' is the reduced gravity. Eliminating u and v yields a single equation for h,

$$\frac{\partial h}{\partial t} + \frac{g'}{r^2\cos\theta}\frac{\partial\left(h-H, \frac{f}{f^2+\varepsilon^2}h\right)}{\partial(\lambda,\theta)} = \varepsilon g'\nabla\cdot\left[\frac{h}{f^2+\varepsilon^2}\nabla(h-H)\right]. \tag{9}$$

Now consider the more general case in which the temperature is given by (5), but still approaches a uniform value (conveniently taken to be zero) as $z \to \infty$. This imposes the condition $F''(\infty) = 0$ on the profile function. The quantities P and D are now determined by the requirements that the horizontal velocity (u,v) vanish as $z \to \infty$ and that there be no flow through the ocean bottom at $z = -H$. The ideal ($\varepsilon = 0$) S-equation (6) takes the form of a "bottom-layer" potential vorticity equation,

$$\frac{\partial q}{\partial t} - \frac{F'(q)}{r^2\sin\theta}\frac{\partial q}{\partial\lambda} + \frac{F''(q)}{r^2 f\cos\theta}\frac{\partial(H,q)}{\partial(\lambda,\theta)} = 0, \tag{10}$$

where

$$q \equiv S - \frac{H}{f}, \tag{11}$$

and $F'''(q)$ is the potential vorticity fT_z at the ocean bottom. The second term in (10) induces a westward propagation of q at speed determined by q itself; the third term propagates q along isobaths in the sense of clockwise propagation around deeps in the southern hemisphere.

Next, assuming $\varepsilon \neq 0$, adopting (7) instead of (5), and following the procedure summarized after (7), we obtain the frictional generalization of (10), namely

$$\frac{\partial q}{\partial t} - \frac{f^3}{\sqrt{f^2+f_0^2}\left(f^2+\varepsilon^2\right)}\frac{F'(q)}{r^2\sin\theta}\frac{\partial q}{\partial\lambda} + \frac{f}{\left(f^2+\varepsilon^2\right)}\frac{F''(q)}{r^2\cos\theta}\frac{\partial(H,q)}{\partial(\lambda,\theta)} = \textit{diffusion} \tag{12}$$

where the right-hand side stands for relatively simple diffusion terms which will not be written out. In the special case

$$F''(\xi) = \begin{cases} 0, & \xi > 0 \\ -g' & \xi < 0 \end{cases} \tag{13}$$

in which the profile function is a step function, (12) reduces exactly to (9) with $q = -h/f$ (the layer-depth potential vorticity).

I have used (12) as the basis for a simple numerical model of bottom water flowing northward in the southwestern Pacific. Observations (Mantyla and Reid 1983, Taft et al. 1991) show that the densest abyssal water enters the North Pacific through the Samoa Passage at 10°S, 170°W. I solve (12) on an open computational domain extending from 25°S to 5°S, and from 180° to 160°W, including the Samoa Passage and several apparently less important passages for the northward moving bottom water. Refer to Figure 2. The temperature profile function is a "smooth step,"

$$F''(\xi) = \frac{1}{2}T_0\left[\tanh(\xi/\Delta) - 1\right] \tag{14}$$

with constant amplitude T_o and "step-width" Δ chosen to agree roughly with the local Levitus data. The boundary conditions are fixed q (i.e., fixed temperature) on the computational domain. The Rayleigh damping coefficient ε is 0.15 times a representative value of f.

The calculation begins from a state (Figure 2, top) in which the cold bottom water is pressed against the southern computational boundary at 25°S. As time increases, this cold water spreads northward, steered by the bathymetry in the potential vorticity equation (12). After 128 days (Figure 3a), the cold, dense water has filled the Tonga Trench, turning westward (with the axis of the trench around the Samoa Islands. By 558 days (Figure 3b), significant flow is also occurring around the eastern side of the islands. By 1192 days (Figure 3c) cold water is spilling through the Samoa Passage and through a shallower passage in the Robbie Ridge at 175°W. The maximum current speed of 9.8 km day^{-1} in the Samoa Passage agrees well with recent direct measurements of the current (Dan Rudnick personal communication)

Figure 4 shows three temperature sections at 1563 days. The north-south section at mid-domain (Figure 4, bottom) passing through the Samoa Passage shows that the cold water has nearly reached the northern computational boundary. A section along 17°S (Figure 4, middle) shows how the cold water has filled up the abyss south of Samoa. The top section in Figure 4 corresponds to the broken line on Figure 2 (bottom) and crosses the axes of all the important passages into the North Pacific. This top section shows the coldest water flowing northward through the Samoa Passage, but cold water is also flowing northward through the passes in the Robbie Ridge, and to the east of the Manihiki Plateau.

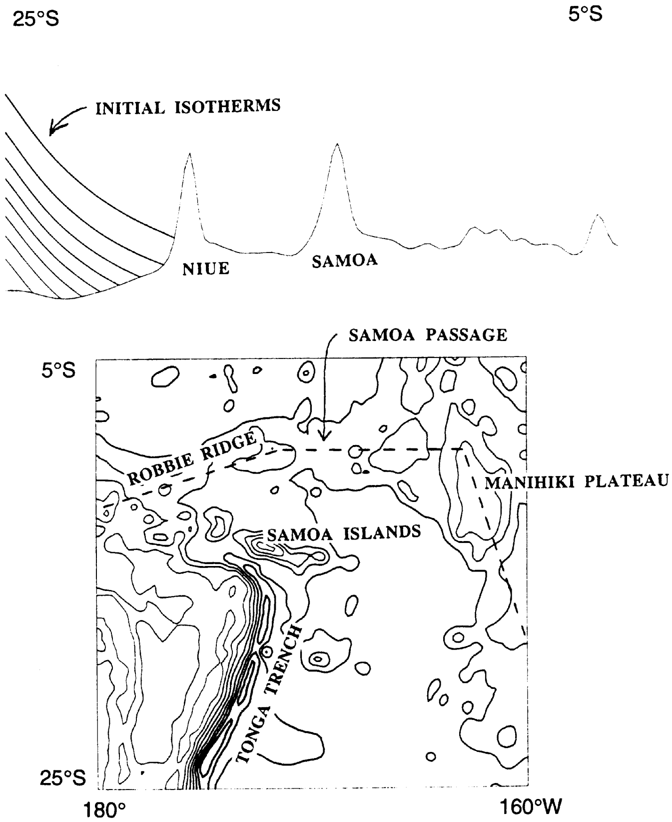

Figure 2. The ocean depth (bottom) and initial temperature (top) in a mid-domain section along 170°W in the preliminary study of the northward spread of bottom water in the southwest Pacific. The computational domain is open, with boundary conditions of prescribed temperature. The model topography is a smoother version of the "etopo-5 topography with a grid spacing of 0.1667 degrees in latitude and longitude.

The initial-value calculation summarized on Figures 2 to 4 is mainly intended to show the feasibility of using reduced-PGE models with realistic bathymetry. I chose the Samoa Passage for its importance as a source of North Pacific abyssal water and because the assumption of a level-of-no-motion far above the bottom is perhaps easier to defend there than in other places. With one-sixth-degree resolution (121 by 121 grids) the calculation required three hours CPU time on a Sparc-120 workstation. Thus even higher spatial resolution and a broader computational domain are quite feasible. I am particularly interested in the influence of spatial resolution (i.e., the very small scales in the bathymetry) on the solutions.

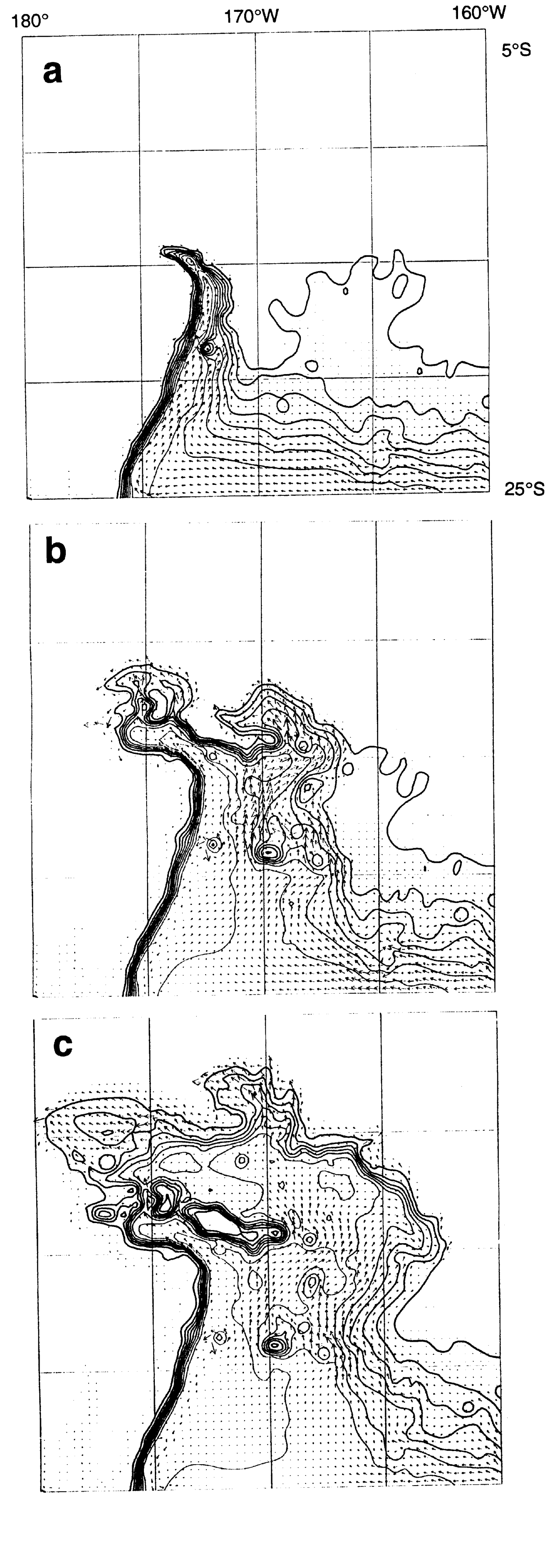

Figure 3. The temperature (contours) and horizontal velocity (arrows) at the ocean bottom at three successive times in the numerical solution of (13) with the bottom topography shown in Figure 2. (a) The flow after 128 days, (b) after 558 days, (c) after 1192 days. ➔

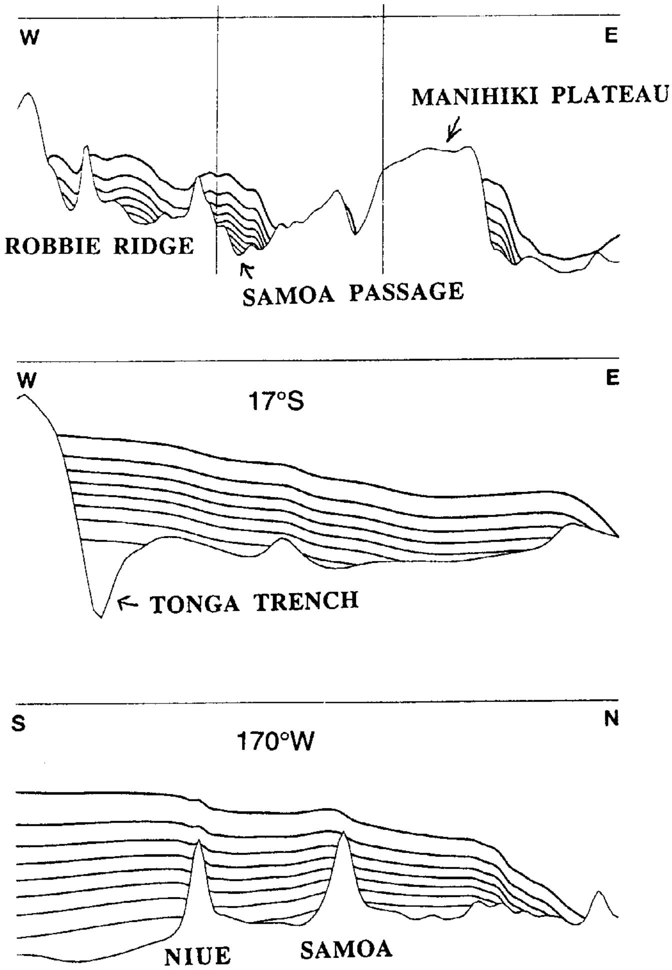

Figure 4. Temperature after 1563 days along 170°W (bottom), 17°S (middle), and (top) along the dashed line shown on Figure 3, which crosses the axes of all the major passages into the North Pacific. The densest water and highest flow speeds occur in the Samoa Passage, but the top section also shows significant deep flow through the pass in the Robbie Ridge, and east of the Manihiki Plateau.

Of course it is a great conceit to imagine that one can explain the observed ocean flow with no inertia whatsoever; with the highly constrained buoyancy (7) required for reduction to two space dimensions; or without separate equations for the temperature and salinity. And I am certainly aware of the widely held opinion that flow through some deep ocean passes is hydraulically (and therefore inertially) controlled. But for me it is very important to start with dynamics, like (12), which is simple enough that its solutions can be physically understood, this simple dynamics will certainly not explain everything that is observed, but what it can explain can at least be understood. However, my general strategy is grounded in the belief that accurate incorporation of realistic bathymetry may actually be more important than much of the "higher order" physics.

Acknowledgment. Work was supported by the National Science Foundation, OCE-92-16412. It is a pleasure to thank Rupert Ford for stimulating discussion and advise.

References

Mantyla, A, and J. Reid, 1983, Abyssal characteristics of the world ocean waters, *Deep-Sea Res.*, 30, 805-833.

Needler, G.T., 1971, Thermocline models with arbitrary barotropic flow, *Deep-Sea Res.*, 18, 895-903.

Salmon, R, 1986, A simplified linear ocean circulation theory, *J. Mar. Res.*, 44, 695-711.

Salmon, R, 1990, The thermocaline as an "internal boundary layer," *J. Mar. Res.* 48, 437-469.

Salmon, R., and R. Hollerbach, 1991, Similarity solutions of the thermocline equations, *J. Mar. Res.*, 49,249-280.

Salmon, R., 1992, A two-layer Gulf Stream over a continental slope, *J. Mar. Res.*, 50, 341-365.

Salmon, R., 1994, Generalized two-layer models of ocean circulation, *J. Mar. Res.*, 52, 865-908.

Taft, B.A., S.P. Hayes, G.E. Friederich, and L.A. Codispoti, 1991, Flow of abyssal water into the Samoa Passage. *Deep-Sea Res.*, 38, suppl. 1, S103-128.

Welander, P., 1971, Some exact solutions to the equations describing an ideal-fluid thermocline. *J. Mar. Res.*, 19, 60-68.