Rossby Waves over a Lattice of Different Seamounts

M. C. Hendershott

Scripps Institution of Oceanography 0209, La Jolla CA 92093

Abstract. Topographic (β=0) Rossby waves over an infinite rectangular lattice of identical seamounts have the form of plane waves propagating in frequency bands close to the natural frequencies of the topographic Rossby waves trapped around one seamount in isolation (Rhines). When the height of the seamounts varies randomly and the seamounts are separated by more than a few seamount radii, then the topographic Rossby wave field is described by a model first introduced by Anderson in another context. In this model, randomness in seamount height converts the extended plane waves into localized modes whose horizontal scale of energy trapping decreases with increasing disorder in seamount height. The numerical results of Mackinnon and Kramer are used to quantify this relationship.

1. Introduction

The ability of Rossby waves to transmit low frequency energy over long distances in the ocean is important directly or implicitly in most quasianalytical theories of the large scale circulation as well as in the interpretation of results obtained from numerical models of ocean circulation. Our qualitative ideas about such waves have grown largely out of analytical or quasianalytical solutions of the linearized shallow water equations or of the quasigeostrophic equations over ocean bottom relief that is either flat or has some very simple form which makes analysis feasible. But basin scale modes and much smaller scale topographic modes can easily have very similar frequencies (Miller, 1986). This observation raises the question of whether and/or when Rossby modes computed over smoothed relief would have recognizable counterparts over realistic relief.

An important aspect of this question has been addressed for surface gravity waves by Devillard et al. (1989), who found that for bottom relief and waves varying in only one direction, any amount of random variation in the underlying bottom relief changed the travelling waves into standing modes spatially localized over a horizontal scale that became ever larger as the random variation of the relief was decreased. The present calculation establishes a similar result for two dimensional topographic Rossby waves.

Even when β=0, Rhines (1970) has shown that energy transmission over unlimited distances via traveling topographic Rossby waves can occur if the bottom relief is periodic in space. It is shown below that if a collection of identical seamounts arrayed on a periodic lattice and separated by more than a few seamount radii is randomly perturbed in seamount height, then these traveling topographic Rossby waves extending over the entire horizontal plane are changed into spatially standing modes localized over a horizontal scale that becomes ever smaller as the random variation of the relief is increased. This is accomplished by transforming the Rossby wave problem into a form previously investigated by Anderson (1958) in another context, and then making use of quantitative numerical studies of the horizontal scale of localization for this model (MacKinnon and Kramer, 1983). The salient results are summarized in Figure (1).

2. Topographic Rossby waves over a Lattice of Seamounts

Over a single isolated seamount of radius $\hat{a}_l$ that is centered at $r_l = (x_l, y_l)$ and whose relief $D(x,y) = D\,(1-h(x,y))$ varies only for $|r-r_l| < \hat{a}_l$ so that

$$h(x,y) = \begin{cases} P_l(|r-r_l|) & |r-r_l| < \hat{a}_l \\ 0 & |r-r_l| > \hat{a}_l \end{cases} \tag{2.1}$$

the streamfunction ψ_{lK} describing purely topographic (β=0) seamount oscillations with natural frequency σ_{lK} satisfies

$$-i\sigma_{lK}\nabla^2\psi_{lK} + J(\psi_{lK}, f_0 P_l) = 0 \tag{2.2}$$

The subscript l indexes the seamount (in preparation for the case of many seamounts) while the subscript K orders modes over the seamount. From (2.2) the sign of σ_{lK} changes with the sign of the relief; $\sigma_{lK} < 0$ for hills.

In cylindrical coordinates r,θ centered at r_l, the eigenfunctions over circularly symmetric relief have the separated form

$$\psi = \begin{cases} e^{in\Theta}\,\varphi_{lK}(r) & r<\hat{a}_l \\ e^{in\Theta}\,\varphi_{lK}(\hat{a}_l)(\hat{a}_l/r)^n & r>\hat{a}_l \end{cases} \tag{2.3}$$

n=1,2,3 ... . The index K is usually a pair of integers indexing variation of ψ_{lK} in the θ,r directions. Thus over the particular relief

$$h(x,y) = \begin{cases} h_0(\,1-(r/\hat{a}_l)^2\,) & r<\hat{a}_l \\ 0 & r>\hat{a}_l \end{cases} \tag{2.4}$$

the eigenfunctions ψ_{lK} are of the form

$$\psi = \begin{cases} A\,e^{in\Theta}\,J_n(\alpha_{nm}r) & r<\hat{a}_l \\ A\,e^{in\Theta}\,J_n(\alpha_{nm}\hat{a}_l)(\hat{a}_l/r)^n & r>\hat{a}_l \end{cases} \tag{2.5}$$

in which A is a constant and the eigenfrequencies σ_{lK} are given by

$$\sigma_{l,nm} = -2nh_l f_0/(\alpha_{nm}\hat{a}_l)^2$$
$$J_{n-1}(\alpha_{nm}\hat{a}_l) = 0 \qquad n,m = 1,2,3 \tag{2.6}$$

The index K in (2.3) is the pair n,m in (2.6). Seamount oscillations over the tophat relief

$$h(x,y) = \begin{cases} h_l & r<\hat{a}_l \\ 0 & r>\hat{a}_l \end{cases} \tag{2.7}$$

are of the form

$$\psi = \begin{cases} e^{in\Theta}(r/\hat{a}_l)^n & r<\hat{a}_l, \quad A=(4n\pi)^{-1/2} \\ e^{in\Theta}(\hat{a}_l/r)^n & r>\hat{a}_l, \quad n=1,2,3... \end{cases} \tag{2.8}$$

and are degenerate in that all modes of angular wavenumber n have the same radial variation, and the eigenfrequency

$$\sigma_{lK} = -f_0 h_l/2 \tag{2.9}$$

is the same for all the modes. Nonetheless the tophat modes will be found useful because their simplicity facilitates estimation of the coupling between seamounts; away from the seamount their dependance on r,θ is the same as in the general case (2.3).

Over isolated seamount l, of arbitrary shape, eigenfunctions ψ_{lK} satisfy the orthogonality relationships

$$\overline{\psi^*_{lL}J(\psi_{lK},f_0P_l)} = -i\sigma_{lK}\,\overline{\nabla\psi^*_{lL}\cdot\nabla\psi_{lK}} = -i\sigma_{lK}\delta_{KL}$$
$$\overline{\psi_{lL}J(\psi_{lK},f_0P_l)} = -i\sigma_{lK}\,\overline{\nabla\psi_{lL}\cdot\nabla\psi_{lK}} = 0 \tag{2.10}$$

in which the overbar here and everafter indicates integration over the entire horizontal plane. The constant A in (2.8) has been chosen in accord with (2.10).

For every eigenfunction ψ_{lK} with frequency σ_{lK} there is another eigenfunction ψ_{lK}* with eigenfrequency -σ_{lK}. A given eigensolution and its complex conjugate are indistinguishable when free but respond differently to forcing. This is most obvious in the case where clockwise rotation of phases about a hill necessarily forces counterclockwise rotation of phases about any neighboring hill, a sense of rotation of phases opposite to that associated with free seamount oscillations at the neighboring hill.

Now let the relief be an ensemble of distinct seamounts:

$$h(x,y) = \sum_s P_s(|r-r_s|) \tag{2.11}$$

Expand the total streamfunction ψ(x,y,t) over this relief in individual seamount eigenfunctions:

$$\psi(x,y,t) = \sum_{lK}\,[a_{lK}(t)\psi_{lK}(x,y) + b_{lK}(t)\psi^*_{lK}(x,y)] \tag{2.12}$$

The total energy E is

$$2E = \rho\overline{\nabla\psi\cdot\nabla\psi^*} = \rho\sum_{lK,nM} a^*_{nM}a_{lK}\,\overline{\nabla\psi^*_{nM}\cdot\nabla\psi_{lK}} + ... \tag{2.13}$$

in which ρ is the mass per unit surface area of the fluid layer. With the normalization of (2.10), $|a_{lK}|^2$ and $|b_{lK}{}^2|$ are the energies associated with ψ_{lK} and ψ_{lK}* in isolation. Insert the expansion (2.12) into the governing equation

$$\partial\nabla^2\psi/\partial t + J(\psi,f_0h) = 0 \tag{2.14}$$

for linearized nondivergent quasigeostrophic flow over relief h given by (2.11) to obtain

$$\sum_{lK} [\![\dot{a}_{lK} \nabla^2 \psi_{lK} + \dot{b}_{lK} \nabla^2 \psi^*_{lK} + a_{lK} \sum_s J(\psi_{lK}, f_0 P_s) + b_{lK} \sum_s J(\psi^*_{lK}, f_0 P_s)]\!] = 0 \tag{2.15}$$

Multiplying (2.15) by ψ_{0M} and ψ_{0M} * separately and then integrating the resulting equations over the horizontal plane yields the following coupled ode's for the amplitudes a_{lK} and b_{lK}:

$$\begin{aligned}
&\overline{\nabla\psi^*_{0M} \cdot \nabla\psi_{0M}}\, \dot{a}_{0M} + \sum_K \sum_{l\neq 0} [\overline{\nabla\psi^*_{0M} \cdot \nabla\psi_{lK}}]\, \dot{a}_{lK} \\
&+ \overline{\nabla\psi^*_{0M} \cdot \nabla\psi^*_{0M}}\, \dot{b}_{0M} + \sum_K \sum_{l\neq 0} [\overline{\nabla\psi^*_{0M} \cdot \nabla\psi^*_{lK}}]\, \dot{b}_{lK} = \\
&(D^{M*M}_{000} + \sum_{s\neq 0} D^{M*M}_{00s})\, a_{0M} + \sum_K \sum_{l\neq 0} [D^{M*K}_{0l0} + D^{M*K}_{0ll}]\, a_{lK} \\
&+ \sum_K \sum_{l\neq 0} \sum_{s\neq 0} (D^{M*K}_{0ls})\, a_{lK} + [D^{M*M*}_{000} + \sum_{s\neq 0} D^{M*M*}_{00s}]\, b_{0M} \\
&+ \sum_K \sum_{l\neq 0} (D^{M*K*}_{0l0} + D^{M*K*}_{0ll})\, b_{lK} + \sum_K \sum_{l\neq 0} \sum_{s\neq 0,l} [D^{M*K*}_{0ls}]\, b_{lK}
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
&[\overline{\nabla\psi_{0M} \cdot \nabla\psi_{0M}}]\, \dot{a}_{0M} + \sum_K \sum_{l\neq 0} \overline{\nabla\psi_{0M} \cdot \nabla\psi_{lK}}\, \dot{a}_{lK} \\
&+ \overline{\nabla\psi_{0M} \cdot \nabla\psi^*_{0M}}\, \dot{b}_{0M} + \sum_K \sum_{l\neq 0} [\overline{\nabla\psi_{0M} \cdot \nabla\psi^*_{lK}}]\, \dot{b}_{lK} = \\
&[D^{MM}_{000} + \sum_{s\neq 0} D^{MM}_{00s}]\, a_{0M} + \sum_K \sum_{l\neq 0} (D^{MK}_{0l0} + D^{MK}_{0ll})\, a_{lK} \\
&+ \sum_K \sum_{l\neq 0} \sum_{s\neq 0,l} [D^{MK}_{0ls}]\, a_{lK} + (D^{MM*}_{000} + \sum_{s\neq 0} D^{MM*}_{00s})\, b_{0M} \\
&+ \sum_K \sum_{l\neq 0} [D^{MK*}_{0l0} + D^{MK*}_{0ll}]\, b_{lK} + \sum_K \sum_{l\neq 0} \sum_{s\neq 0,l} (D^{MK*}_{0ls})\, b_{lK}
\end{aligned} \tag{2.17}$$

In (2.16) and (2.17) terms refering to the seamount labeled 0 have been isolated, and the notation

$$D^{M*K}_{0ls} = \overline{\psi^*_{0M} J(\psi_{lK}, f_0 P_s)} \tag{2.18}$$

has been introduced. In (2.16) and (2.17) only, the square brackets [] enclose terms which are found upon detailed calculation to vanish. The remaining nonzero terms may be grouped as follows:

$$S_{0M} = \sum_{s=0} i D^{M*M}_{00s} \tag{2.19}$$

$$\phi^{MK}_{0l} = \sum_{s\neq 0,l} i\, D^{M*K}_{0ls} \tag{2.20}$$

$$\Delta^{MK}_{0l} = \overline{\nabla\psi_{0M} \cdot \nabla\psi_{lK}} \tag{2.21}$$

and we have the particular values

$$\begin{aligned}
&\overline{\nabla\psi^*_{0M} \cdot \nabla\psi_{0M}} = 1 \\
&D^{M*M}_{000} = -i\sigma_{0M} \overline{\nabla\psi^*_{0M} \cdot \nabla\psi_{0M}} = 1, \\
&D^{M*K*}_{0ll} + D^{M*K*}_{0l0} = -i(\sigma_{0M} - \sigma_{lK})(\Delta^{MK}_{0l})^* \\
&\Delta^{MK}_{00} = 0, \qquad D^{M*M}_{000} = -i\sigma_{0M}
\end{aligned} \tag{2.22}$$

S_{0M} in (2.19) is a small correction to the eigenfrequency σ_{0M} of seamount oscillation ψ_{0M} over seamount 0 due to the presence of all the other seamounts. ϕ_{0l}^{MK} in (2.20) is the interaction between eigensolution pairs at sites 0 and l because of their overlap with all other seamounts. Δ_{0l}^{MK} in (2.21) is the direct interaction between complex conjugate eigensolution pairs at sites 0 and l.

With the introduction of (2.19) - (2.22), (2.16) - (2.17) become

$$\begin{aligned}
&i\dot{a}_{0M} + i \sum_K \sum_{l\neq 0} (\Delta^{MK}_{0l})^*\, \dot{b}_{lK} = (\sigma_{0M} + S_{0M})\, a_{0M} \\
&+ \sum_K \sum_{l\neq 0} (\phi^{MK}_{0l})\, a_{lK} + \sum_K \sum_{l\neq 0} (\sigma_{0M} - \sigma_{lK})(\Delta^{MK}_{0l})^*\, b_{lK}
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
&i\dot{b}_{0M} + i \sum_K \sum_{l\neq 0} (\Delta^{MK}_{0l})\, \dot{a}_{lK} = -(\sigma_{0M} + S_{0M})\, b_{0M} \\
&- \sum_K \sum_{l\neq 0} (\phi^{MK}_{0l})^*\, b_{lK} - \sum_K \sum_{l\neq 0} (\sigma_{0M} - \sigma_{lK})(\Delta^{MK}_{0l})\, a_{lK}
\end{aligned} \tag{2.24}$$

3. Topographic Rossby waves over a Lattice of Randomly Differing Seamounts

Rhines (1970) has shown that if all the seamounts are identical and they are arrayed in a rectangular lattice, then even in the absence of β, energy initially in topographic Rossby waves can propagate indefinitely. If the distance R between seamounts somewhat exceeds the seamount radius â, then detailed calculations (some of which are shown subsequently) show that the correction and coupling terms (2.19)-(2.21) decay as positive powers of a/R. If we label lattice positions by x,y (in units of R), then the solutions found by Rhines (1970) have the form

$$\begin{aligned} a_{xy,M} &= e^{i(\mu x + \nu y - \sigma(\mu,\nu))} \\ \sigma(\mu,\nu) &\approx \sigma_{0M}[1 + O(a/R)^{2n+2}(\cos(\mu x) + \cos(\nu y))] \end{aligned} \tag{3.1}$$

where n is the angular order of an individual seamount oscillation with frequency σ_{0M}. Plane waves thus propagate within frequency bands that are centered about the frequencies σ_{0M} of isolated seamount oscillations and have widths the order of $\sigma_{0M}(a/R)^{2n+1}$.

The question now to be addressed is 'what happens to these propagating topographic modes when the seamounts are of slightly differing heights h_l?' We thus consider a periodic lattice of seamounts whose individual seamount oscillation frequencies are randomly distributed about a central value σ_*. The width of this distribution will be supposed to be small relative to the size of σ_*. For motions at a fixed frequency ω, (2.23)-(2.24) may be compactly written as

$$\left[\begin{bmatrix}\sigma & 0\\ 0 & -\sigma\end{bmatrix} + \begin{bmatrix}S & 0\\ 0 & -S\end{bmatrix} + \begin{bmatrix}\phi & 0\\ 0 & -\phi^*\end{bmatrix}\right]\begin{bmatrix}a\\ b\end{bmatrix} = $$
$$+\begin{bmatrix}0 & \sigma\Delta^* - \Delta^*\sigma\\ \Delta\sigma - \sigma\Delta & 0\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \omega\begin{bmatrix}I & \Delta^*\\ \Delta & I\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} \tag{3.2}$$

in which σ is the diagonal matrix of individual site eigenfrequencies σ_{0M}, and S, ϕ, and Δ are matrices whose elements are given by (2.19)-(2.21).

As long as the disorder in seamount height is sufficiently small that individual seamount oscillation frequencies associated with different angular or radial orders do not overlap, we may retain but one mode per site (typically the mode having the smallest angular wavenumber n=1 and consequently the greatest frequency). The elements of S, ϕ, and Δ may be shown to decay as positive powers of (a/R), so that (3.2) may be simplified by left multiplication by the factor

$$\begin{bmatrix}I & \Delta^*\\ \Delta & I\end{bmatrix}^{-1} = \begin{bmatrix}(1-\Delta^*\Delta)^{-1} & -(1-\Delta^*\Delta)^{-1}\Delta^*\\ -(1-\Delta^*\Delta)^{-1}\Delta & (1-\Delta^*\Delta)^{-1}\end{bmatrix}$$
$$\simeq \begin{bmatrix}(1+\Delta^*\Delta) & -\Delta^*\\ -\Delta & (1+\Delta^*\Delta)\end{bmatrix} \tag{3.3}$$

The result is

$$\begin{bmatrix}(\sigma + S + \phi + \Delta^*\sigma\Delta) & \sigma\Delta^*\\ -\sigma\Delta & -(\sigma + S + \phi^* + \Delta\sigma\Delta^*)\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \omega\begin{bmatrix}a\\ b\end{bmatrix} \tag{3.4}$$

Here, terms such as $\Delta\Delta^*\Delta$ have been neglected relative to terms such as $\Delta\Delta^*$, and terms such as Δ^*S and $\Delta^*\phi$ have been neglected relative to terms such as $\Delta^*\sigma$.

Now we consider motions at a frequency ω that is close to the natural frequencies σ_0 of individual seamounts in the ensemble (the single subscript now indexes seamounts). The dominant terms of the diagonal elements of (3.3) are thus the diagonal matrix σ, so that the linear equations for a may be close to resonance but those for b are far from resonance. Consequently we may take

$$b \approx -(\omega I + \sigma)^{-1}\sigma\Delta a \tag{3.5}$$

Hence (3.3) finally becomes

$$\begin{aligned} \omega I a &= (\sigma + S)a + Va \\ V &= (\phi + \Delta^*\sigma\Delta - \sigma\Delta^*(\omega I + \sigma)^{-1}\sigma\Delta) \\ &\approx (\phi + \sigma_*\Delta^*\Delta/2) \end{aligned} \tag{3.6}$$

in which σ_* denotes the average of the individual free seamount oscillation frequencies σ_0. The terms neglected in (3.5) would contribute minor modifications to V and S in (3.6).

We now evaluate the elements of ϕ and of $\Delta^*\Delta$ appearing in (3.6). We have

$$\begin{aligned} \phi_{0l} &= \sum_{s\neq 0,l} \overline{i\psi_0^* J(\psi_l^*, f_0 h_s)} = \sum_{s\neq 0,l} \overline{-if_0 h_s J(\psi_l^*, \psi_0^*)} \\ &\approx \sum_{s\neq 0,l} (-\sigma_s \hat{a}_s^2 \hat{a}_0 \hat{a}_l) \frac{1}{(x_s - x_l - i(y_s - y_l))^2} \frac{1}{(x_s - x_0 + i(y_s - y_0))^2} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \Delta_{0l}^* &= \overline{\nabla\psi_0^*\nabla\psi_l^*} = -(i/\sigma_l)\overline{\psi_0^* J(\psi_l^*, f_0 h_l)} \\ &= (i/\sigma_l)\overline{f_0 h_l J(\psi_l^*, \psi_0^*)} = -\hat{a}_0\hat{a}_l \frac{1}{(x_0 - x_l + i(y_0 - y_l))} \end{aligned} \tag{3.8}$$

The evaluation of V in (3.6) thus requires evaluation of the sum

$$S_{0l}(x,y) = \sum_{x,y\neq x_0,x_l} \frac{1}{(x - x_0 - i(y - y_0))^2} \frac{1}{(x - x_l - i(y - y_l))^2} \tag{3.9}$$

This will be illustrated for the particular case $y_0 = y_l = 0$. It is convenient to write (3.9) as

$$S_{0l} = R^{-4} S_{\mu\nu} = R^{-4} \sum_{n,m \neq \nu,0;\mu,0} \frac{1}{(n-\nu-im)^2} \frac{1}{(n-\mu+im)^2} \tag{3.10}$$

Note that

$$S_{\nu\mu} = \frac{d}{d\nu}\frac{d}{d\nu}\sum \frac{1}{(n-\nu-im)}\frac{1}{n-\mu+im)}$$

$$= \frac{d}{d\nu}\frac{d}{d\nu}\sum \left[\left(\frac{n\ \nu}{(n-\nu)^2+m^2} - \frac{n-\mu}{(n\ \mu)^2+m^2} \right) \left(\frac{\nu-\mu\ 2im}{(\nu-\mu)^2+4m^2} \right) + \left(\frac{1}{(n-\nu)^2+m^2} - \frac{1}{(n\ \mu)^2+m^2} \right) \left(\frac{2im}{\nu-\mu+2im} \right) \right] \tag{3.11}$$

Since the summation indices exclude n,m $=$ υ,0; μ,0, the very last sum in (3.11) may be written as

$$\sum_m \left(\frac{2im}{\nu-\mu+2im} \right) \left(\sum_{all\, n} \frac{1}{(n\ \mu)^2+m^2} - \frac{1}{m^2} - \frac{1}{(\nu-\mu)^2+m^2} \right) \tag{3.12}$$

In this form it is clear that this sum exactly equals the next to last sum in (3.11) so that the last two terms of (3.11) cancel. The imaginary part of the summand in first two sums in (3.11) is odd in m and hence the corresponding sum vanishes. The remaining summands are odd in n-ν and n-μ. Since the corresponding sums exclude n,m = ν,0; μ,0, (3.11) may finally be written as

$$\frac{d}{d\nu}\frac{d}{d\mu}\frac{2}{(\mu-\nu)^2} = \frac{12}{(\mu-\nu)^4} \tag{3.13}$$

so that the sum (3.10) is

$$S_{0l} = \frac{12}{R^4(\mu-\nu)^4} = \frac{12}{(x_0-x_l)^4} \tag{3.14}$$

On account of the rapid decrease of S_{0l} with site separation, we truncate the sum over all seamounts that is implicit in the matrix equation (3.6) to a sum over nearest geographical neighboring sites, and so finally obtain an approximate version of (3.6);

$$\omega a_0 = (\sigma_0 + S_0)a_0 + V(a_e + a_w + a_n + a_s) \tag{3.15}$$
$$V = 6\sigma_*(a/R)^4$$

now notated in an obvious geographical manner in which site e is east of site o ... .

4. The Anderson Model

(3.15) has the form of model employed by Anderson (1958) to discuss the nature of single electron wavefunctions in a spatially random potential. In the numerical study of MacKinnon and Kramer (1983) that model is studied in the form

$$(\omega - \sigma_*)a_0 = (\sigma_0 + S_0 - \sigma_*)a_0 + V(a_e + a_w + a_n + a_s) \tag{4.1}$$

The effective individual site frequencies σ_0+S_0 are supposed randomly distributed over the range $\sigma_*\pm\Delta\sigma$. The interaction potential V is taken constant. It is convenient to discuss the results in terms of a disorder parameter W defined by

$$W = \Delta\sigma/V = \frac{\Delta\sigma}{\sigma_*}\frac{1}{6(a/R)^4} \tag{4.2}$$

Over a given relief, the topographic Rossby normal modes and their natural frequencies could in principle be found by solving the eigenvalue problem (3.6) or its approximate form (4.1). But we wish to study the properties of normal modes over an ensemble of reliefs characterized by $\Delta\sigma$ or, equivalently, W. What can be done is most simply illustrated for the case of a one dimensional chain of seamounts labeled i=1,2, ... , for which (4.1) becomes

$$(\omega - \sigma_*)a_i = (\sigma_i + S_i - \sigma_*)a_i + V(a_{i+1} + a_{i-1}) \tag{4.3}$$

Rather than considering the eigenvalue problem, as part of whose solution the natural frequencies ω are determined, we fix the frequency ω near σ_*, specify a_1 and a_2, and solve for a_i, i=3,4,5 ... . If the individual site frequencies $q+S$ were all the same, then the solution would be a plane wave whose amplitude would neither grow or decay along the chain. But if the normal modes of the chain are all evansescent, then solutions of this 'Cauchy' problem with σ_i+S_i specified randomly within $\sigma_*\pm\Delta\sigma$ will ultimately grow exponentially along the chain for almost any initial values a_1 and a_2. Such exponential growth characterizes not only solutions of the one dimensional problem, but also solutions of the two dimensional problem of a long 'bar' of lattice sites. Numerical estimates of the growth rate must necessarily be made by solving such a Cauchy problem for bars of finite width; MacKinnon and Kramer (1983) carry out such studies and explain how the results are to be extrapolated to a bar of infinite width.

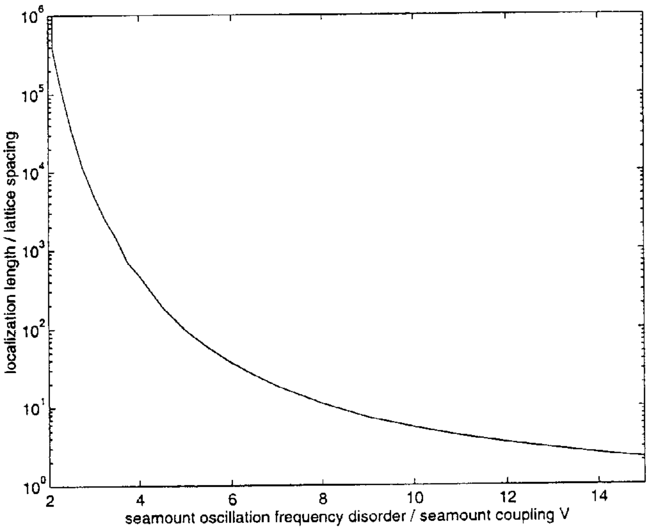

Figure 1. Localization length ξ/R (units of lattice spacing R) vs disorder $W = (\Delta\sigma/\sigma_*)/|6(a/R)^{4}|$.

They find that the exponential growth scale or 'localization length' ξ (units of lattice spacing R) increases as W is decreased. Their numerical results are summarized in Figure (1). Thus if, for example, the ratio $\hat{a}/R$ of seamount radius to lattice spacing is 1/4, then a disorder $\Delta\sigma/\sigma_*$ of about 25% leads to a localization length of about five lattice spacings. If the lattice spacing is increased so that $\hat{a}/R$ is 1/6, then a disorder of about 4% results in a similar localization length. Identification of this exponential growth rate with the localization scale of the modes over the full lattice is discussed in the one dimensional case by Crisanti et al. (1993; the numerical results of MacKinnon and Kramer (1983) in two dimensions correspond to the largest possible horizontal localization scale of the modes over the full lattice.

5. Discussion

The Anderson model (4.1) was the result of supposing that the topographic Rossby modes over the entire seamount lattice are primarily composed only of individual seamount oscillations of angular order n=1. If the seamounts are not too dissimilar, then there will also be topograhic Rossby modes over the full lattice that are primarily composed only of individual seamount oscillations of angular order n=2, 3 ... as well. But for sufficiently high angular order or sufficiently great seamount disorder, indivdual seamount oscillations of different angular orders at different seamounts may have very similar frequencies, so that coupling between different angular orders can no longer be neglected. Thus, although the foregoing analysis shows that the highest frequency lattice modes are localized by disorder in seamount height, the assumption of small seamount height disorder made in going from (3.2) to (4.1) is violated at sufficiently small frequencies. The ultimate consequences of this are not presently understood.

The observation that the degree of localization induced by a given disorder in seamount frequency increases as the coupling V becomes smaller indicates that the inclusion of stratification or a free surface would have increased the degree of localization because individual seamount oscillations would then decay exponentially away from lattice sites.

The foregoing results suppose that β=0. If all the seamounts are identical save one and if the natural frequency of seamount oscillations of, say, angular order n=1 about that seamount in isolation falls outside the bands within which plane topographic Rossby waves propagate over the lattice of identical seamounts, then a disturbance initially localized at that seamount will not ultimately propagate entirely away. It may be shown that this continues to be true for sufficiently small but nonzero β. This suggests, but does not prove, a similar result for more general relief.

Acknowledgements. A. Miller and R. Salmon provided helpful discussions. C. D. Winant helped me with the preparation of the manuscript. This work was supported by the National Science Foundation, grant OCE88-00080.

REFERENCES

Anderson, P. W., 1958: Absence of Diffusion in Certain random Lattices. *Phys. Rev.*, **109**, 1492-1505.

Crisanti, A., G. Paladin and A. Vulpiani, 1993. Products of Random Matrices. Springer-Verlag, 161 pp.

Devillard, P, F. Dunlop and B. Souillard, 1988. *J. Fluid Mech.*, **186**, 521-538.

MacKinnon, A. and B. Kramer, 1983: The Scaling Theory of Electrons in Disordered Solids: Additional Numerical Results. *Zeit. für Physik*, **53**, 1-13.

Miller, A. M. , 1986. Nov-divergent planetary oscillations in mid-latitude ocean basins with continental shelves. *J. Phys. Oceanogr.*, **16**, 1914-1928.

Rhines, P. B., 1970: Wave Propagation in a Periodic Medium with Application to the Ocean. *Rev. Geophys. Space Phy.*, **8**, 303-319.